

# Extension of Lipschitz Functions Defined on Metric Subspaces of Homogeneous Type

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## ABSTRACT

If a metric subspace  $M^o$  of an arbitrary metric space  $M$  carries a doubling measure  $\mu$ , then there is a simultaneous linear extension of all Lipschitz functions on  $M^o$  ranged in a Banach space to those on  $M$ . Moreover, the norm of this linear operator is controlled by logarithm of the doubling constant of  $\mu$ .

*Key words:* metric space of homogeneous type, Lipschitz function, linear extension.

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## 1. Formulation of the main result

Let  $(M, d)$  be a metric space and  $X$  be a Banach space. The space  $\text{Lip}(M, X)$  consists of all  $X$ -valued Lipschitz functions on  $M$ . The Lipschitz constant

$$L(f) := \sup_{m \neq m'} \left\{ \frac{\|f(m) - f(m')\|}{d(m, m')} : m, m' \in M \right\}$$

of a function  $f$  from this space is therefore finite and the function  $f \mapsto L(f)$  is a Banach seminorm on  $\text{Lip}(M, X)$ .

Let  $M^o$  be a metric subspace of  $M$ , i.e.,  $M^o \subset M$  is a metric space endowed with the induced metric  $d|_{M^o \times M^o}$ .

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**Convention.** We mark all objects related to the subspace  $M^o$  by the upper “o”.

A linear operator  $E : \text{Lip}(M^o, X) \rightarrow \text{Lip}(M, X)$  is called a *simultaneous extension* if for all  $f \in \text{Lip}(M^o, X)$

$$Ef|_{M^o} = f$$

and, moreover, the norm

$$\|E\| := \sup \left\{ \frac{L(Ef)}{L(f)} : f \in \text{Lip}(M^o, X) \right\}$$

is finite.

To formulate the main result we also need

**Definition 1.1.** A Borel measure  $\mu$  on a metric space  $(M, d)$  is said to be doubling if the  $\mu$ -measure of every open ball

$$B_R(m) := \{ m' \in M : d(m, m') < R \}$$

is strictly positive and finite and the doubling constant

$$D(\mu) := \sup \left\{ \frac{\mu(B_{2R}(m))}{\mu(B_R(m))} : m \in M, R > 0 \right\} \tag{1}$$

is finite.

A metric space carrying a fixed doubling measure is called of homogeneous type.

Our main result is

**Theorem 1.2.** *Let  $M^o$  be a metric subspace of an arbitrary metric space  $(M, d)$ . Assume that  $(M^o, d^o)$  is of homogeneous type and  $\mu^o$  is the corresponding doubling measure. Then there exists a simultaneous extension  $E : \text{Lip}(M^o, X) \rightarrow \text{Lip}(M, X)$  satisfying*

$$\|E\| \leq c(\log_2 D(\mu^o) + 1) \tag{2}$$

with some numerical constant  $c > 1$ .

Let us discuss relations of this theorem to some known results. First, a similar result holds for an arbitrary subspace  $M^o$  provided that the ambient space  $M$  is of *pointwise homogeneous type*, see [1, Theorem 2.21; 2, Theorem 1.14]. The class of metric spaces of pointwise homogeneous type contains, in particular, all metric spaces of homogeneous type, Riemannian manifolds  $M_\omega \cong \mathbb{R}^n \times \mathbb{R}_+$  with the path metric defined by the Riemannian metric

$$ds^2 := \omega(x_{n+1})(dx_1^2 + \dots + dx_{n+1}^2), \quad (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous nonincreasing function (e.g., the hyperbolic spaces  $\mathbb{H}^n$  are in this class), and finite direct products of these objects.

The following problem is of a considerable interest.

**Problem 1.3.** *Is it true that Theorem 1.2 is valid for  $M^o (\subset M)$  isometric to a subspace of a metric space  $(\hat{M}, \hat{d})$  of pointwise homogeneous type with  $\|E\| \leq c(\hat{M})$ ? (Here  $c(\hat{M})$  depends on some characteristics of  $\hat{M}$  only.)*

It is proved in [2] that as such  $M^o$  one can take, e.g., finite direct products of Gromov hyperbolic spaces of bounded geometry and that the answer in Problem 1.3 is positive in this case.

