Root Arrangements of Hyperbolic Polynomial-like Functions

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ABSTRACT

A real polynomial P of degree n in one real variable is *hyperbolic* if its roots are all real. A real-valued function P is called a *hyperbolic polynomial-like function (HPLF)* of degree n if it has n real zeros and $P^{(n)}$ vanishes nowhere. Denote by $x_k^{(i)}$ the roots of $P^{(i)}$, $k = 1, \ldots, n-i$, $i = 0, \ldots, n-1$. Then in the absence of any equality of the form

$$x_i^{(j)} = x_k^{(l)} \tag{1}$$

one has

$$\forall i < j \qquad x_k^{(i)} < x_k^{(j)} < x_{k+j-i}^{(i)} \tag{2}$$

(the Rolle theorem). For $n \geq 4$ (resp. for $n \geq 5$) not all arrangements without equalities (1) of $n(n+1)/2$ real numbers $x_k^{(i)}$ and compatible with (2) are realizable by the roots of hyperbolic polynomials (resp. of HPLFs) of degree n and of their derivatives. For $n = 5$ and when $x_1^{(1)} < x_2^{(1)} < x_1^{(3)} < x_2^{(3)} < x_3^{(1)} < x_4^{(1)}$ we show that from the 40 arrangements without equalities (1) and compatible with (2) only 16 are realizable by HPLFs (from which 6 by perturbations of hyperbolic polynomials and none by hyperbolic polynomials).

Key words: hyperbolic polynomial, polynomial-like function, root arrangement, configuration vector.

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1. Introduction

1.1. Hyperbolic polynomials, polynomial-like functions, and their root arrangements

Consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $x, a_i \in \mathbf{R}$. A polynomial of this family is called *(strictly) hyperbolic* if all its roots are real (real and distinct). It is clear that if P is (strictly) hyperbolic, then such are $P^{(1)}, \dots, P^{(n-1)}$ as well. Hyperbolic are the polynomials of all known orthogonal families (e.g., the Legendre, Laguerre, Hermite, Tchebyshev polynomials).

A problem of interest in (real) algebraic geometry is the dependence of the roots of the first, second, etc. derivatives of a polynomial on the roots of the polynomial itself. For complex polynomials one has the Gauß-Lucas theorem which says that the roots of the derivative belong to the convex hull of the roots of the polynomial. But in the case of hyperbolic polynomials there are also some properties specific to this class. For instance, one has the following property.

Property 1.1 (see [1] or [6]). *If one of the roots of a hyperbolic polynomial moves to the right (resp. to the left) while the other roots remain fixed, then every root of every derivative of the polynomial moves to the right (resp. to the left) or remains fixed.*

The roots of the first (resp. of the second) derivative have a geometric interpretation because they define the critical (resp. the inflection) points. If one is interested in these points in the graph of the first (resp. of the second) derivative, then one has to study the third (resp. the fourth) derivative, thus one is led in a natural way to the study of the arrangements of the roots of a hyperbolic polynomial and of all its derivatives.

Notation 1.2. Denote by $x_1 \leq \cdots \leq x_n$ the roots of P and by $x_1^{(k)} \leq \cdots \leq x_{n-k}^{(k)}$ the ones of $P^{(k)}$. We set $x_j^{(0)} = x_j$. In the examples we never go beyond degree 5 and to avoid double indices we use also the notation f_j, s_j, t_j, l_j for the roots respectively of $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, $P^{(4)}$. The letters are chosen to match "first", "second", "third" and "last".

Definition 1.3. The *arrangement* (or *configuration*) defined by the roots of P, $P^{(1)}$, $\dots$, $P^{(n-1)}$ is the complete system of strict inequalities and equalities that hold for these roots. To explicit an arrangement one can write the roots in a chain in which any two consecutive roots are connected with a sign $<$ or $=$. An arrangement is called *non-degenerate* if there are no equalities between any two of the roots, i.e., no equalities of the form $x_i^{(j)} = x_q^{(r)}$ for any indices i, j, q, r. A *partial arrangement* is the arrangement of the roots of only part of the derivatives $P^{(k)}$, $k = 0, 1, \dots, n-1$.