Second, as a consequence of Theorem 1.2 we obtain a deep extension result due to Lee and Naor, see [4, Theorem 1.6]. The latter asserts that a simultaneous extension  $E : \text{Lip}(M^o, X) \rightarrow \text{Lip}(M, X)$  exists whenever the subspace  $(M^o, d^o)$  of  $(M, d)$  has the finite doubling constant  $\delta(M^o)$  and, moreover,

$$\|E\| \leq c \log_2 \delta(M^o)$$

with some numerical constant  $c > 1$ .

Let us recall that the doubling constant  $\delta(M)$  of a metric space  $(M, d)$  is the infimum of integers  $N$  such that every closed ball of  $M$  of radius  $R$  can be covered by  $N$  closed balls of radius  $R/2$ . The space  $M$  is said to be *doubling* if  $\delta(M) < \infty$ .

To derive the Lee-Naor theorem from our main result we first note that without loss of generality one may assume that  $(M^o, d^o)$  is complete. By the Koniagin-Vol'berg theorem [6] (see also [5]) a complete doubling space  $M$  carries a doubling measure  $\mu$  such that

$$\log_2 D(\mu) \leq c \log_2 \delta(M)$$

where  $c \geq 1$  is a numerical constant. Together with (2) this implies the Naor-Lee result.

On the other hand, it was noted in [3] that if  $M$  carries a doubling measure  $\mu$ , then this space is doubling and

$$\log_2 \delta(M) \leq c \log_2 D(\mu) \tag{3}$$

with some numerical constant  $c > 1$ . Hence, Theorem 1.2 is, in turn, a consequence of (3) and the Lee-Naor theorem. However, the rather elaborated proof of the latter result is nonconstructive. (It exploits an appropriate stochastic metric decomposition of  $M \setminus M^o$ .) In contrast, our proof is constructive and is based on a simple average procedure. Therefore our proof can be also seen as a streamlining constructive method of the proof of the Lee-Naor theorem.

## 2. Proof of Theorem 1.2

We begin with the following remark reducing the required result to a special case.

Let  $M$  and  $M^o$  be isometric to subspaces of a new metric space  $\hat{M}$  and its subspace  $\hat{M}^o$ , respectively. Assume that there exists a simultaneous extension  $\hat{E} :$

$\text{Lip}(\hat{M}^\circ, X) \rightarrow \text{Lip}(\hat{M}, X)$ . Then, after identification of  $M^\circ$  and  $M$  with the corresponding isometric subspaces of  $\hat{M}$ , the operator  $\hat{E}$  gives rise to a simultaneous extension  $E : \text{Lip}(M^\circ, X) \rightarrow \text{Lip}(M, X)$  satisfying

$$\|E\| \leq \|\hat{E}\|.$$

If, in addition,  $\|\hat{E}\|$  is bounded by the right-hand side of (2), then the desired result immediately follows.

We choose as the above pair  $\hat{M}^\circ \subset \hat{M}$  metric spaces denoted by  $M_N^\circ$  and  $M_N$  where  $N \geq 1$  is a fixed integer and defined as follows.

The underlying sets of these spaces are

$$M_N := M \times \mathbb{R}^N, \quad M_N^\circ := M^\circ \times \mathbb{R}^N;$$

a metric  $d_N$  on  $M_N$  is given by

$$d_N((m, x), (m', x')) := d(m, m') + |x - x'|_1$$

where  $m, m' \in M$  and  $x, x' \in \mathbb{R}^N$ , and  $|x|_1 := \sum_{i=1}^N |x_i|$  is the  $l_1^N$ -metric of  $x \in \mathbb{R}^N$ . Further,  $d_N^\circ$  denotes the metric on  $M_N^\circ$  induced by  $d_N$ .