The classical Rolle theorem implies that the roots of P and of its derivatives satisfy the following inequalities:

$$\forall\, i < j, \qquad x_k^{(i)} \leq x_k^{(j)} \leq x_{k+j-i}^{(i)} \tag{3}$$

One has also the self-evident condition:

$$\big((x_k^{(i)} = x_k^{(i+1)}) \quad \text{or} \quad (x_{k+1}^{(i)} = x_k^{(i+1)})\big) \quad \Rightarrow \quad (x_k^{(i)} = x_k^{(i+1)} = x_{k+1}^{(i)}) \tag{4}$$

Remark 1.4. In what follows when speaking about root arrangements we always assume that they satisfy conditions (3) and (4).

The Rolle theorem provides only necessary conditions, i.e., not all arrangements are realizable by the roots of hyperbolic polynomials and of their derivatives. In previous papers (see [2, 3, 7]) we ask the question:

> *Which of the arrangements of the $n(n+1)/2$ real numbers $x_j^{(k)}$, $k = 0, \dots, n-1$, $j = 1, \dots, n-k$, satisfying conditions* (3) *and* (4)*, can be realized by the roots of hyperbolic polynomials of degree n and of their derivatives?*

For $n = 1, 2, 3$ this is the case of all arrangements (degenerate or not). For $n \geq 4$ this is no longer like this. E.g., for $n = 4$ only 10 out of 12 non-degenerate arrangements are realizable by hyperbolic polynomials (see [1] or [3]; this fact is closely related to Property 1.1); for $n = 5$ these numbers equal respectively 116 and 286 (see [2]).

Remark 1.5. For any n the number $N(n)$ of non-degenerate arrangements compatible with (3) equals (see [9])

$$N(n) = \binom{n+1}{2}!\, \frac{1!\, 2! \cdots (n-1)!}{1!\, 3! \cdots (2n-1)!}$$

An obvious reason why this proportion of realizability is to drop further when n increases is the lack of dimension — a root arrangement of a hyperbolic polynomial and its derivatives is defined by $n-2$ coefficients (By a shift of the origin of the x-axis one can transform a_1 into 0; if after this one has $a_2 \neq 0$, then a subsequent change of the scope of the x-axis transforms a_2 into -1.) while the total number of roots is $n(n+1)/2$. (By similar transformations one can obtain the conditions $x_1 = 0$, $x_n = 1$, i.e., one can again kill two parameters.)

However, the Rolle theorem is formulated not only for hyperbolic polynomials, but for smooth functions. Therefore one can introduce the following generalization of a hyperbolic polynomial in the tentative to realize all non-degenerate arrangements.

Definition 1.6. A polynomial-like function (PLF) of degree n is a C^∞-smooth function whose n-th derivative vanishes nowhere. (Hence, a PLF has at most n real roots counted with the multiplicities.) A PLF of degree n is called *(strictly) hyperbolic* if it has exactly n real (and distinct) roots. In what follows all PLFs are presumed hyperbolic.

Example 1.7. Prove that the function $f(x) := e^x - \frac{x^4}{24} - \frac{x^3}{4} - \frac{x^2}{2} - x - 1$ is a hyperbolic PLF of degree 5. Indeed, one has $f^{(5)} = e^x$ which vanishes nowhere, hence, f is a PLF of degree 5. Moreover, one has $f(0) = f^{(1)}(0) = f^{(2)}(0) = 0$, $f^{(3)}(0) = -\frac{1}{2}$, $\lim_{x\to\pm\infty} f(x) = \pm\infty$, i.e., f has a triple root at 0, a simple positive and a simple negative root. Hence, f is hyperbolic. If $\varepsilon > 0$ is small enough, then the function $f + \varepsilon x$ is a strictly hyperbolic PLF of degree 5. (The triple root at 0 splits into three simple roots.)

In paper [4] we show that that for $n = 4$ PLFs realize all arrangements, and that one can choose these PLFs to be either hyperbolic polynomials of degree 4 or non-hyperbolic polynomials of degree 6. In particular, the two non-degenerate arrangements not realizable by hyperbolic polynomials are realizable by perturbations of such; these perturbations are polynomials of degree 6.

Remark 1.8. Suppose that a strictly hyperbolic PLF f and its derivatives realize a given non-degenerate arrangement. One can approximate $f^{(n)}$ by polynomials and keep the same constants of integration thus obtaining polynomials realizing the same arrangement. Therefore all non-degenerate arrangements realizable by PLFs are realizable by polynomials as well (but not necessarily hyperbolic).

In paper [5] we show that for $n = 5$ there are non-degenerate arrangements which are not realizable by PLFs. As PLFs belong to an infinite-dimensional space, dimension is not the only obstacle towards realizability of arrangements by hyperbolic polynomials (or by PLFs).

In the present paper we continue the study of the question for $n = 5$ which arrangements are realizable by PLFs. The partial arrangements of the roots of the first and third derivatives of a PLF define four possible cases two of which are symmetric, see Subsection 1.3. The present paper offers the thorough study of one of the other two cases.

In paper [8] PLFs of degree 3 are considered (The authors call them *pseudopolynomials.*) and necessary and sufficient conditions are given for the numbers $x_1 < x_2 < x_3$, $y_1 < y_2$, and z_1 to be roots respectively of a PLF and of its first and second derivatives. In the present paper we use some of the ideas from [8].

1.2. Configuration vectors

We use *configuration vectors (CVs)* to define arrangements. On a CV the positions of the roots of P, $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, $P^{(4)}$ are denoted by 0, f, s, t, l.

We use two ways to present a CV — on a line and by a partially filled matrix. When the presentation is on a line, coinciding roots (if any) are put in square brackets. E.g., for $n = 5$ the CV (compatible with (3) and (4))

$$([0f0], s, f, t, l, 0, s, f, t, s, [0f0])$$

indicates that one has

$$x_1 = f_1 = x_2 < s_1 < f_2 < t_1 < l_1 < x_3 < s_2 < f_3 < t_2 < s_3 < x_4 = f_4 = x_5.$$

We present by matrices only non-degenerate arrangements. To present several arrangements at once we use the sign $\vdots$ with the meaning that whenever two or three roots are surrounded by such signs, then any permutation of the surrounded roots is allowed. E.g., the matrix

$$
\begin{array}{ccccccccccccccc}
 & & & & & & \vdots & l & \vdots & & & & & & \\
 & & & & & t & \vdots & & \vdots & t & & & & & \\
 & & & s & & & \vdots & s & \vdots & & & s & & & \\
 & f & & & f & & \vdots & & \vdots & & f & & & f & \\
0 & & 0 & & & & \vdots & 0 & \vdots & & & & 0 & & 0
\end{array}
\tag{5}
$$

denotes each of the six CVs $(0, f, 0, s, f, t, \mathcal{P}, t, f, s, 0, f, 0)$ where $\mathcal{P}$ is any of the six permutations of 0, s and l.

We use also *partial CVs* when we need to denote the partial arrangement (presumed non-degenerate) defined by the relative position of the roots of only two of the derivatives. E.g., the notation $(ffttff)$ means that one has $f_1 < f_2 < t_1 < t_2 < f_3 < f_4$. By $\{(ffttff)\}$ we denote the subset of all non-degenerate arrangements for which the last chain of inequalities holds.

In what follows we identify for convenience arrangements with the CVs defining them (represented on a line or by a matrix). The following lemma can be proved by full analogy with Lemma 4.2 from [7]: (The latter is formulated for hyperbolic polynomials, not for PLFs.)

Lemma 1.9. *A root of multiplicity m of a PLF f of degree n, $0 \leq m < i+1 \leq n$, is at most a simple root of $f^{(i)}$.*

Definition 1.10. A degenerate arrangement (V) is *adjacent* to the arrangement (W) if (W) is obtained from (V) by replacing one or several equalities between roots by strict inequalities.