Finally, we define a Borel measure  $\mu_N^\circ$  on  $M_N^\circ$  as the tensor product of the measure  $\mu^\circ$  and the Lebesgue measure  $\lambda_N$  on  $\mathbb{R}^N$ :

$$\mu_N^\circ := \mu^\circ \otimes \lambda_N.$$

We extend this measure to the  $\sigma$ -algebra consisting of subsets  $S \subset M_N$  such that  $S \cap M_N^\circ$  is a Borel subset of  $M_N^\circ$ . Namely, we set for these  $S$

$$\bar{\mu}_N(S) := \mu_N^\circ(S \cap M_N^\circ).$$

It is important for the subsequent part of the proof that every open ball  $B_R((m, x)) \subset M_N$  belongs to this  $\sigma$ -algebra. In fact, its intersection with  $M_N^\circ$  is a Borel subset of this space, since the function  $(m', x') \mapsto d_N((m, x), (m', x'))$  is continuous on  $M_N^\circ$ . Hence,

$$\bar{\mu}_N(B_R((m, x))) = \mu_N^\circ(B_R((m, x)) \cap M_N^\circ). \tag{4}$$

**Auxiliary results.** The measure  $\mu_N^\circ$  is clearly doubling. Therefore its *dilation function* given for  $l \geq 1$  by

$$D_N^\circ(l) := \sup \left\{ \frac{\mu_N^\circ(B_{lR}^\circ(\hat{m}))}{\mu_N^\circ(B_R^\circ(\hat{m}))} : \hat{m} \in M_N^\circ \text{ and } R > 0 \right\}$$

is finite.

Hereafter we denote by  $\hat{m}$  the pair  $(m, x)$  with  $m \in M$  and  $x \in \mathbb{R}^N$ , and by  $B_R^\circ(\hat{m})$  the open ball in  $M_N^\circ$  centered at  $\hat{m} \in M_N^\circ$  and of radius  $R$ . The open ball  $B_R(\hat{m})$  of  $M_N$  relates to that by

$$B_R^\circ(\hat{m}) = B_R(\hat{m}) \cap M_N^\circ$$

provided  $\hat{m} \in M_N^o$ .

In [1] the value  $D_N^o(1 + 1/N)$  is proved to be bounded by some numerical constant for all sufficiently large  $N$ . In the argument presented below we require a similar estimate for a (modified) dilation function  $D_N$  for the extended measure  $\bar{\mu}_N$ . This is given for  $l \geq 1$  by

$$D_N(l) := \sup \left\{ \frac{\bar{\mu}_N(B_{lR}(\hat{m}))}{\bar{\mu}_N(B_R(\hat{m}))} \right\} \tag{5}$$

where the supremum is taken over all  $R$  satisfying

$$R > 4d(\hat{m}, M_N^o) := 4 \inf \{ d_N(\hat{m}, \hat{m}') : \hat{m}' \in M_N^o \} \tag{6}$$

and then over all  $\hat{m} \in M_N$ .

Due to (4) and (6) the denominator in (5) is not zero and  $D_N(l)$  is well defined.

Comparison of the above dilation functions shows that  $D_N^o(l) \leq D_N(l)$ . Nevertheless, the converse is also true for  $l$  close to 1.

**Lemma 2.1.** *Assume that  $N$  and the doubling constant  $D := D(\mu^o)$ , see (1), are related by*

$$N \geq [3 \log_2 D] + 5. \tag{7}$$

*Then the following is true:*

$$D_N(1 + 1/N) \leq \frac{6}{5} e^4.$$

*Proof.* In accordance with the definition of  $D_N$ , see (5), we must estimate the function

$$\frac{\bar{\mu}_N(B_{R_N}(\hat{m}))}{\bar{\mu}_N(B_R(\hat{m}))} \quad \text{where} \quad R_N := \left(1 + \frac{1}{N}\right)R. \tag{8}$$

Since the points  $\hat{m}'$  of the ball  $B_{R_N}(\hat{m})$  of  $M_N$  satisfy the inequality

$$d(m, m') + |x - x'|_1 < R_N,$$

the Fubini theorem and (4) yield

$$\bar{\mu}_N(B_{R_N}(\hat{m})) = \gamma_N \int_{M^o \cap B_{R_N}(m)} (R_N - d(m, m'))^N d\mu^o(m'). \tag{9}$$

Here  $\gamma_N$  is the volume of the unit  $l_1^N$ -ball.

We must estimate the integral in (9) from above under the condition

$$d_N(\hat{m}, M_N^o) < R/4. \tag{10}$$

To this end split the integral into one over  $B_{3R/4}(m) \cap M^o$  and one over the remaining part  $(B_{R_N}(m) \setminus B_{3R/4}(m)) \cap M^o$ . Denote these integrals by  $I_1$  and  $I_2$ . For  $I_2$  we get

$$I_2 \leq \gamma_N (R_N - 3R/4)^N \mu^o(B_{R_N}(m) \cap M^o). \tag{11}$$

Further, from (10) we clearly have

$$d(m, M^o) < R/4.$$

Pick a point  $\tilde{m} \in M^o$  so that

$$d(m, M^o) \leq d(m, \tilde{m}) < R/4.$$

Then we have the following embeddings

$$B_{R_N/4}^o(\tilde{m}) \subset B_{R_N/2}(m) \cap M^o \subset B_{R_N}(m) \cap M^o \subset B_{5R_N/4}^o(\tilde{m}).$$

Applying the doubling inequality for the measure  $\mu^o$ , see (1), we then obtain

$$\mu^o(B_{R_N}(m) \cap M^o) \leq D^3 \mu^o(B_{R_N/2}(m) \cap M^o).$$

Moreover, due to (7)

$$D^3 < 2^{[3 \log_2 D]+1} \leq 2^{N-4}.$$

Combining the last two inequalities with (11) we have

$$I_2 \leq \gamma_N 2^{-N-4} \left(1 + \frac{4}{N}\right)^N R^N \mu^o(B_{R_N/2}(m) \cap M^o). \tag{12}$$

To estimate the integral  $I_1$  we rewrite its integrand as follows:

$$(R_N - d(m, m'))^N = \left(1 + \frac{1}{N}\right)^N (R - d(m, m'))^N \left(1 + \frac{d(m, m')}{(N+1)(R - d(m, m'))}\right)^N.$$

Since  $m' \in B_{3/4R}(m)$ , the last factor is at most  $\left(1 + \frac{3R/4}{(N+1)R/4}\right)^N = \left(1 + \frac{3}{N+1}\right)^N$ . This yields

$$\begin{aligned} I_1 &\leq \gamma_N \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{3}{N+1}\right)^N \int_{B_{3R/4}(m) \cap M^o} (R - d(m, m'))^N d\mu^o(m') \\ &\leq e^4 \bar{\mu}_N(B_R(\hat{m})). \end{aligned}$$

Hence for the part of fraction (8) related to  $I_1$  we have

$$\tilde{I}_1 := \frac{I_1}{\bar{\mu}_N(B_R(\hat{m}))} \leq e^4. \tag{13}$$

To estimate the remaining part  $\tilde{I}_2 := \frac{I_2}{\bar{\mu}_N(B_R(\hat{m}))}$  we note that its denominator is greater than

$$\gamma_N \int_{M^o \cap B_{R_N/2}(m)} (R - d(m, m'))^N d\mu^o(m').$$

Since here  $d(m, m') \leq R_N/2$ , this, in turn, is bounded from below by

$$\gamma_N 2^{-N} \left(1 - \frac{1}{N}\right)^N R^N \mu^o(B_{R_N/2}(m) \cap M^o).$$

Combining this with (12) and noting that  $N \geq 5$  we get

$$\tilde{I}_2 \leq 2^{-4} \left(1 - \frac{1}{N}\right)^{-N} \left(1 + \frac{4}{N}\right)^N < \frac{1}{5} e^4.$$