Example 1.11. For $n = 4$ the arrangement $([0f0], s, f, [t0], s, f, 0)$ is adjacent to and only to the following five arrangements: $(0, f, 0, s, f, [t0], s, f, 0)$, $([0f0], s, f, t, 0, s, f, 0)$, $([0f0], s, f, 0, t, s, f, 0)$, $(0, f, 0, s, f, t, 0, s, f, 0)$, and $(0, f, 0, s, f, 0, t, s, f, 0)$. Only the last two of them are non-degenerate.

Proposition 1.12. *If a degenerate arrangement (V) is realizable by a PLF f of degree n, then all arrangements to which (V) is adjacent and with the same multiplicities of the roots of the PLF are realizable by PLFs which are perturbations of f.*

Proof. 1.° It suffices to prove the proposition for the arrangements obtained from (V) by replacing just one equality between roots by an inequality $<$ or $>$. Observe first that by Lemma 1.9 there are cases in which this cannot be any equality. Indeed, if $x_{i_0}^{(0)} = \cdots = x_{i_{m-1}}^{(m-1)}$ is a root of f of multiplicity $m > 2$, and if in this chain of equalities one wants to replace only one equality by an inequality, then this can be only the last one.

2.º Suppose that the arrangement (V) contains the following chain of equalities: $x_{i_1}^{(j_1)} = \cdots = x_{i_q}^{(j_q)}$. (The indices j_ν need not be consecutive integers.) Suppose without loss of generality that $x_{i_1}^{(j_1)} = 0$ and that one wants to change the equality $x_{i_s}^{(j_s)} = x_{i_{s+1}}^{(j_{s+1})}$ by the inequality $x_{i_s}^{(j_s)} > x_{i_{s+1}}^{(j_{s+1})}$ or $x_{i_s}^{(j_s)} < x_{i_{s+1}}^{(j_{s+1})}$. The new arrangement thus obtained is denoted by (W).

3.º Denote by χ a germ of a C^∞-function at 0 such that $\chi^{(r)}(0) = 0$ for $r = 0, 1, \dots, j_{s+1}-1, j_{s+1}+1, j_{s+1}+2, \dots, n$, $\chi^{(j_{s+1})}(0) = 1$. Denote by ψ a C^∞-function with compact support such that $\psi \geq 0$, $\psi(x) \equiv 1$ for $|x| \leq \frac{\eta}{2}$ and $\psi(x) \equiv 0$ for $|x| \geq \eta$ where $\eta > 0$. Then for η small enough the support of ψ contains no zeros of f, $f^{(1)}$, $\dots$, $f^{(n-1)}$ other than 0.

Hence, the perturbation realizing the arrangement (W) can be chosen of the form $f + \varepsilon\psi\chi$ where $\varepsilon \in \mathbf{R}$ is small enough and the choice of the sign of ε results in the choice of the sign of the inequality ($<$ or $>$). $\square$

1.3. Aim, scope and basic results of the present paper

In what follows we focus on non-degenerate arrangements. The following four partial arrangements are possible between the roots of $P^{(1)}$ and $P^{(3)}$ where P is a PLF:

$$(ffttff), \qquad (ftftff), \qquad (fftftf), \qquad (ftfftf). \tag{6}$$

The present paper is devoted to the first of these cases.

Remarks 1.13. (i) It would be hard to imagine an entire study of the case $n \geq 6$ due to $N(6) = 33592$ (see Remark 1.5) unless some general rules and theorems about realizability of arrangements are proved. On the other hand, the case $n = 4$ is thoroughly studied and there are only 12 non-degenerate arrangements there. Therefore the case $n = 5$ is the first truly interesting case to study. To subdivide the study into several cases is reasonable because of the great number (namely, 286) of arrangements, see Remark 1.5.