Hence the fraction (8) is bounded by  $\tilde{I}_1 + \tilde{I}_2 \leq \frac{6}{5} e^4$ , see (13), and this immediately implies the required estimate of  $D_N(1 + 1/N)$ .  $\square$

In the next lemma we estimate  $\bar{\mu}_N$ -measure of the spherical layer  $B_{R_2}(\hat{m}) - B_{R_1}(\hat{m})$ ,  $R_2 \geq R_1$ , by a kind of a surface measure. For its formulation we set

$$A_N := \frac{12}{5} e^4 N. \tag{14}$$

**Lemma 2.2.** *Assume that*

$$N \geq [3 \log_2 D] + 6.$$

*Then for all  $\hat{m} \in M_N$  and  $R_1, R_2 > 0$  satisfying*

$$R_2 \geq \max\{R_1, 8d_N(\hat{m}, M_N^o)\}$$

*the following is true*

$$\bar{\mu}_N(B_{R_2}(\hat{m}) \setminus B_{R_1}(\hat{m})) \leq A_N \frac{\bar{\mu}_N(B_{R_2}(\hat{m}))}{R_2} (R_2 - R_1).$$

*Proof.* By definition  $M_N = M_{N-1} \times \mathbb{R}$  and  $\bar{\mu}_N = \bar{\mu}_{N-1} \otimes \lambda_1$ . Then by the Fubini theorem we have for  $R_1 \leq R_2$  with  $\hat{m} = (\tilde{m}, t)$

$$\begin{aligned} \bar{\mu}_N(B_{R_2}(\hat{m})) - \bar{\mu}_N(B_{R_1}(\hat{m})) &= 2 \int_{R_1}^{R_2} \bar{\mu}_{N-1}(B_s(\tilde{m})) ds \\ &\leq \frac{2R_2 \bar{\mu}_{N-1}(B_{R_2}(\tilde{m}))}{R_2} (R_2 - R_1). \end{aligned}$$

We claim that for arbitrary  $l > 1$  and  $R \geq 8d_N(\hat{m}, M_N^o) := 8d_{N-1}(\tilde{m}, M_{N-1}^o)$

$$R \bar{\mu}_{N-1}(B_R(\tilde{m})) \leq \frac{l D_{N-1}(l)}{l-1} \bar{\mu}_N(B_R(\hat{m})). \tag{15}$$

Together with the previous inequality this will yield

$$\bar{\mu}_N(B_{R_2}(\hat{m})) - \bar{\mu}_N(B_{R_1}(\hat{m})) \leq \frac{2l D_{N-1}(l)}{l-1} \cdot \frac{\bar{\mu}_N(B_{R_2}(\hat{m}))}{R_2} (R_2 - R_1).$$

Finally choose here  $l = 1 + \frac{1}{N-1}$  and use Lemma 2.1. This will give the required inequality.

Hence, it remains to establish (15). By the definition of  $D_{N-1}(l)$  we have for  $l > 1$  using the previous lemma

$$\begin{aligned} \bar{\mu}_N(B_{lR}(\hat{m})) &= 2l \int_0^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m})) \, ds \leq 4l \int_{R/2}^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m})) \, ds \\ &\leq 4lD_{N-1}(l) \int_{R/2}^R \bar{\mu}_{N-1}(B_s(\tilde{m})) \, ds \leq 2lD_{N-1}(l)\bar{\mu}_N(B_R(\hat{m})). \end{aligned}$$

On the other hand, replacing  $[0, R]$  by  $[l^{-1}R, R]$  we also have

$$\bar{\mu}_N(B_{lR}(\hat{m})) \geq 2l\bar{\mu}_{N-1}(B_R(\tilde{m}))(R - l^{-1}R) = 2(l - 1)R\bar{\mu}_{N-1}(B_R(\tilde{m})).$$

Combining the last two inequalities we get (15). □

**Extension operator.** We define the required simultaneous extension

$$E : \text{Lip}(M_N^o, X) \rightarrow \text{Lip}(M_N, X)$$

using the standard average operator *Ave* defined on continuous and locally bounded functions  $g : M_N^o \rightarrow X$  by

$$\text{Ave}(g; \hat{m}, R) := \frac{1}{\bar{\mu}_N(B_R(\hat{m}))} \int_{B_R(\hat{m})} g \, d\bar{\mu}_N.$$

To be well-defined the domain of integration  $B_R(\hat{m}) \cap M_N^o$  should be of strictly positive  $\bar{\mu}_N$ -measure (i.e.,  $\mu_N^o$ -measure). This condition is fulfilled in the case presented now. Namely, we define the simultaneous extension  $E$  on functions  $f \in \text{Lip}(M_N^o, X)$  by

$$(Ef)(\hat{m}) := \begin{cases} f(\hat{m}) & \text{if } \hat{m} \in M_N^o, \\ \text{Ave}(f; m, R(\hat{m})) & \text{if } \hat{m} \notin M_N^o \end{cases} \tag{16}$$

where we set

$$R(\hat{m}) := 8d_N(\hat{m}, M_N^o).$$

The required estimate of  $\|E\|$  is presented below. To formulate the result we set

$$K_N(l) := A_N D_N(l)(4l + 1) \tag{17}$$

where the first of two factors are defined by (14) and (5).

**Proposition 2.3.** *The following inequality,*

$$\|E\| \leq 20A_N + \max\left(\frac{4l + 1}{2(l - 1)}, K_N(l)\right),$$

*is true provided  $l := 1 + 1/N$ .*



Before we begin the proof let us derive from here the desired result. Namely, choose

$$N := [3 \log_2 D] + 6$$

and use Lemma 2.1 and (14) to estimate  $D_N(1 + 1/N)$  and  $A_N$ . Then we get

$$\|E\| \leq C(\log_2 D + 2)$$

with some numerical constant  $C$ . This clearly gives (2).

*Proof.* We have to show that for every  $\hat{m}_1, \hat{m}_2 \in M_N$

$$\|(Ef)(\hat{m}_1) - (Ef)(\hat{m}_2)\|_X \leq K \|f\|_{\text{Lip}(M_N, X)} d_N(\hat{m}_1, \hat{m}_2) \tag{18}$$

where  $K$  is the constant in the inequality of the proposition.

It suffices to consider only two cases:

(i)  $\hat{m}_1 \in M_N^o$  and  $\hat{m}_2 \notin M_N^o$ ,

(ii)  $\hat{m}_1, \hat{m}_2 \notin M_N^o$ .

We assume without loss of generality that

$$\|f\|_{\text{Lip}(M_N^o, X)} = 1$$

and simplify the computations by introducing the following notations:

$$R_i := d_N(\hat{m}_i, M_N^o), \quad B_{ij} := B_{8R_j}(\hat{m}_i), \quad v_{ij} := \bar{\mu}_N(B_{ij}), \quad 1 \leq i, j \leq 2. \tag{19}$$

We assume also for definiteness that

$$0 < R_1 \leq R_2.$$

By the triangle inequality we then have

$$0 \leq R_2 - R_1 \leq d_N(\hat{m}_1, \hat{m}_2). \tag{20}$$

Further, by Lemma 2.2 the quantities introduced satisfy the following inequality:

$$v_{i2} - v_{i1} \leq \frac{A_N v_{i2}}{R_2} (R_2 - R_1). \tag{21}$$

Let now  $\hat{m}^*$  be such that  $d_N(\hat{m}_1, \hat{m}^*) < 2R_1$ . Set

$$\hat{f}(\hat{m}) := f(\hat{m}) - f(\hat{m}^*). \tag{22}$$

From the triangle inequality we then obtain

$$\max\{\|\hat{f}(\hat{m})\|_X : \hat{m} \in B_{i2} \cap M_N^o\} \leq 10R_2 + (i - 1)d_N(\hat{m}_1, \hat{m}_2). \tag{23}$$

(Here  $i = 1, 2$ .)