(ii) The cases $(ftftff)$ and $(fftftf)$ can be studied by analogy (the symmetry between these two cases is defined by the change of variable $x \mapsto -x$). This change of variable allows one to study in the cases $(ffttff)$ and $(ftfftf)$ only the non-degenerate arrangements with $l_1 < x_3$ or with $x_3 < l_1$. This is what we often do in the paper. We say that two arrangements are *symmetric* (to one another) if the symmetry is induced by the change $x \mapsto -x$. Example: for $n = 4$ such are the arrangements $(0, f, s, 0, t, f, 0, s, f, 0)$ and $(0, f, s, 0, f, t, 0, s, f, 0)$.

(iii) For each of the four cases (see (6)) the number of all non-degenerate arrangements and the number of the ones realizable by hyperbolic polynomials are given in the following table (see [2, Observations 24, 25 and Lemmas 40–43]):

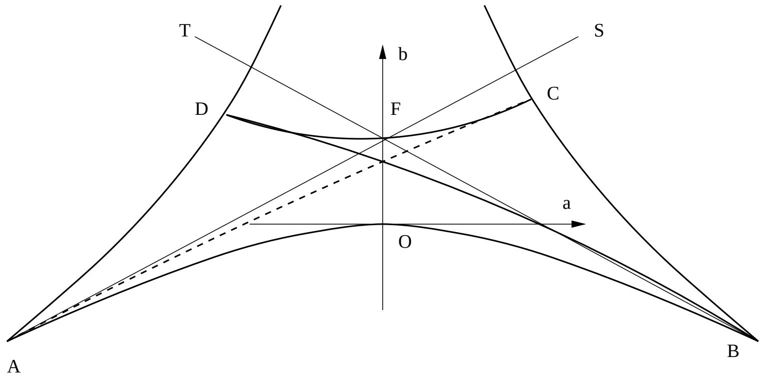

Figure 1: The hyperbolicity domain in degree 5.

Case	All arrangements	Realizable by hyperbolic polynomials
$(ffttff)$	40	0
$(ftftff)$ or $(fftftf)$	72	25
$(ftfftf)$	102	66

(iv) Our interest in the first of the cases is motivated by the absence of non-degenerate arrangements realizable by hyperbolic polynomials, see the table (so one expects to find many examples of non-degenerate arrangements not realizable by PLFs of degree 5). Geometrically this can be explained by fig. 1. On the figure we show the hyperbolicity domain Π of the family of polynomials $P = x^5 - x^3 + ax^2 + bx + c$, i.e., the set of values of (a, b, c) for which the polynomial is hyperbolic. (The reader will find more details about Π in [2].) The axis Oc is vertical, i.e., perpendicular to the plane of the sheet. The hyperbolicity domain is the curvilinear tetrahedron $ABCD$. (The concavity of the faces is everywhere towards its interior.)

The set $D(1, 3)$ of values of (a, b, c) for which the derivatives $P^{(1)}$ and $P^{(3)}$ have a common root (we call such sets *discriminant sets*) is the union of two vertical planes (i.e., parallel to Oc) represented on the figure by the lines BT and AS. They intersect along a vertical line which has a single point in common with Π — the point F. Such a situation might appear to be quite non-generic. This impression is false. Indeed, discriminant sets are defined by algebraic equations whose coefficients are relatively small integers, so it makes no sense to speak about genericity.

The two planes from the set $D(1, 3)$ divide the space into four sectors. These are the four sets defined by the partial arrangements (6). The case $(ffttff)$ is the sector above (which intersects with Π only at the point F), the case $(ftfftf)$ is the sector below.

The basic result of the present paper is the following

Theorem 1.14. (i) *In the case* $(ffttff)$ *exactly* 16 *out of the* 40 *non-degenerate arrangements are realizable by PLFs of degree* 5. *From them exactly* 6 *arrangements (represented by matrix* (5)*) are realizable by perturbations of hyperbolic polynomials. The other* 10 *are the* 2, 1, 2 *arrangements represented respectively by matrices* (7), (11), (12), *and their symmetric ones.*

(ii) *The remaining* 24 *arrangements (which are not realizable by PLFs of degree* 5*) are the* 8, 2, *and* 2 *arrangements represented respectively by matrices* (15), (16), *and* (17), *and their symmetric ones.*

2. Proof of Theorem 1.14

The proof of Theorem 1.14 occupies almost the whole of the rest of the paper. In subsection 2.1 we prove the realizability by the roots of PLFs and their derivatives of the 16 arrangements mentioned in part 1) of the theorem. Lemma 2.1 claims that exactly 6 of them are realizable by perturbations of hyperbolic polynomials. Lemmas 2.3, 2.6 and 2.8 claim the realizability respectively of 4, 2, and 4 of the remaining 10 arrangements by PLFs (which, by Lemma 2.1, are not perturbations of hyperbolic polynomials).