We now prove (18) for  $\hat{m}_1 \in M_N^o$  and  $\hat{m}_2 \notin M_N^o$ . We begin with the evident inequality

$$\|(Ef)(\hat{m}_2) - (Ef)(\hat{m}_1)\|_X = \frac{1}{v_{22}} \left\| \int_{B_{22}} \hat{f}(\hat{m}) d\bar{\mu}_N \right\|_X \leq \max_{B_{22} \cap M_N^o} \|\hat{f}\|_X,$$

see (19) and (22). Applying (23) with  $i = 2$  we then bound this maximum by  $10R_2 + d_N(\hat{m}_1, \hat{m}_2)$ . But  $\hat{m}_1 \in M_N^o$  and so

$$R_2 = d_N(\hat{m}_2, M_N^o) \leq d_N(\hat{m}_1, \hat{m}_2);$$

therefore (18) holds in this case with  $K = 11$ .

The remaining case  $\hat{m}_1, \hat{m}_2 \notin M_N^o$  requires some additional auxiliary results. For their formulations we first write

$$(Ef)(\hat{m}_1) - (Ef)(\hat{m}_2) := D_1 + D_2 \tag{24}$$

where

$$\begin{aligned} D_1 &:= \text{Ave}(\hat{f}; \hat{m}_1, 8R_1) - \text{Ave}(\hat{f}; \hat{m}_1, 8R_2), \\ D_2 &:= \text{Ave}(\hat{f}; \hat{m}_1, 8R_2) - \text{Ave}(\hat{f}; \hat{m}_2, 8R_2), \end{aligned} \tag{25}$$

see (16) and (22).

**Lemma 2.4.** *We have*

$$\|D_1\|_X \leq 20A_N d_N(\hat{m}_1, \hat{m}_2).$$

Recall that  $A_N$  is the constant defined by (14).

*Proof.* By (25), (22), and (19),

$$\begin{aligned} D_1 &= \frac{1}{v_{11}} \int_{B_{11}} \hat{f} d\bar{\mu}_N - \frac{1}{v_{12}} \int_{B_{12}} \hat{f} d\bar{\mu}_N \\ &= \left( \frac{1}{v_{11}} - \frac{1}{v_{12}} \right) \int_{B_{11}} \hat{f} d\bar{\mu}_N - \frac{1}{v_{12}} \int_{B_{12} \setminus B_{11}} \hat{f} d\bar{\mu}_N. \end{aligned}$$

This immediately implies that

$$\|D_1\|_X \leq 2 \cdot \frac{v_{12} - v_{11}}{v_{12}} \cdot \max_{B_{12} \cap M_N^o} \|\hat{f}\|_X.$$

Applying now (21) and (20), and then (23) with  $i = 1$  we get the desired estimate.  $\square$

To obtain a similar estimate for  $D_2$  we will use the following two facts.

**Lemma 2.5.** *Assume that for a given  $l > 1$*

$$d_N(\hat{m}_1, \hat{m}_2) \leq 8(l-1)R_2. \quad (26)$$

*Let for definiteness*

$$v_{22} \leq v_{12}. \quad (27)$$

*Then we have*

$$\bar{\mu}_N(B_{12} \Delta B_{22}) \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \quad (28)$$

*(Here  $\Delta$  denotes symmetric difference of sets.)*

*Proof.* Set

$$R := 8R_2 + d_N(\hat{m}_1, \hat{m}_2).$$

Then  $B_{12} \cup B_{22} \subset B_R(\hat{m}_1) \cap B_R(\hat{m}_2)$ , and

$$\begin{aligned} \bar{\mu}_N(B_{12} \Delta B_{22}) &\leq (\bar{\mu}_N(B_R(\hat{m}_1)) - \bar{\mu}_N(B_{8R_2}(\hat{m}_1))) \\ &\quad + (\bar{\mu}_N(B_R(\hat{m}_2)) - \bar{\mu}_N(B_{8R_2}(\hat{m}_2))). \end{aligned}$$