In subsection 2.2 we prove that the 24 arrangements from part (ii) of Theorem 1.14 are not realizable by the roots of PLFs and their derivatives. Matrices (15) and (16) at the beginning of subsection 2.2 represent respectively 8 and 2 non-degenerate arrangements. These 10 arrangements and their symmetric ones (hence, 20 arrangements altogether) are not realizable by the roots of PLFs and their derivatives; this follows from the results in [5]. The non-realizability of the remaining 4 arrangements is claimed by Lemma 2.10. The proof of that lemma is long. Therefore it is subdivided into several parts and is preceded by a plan.

2.1. Proof of the existence part

Lemma 2.1. (i) *There are exactly* 24 *non-degenerate arrangements to which the following arrangement is adjacent:*

$$(A)\,:\,([0f0],s,[ft],[0sl],[ft],s,[0f0]).$$

They are all realizable by perturbations of hyperbolic polynomials.

(ii) *Exactly six out of these* 24 *non-degenerate arrangements belong to the set* $\{(ffttff)\}$. *These six arrangements are defined by matrix* (5).

(iii) *Out of the remaining* 18 *arrangements, exactly* 8 *are not realizable by hyperbolic polynomials, see* [2, Observation 33].

Proof. Part (ii) of the lemma is to be checked straightforwardly, and part (iii) needs no proof. Prove part (i).

1.º The non-degenerate arrangements to which (A) is adjacent are obtained by replacing each of the two groups $[0f0]$ by $0, f, 0$, of each of the two groups $[ft]$ either by f, t or by t, f, (This gives four possibilities.) and of $[0sl]$ by one of the six permutations of 0, s and l. This gives 24 non-degenerate arrangements.

2.º Consider the polynomial $P_* := x^5 - x^3 + \frac{x}{4} = x(x^2 - \frac{1}{2})^2$. It realizes arrangement (A) and defines the point F on fig. 1. For the roots of P_* and its derivatives one has

$$x_{1,2} = f_1 = \frac{1}{\sqrt{2}}, \qquad x_3 = s_2 = l_1 = 0, \qquad f_{2,3} = t_{1,2} = \pm\frac{1}{\sqrt{10}}$$
$$x_{4,5} = f_4 = -\frac{1}{\sqrt{2}}, \qquad s_{1,3} = \pm\sqrt{\frac{3}{10}}$$

It follows from Proposition 1.12 that all arrangements satisfying the following three conditions are realizable:

(a) arrangement (A) is adjacent to them;

(b) the groups $[ft]$ and $[0sl]$ in arrangement (A) are replaced respectively by (f, t) or (t, f) (independently for each of the two groups $[ft]$) and by any of the 6 permutations of 0, s, l;

(c) the groups $[0f0]$ from arrangement (A) remain the same.

To prove the realizability of the 24 arrangements from the lemma one has after this to make the double roots $x_1 = x_2$ and $x_4 = x_5$ bifurcate. This can be done by adding an affine function $-\varepsilon(x - x_3)$, where $\varepsilon > 0$ is small enough. □

Remark 2.2. In what follows we often draw the graphs of a function (in most cases this is a PLF) and of its derivatives one upon the other. On the figures we use the notation x_j (and similarly f_j, s_j, t_j) both in the sense "the roots of the function g" and in the sense "the point with coordinate x_j on the x-axis in the picture representing the graph of g". This might sometimes oblige the reader when looking at a figure to search different roots, say, f_3 and s_2, on different x-axes — the ones of $g^{(1)}$ and of $g^{(2)}$.