Estimating the terms on the right-hand side by Lemma 2.2 we bound them by

$$A_N \frac{\bar{\mu}_N(B_R(\hat{m}_1))}{R} (R - 8R_2) + A_N \frac{\bar{\mu}_N(B_R(\hat{m}_2))}{R} (R - 8R_2).$$

Moreover,  $8R_2 \leq R \leq 8lR_2$  and  $R - 8R_2 := d_N(\hat{m}_1, \hat{m}_2)$ , see (26); taking into account (5), (19), and (27) we therefore have

$$\bar{\mu}_N(B_{12} \Delta B_{22}) \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \quad \square$$

**Lemma 2.6.** *Under the assumptions of the previous lemma we have*

$$v_{12} - v_{22} \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \quad (29)$$

*Proof.* By (19) the left-hand side is bounded by  $\bar{\mu}_N(B_{12} \Delta B_{22})$ .  $\square$

We now estimate  $D_2$  from (25) beginning with

**Lemma 2.7.** *Under the conditions of Lemma 2.5 we have*

$$\|D_2\|_X \leq K_N(l) d_N(\hat{m}_1, \hat{m}_2)$$

where  $K_N(l) := A_N D_N(l)(4l + 1)$ .

*Proof.* By the definition of  $D_2$  and our notation, see (25), (22), and (19),

$$\begin{aligned} \|D_2\|_X &:= \left\| \frac{1}{v_{12}} \int_{B_{12}} \hat{f} d\bar{\mu}_N - \frac{1}{v_{22}} \int_{B_{22}} \hat{f} d\bar{\mu}_N \right\|_X \\ &\leq \frac{1}{v_{12}} \int_{B_{12} \Delta B_{22}} \|\hat{f}\|_X d\bar{\mu}_N + \left| \frac{1}{v_{12}} - \frac{1}{v_{22}} \right| \int_{B_{22}} \|\hat{f}\|_X d\bar{\mu}_N := J_1 + J_2 . \end{aligned}$$

By (28), (26), and (23)

$$\begin{aligned} J_1 &\leq \frac{1}{v_{12}} \bar{\mu}_N(B_{12} \Delta B_{22}) \sup_{(B_{12} \Delta B_{22}) \cap M_N^c} \|\hat{f}\|_X \\ &\leq \frac{A_N D_N(l)}{4R_2} d_N(\hat{m}_1, \hat{m}_2) (d_N(\hat{m}_1, \hat{m}_2) + 10R_2) \\ &\leq A_N D_N(l) (2l + 1/2) d_N(\hat{m}_1, \hat{m}_2) . \end{aligned}$$

Also, (29), (23), and (26) yield

$$J_2 \leq A_N D_N(l) (2l + 1/2) d_N(\hat{m}_1, \hat{m}_2) .$$

Combining these we get the required estimate. □

It remains to consider the case of  $\hat{m}_1, \hat{m}_2 \notin M_N$  satisfying the inequality

$$d_N(\hat{m}_1, \hat{m}_2) > 8(l - 1)R_2$$

converse to (26). Now the definition (25) of  $D_2$  and (23) imply that

$$\|D_2\|_X \leq 2 \sup_{(B_{12} \cup B_{22}) \cap M_N^c} \|\hat{f}\|_X \leq 2(10R_2 + d_N(\hat{m}_1, \hat{m}_2)) \leq \frac{4l + 1}{2(l - 1)} d_N(\hat{m}_1, \hat{m}_2) .$$

Combining this with the inequalities of Lemmas 2.4 and 2.7 and equality (24) we obtain the required estimate of the Lipschitz norm of the extension operator  $E$ :

$$\|E\| \leq 20A_N + \max\left(\frac{4l + 1}{2(l - 1)}, K_N(l)\right)$$

where  $K_N(l)$  is the constant in (17). □

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