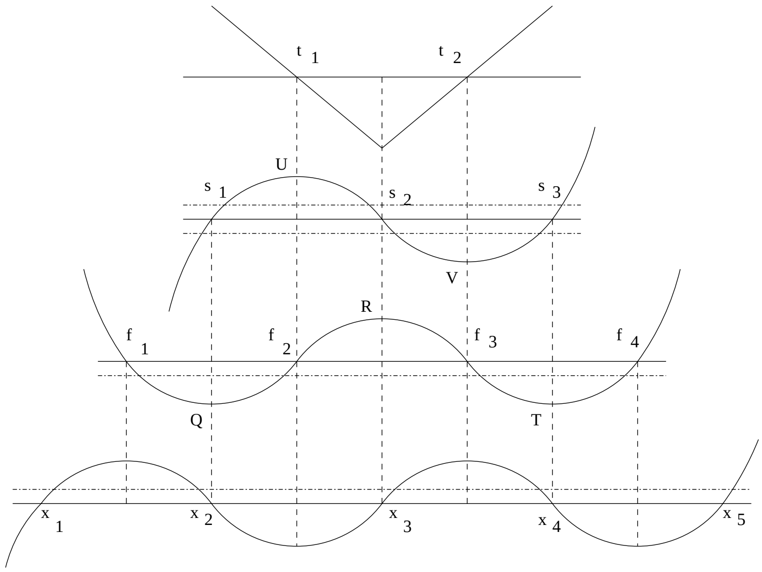

Figure 2: Construction of the PLF realizing arrangement (7).

Lemma 2.3. *The following two non-degenerate arrangements and their symmetric ones are realizable by PLFs of degree* 5*:*

$$\begin{array}{cccccccccccccccc}
 & & & & & & \vdots & l & \vdots & & & & & & & \\
 & & & & & t & \vdots & & \vdots & & t & & & & & \\
 & & & s & & & \vdots & s & \vdots & & & & & s & & \\
 & f & & & f & & \vdots & & \vdots & & & f & & & f & \\
0 & & 0 & & & & \vdots & & \vdots & 0 & & & 0 & & & 0
\end{array} \tag{7}$$

Proof. 1.^{o} To construct PLFs which realize the two arrangements from matrix (7) we first construct an odd C^3-smooth function g which realizes the arrangement $(0, f, [0s], [ft], [0sl], [ft], [0s], f, 0)$. Properly speaking, the presence of the letter l is not justified here given that the function is only C^3-smooth. We set $l_1 = 0$, the sense of this equality will become at least partially clear in 2.^{o} and completely clear in 6.^{o}. The graphs of g, $g^{(1)}$, $g^{(2)}$, and $g^{(3)}$ are drawn one above the other on fig. 2. The x-axis for each of the four graphs are the solid lines.

2.^{o} One has $g^{(4)}(x) \equiv -\frac{1}{2}$ for $x < 0$ and $g^{(4)}(x) \equiv \frac{1}{2}$ for $x > 0$, (This explains why we set $l_1 = 0$.) and one can think of $g^{(5)}$ as of $\delta(x)$ (the delta-function). The graph of $g^{(3)}$ is piecewise-linear and symmetric w.r.t. the line s_2R, the one of $g^{(2)}$

consists of parts of two parabolas. The axes of symmetry of the entire parabolas are the lines t_1Uf_2 and t_2Vf_3, the point s_2 is the center of symmetry of the graph of $g^{(2)}$.

The graph of $g^{(1)}$ consists of the graphs of two cubic functions defined respectively for $x \leq 0$ and $x \geq 0$. When these two functions are defined on $\mathbf{R}$, then their graphs have respectively the points f_2 and f_3 as centers of symmetry. The graph of $g^{(1)}$ has the line s_2Rx_3 as axis of symmetry.

3.$^{\text{o}}$ One can think of g as of the result of a five-fold integration of $\delta(x)$ with suitable constants $c_1, \ldots, c_5$ of integration. Decrease c_5 so that the graph of g still cut the x-axis in five distinct points. (The graphs of $g^{(4)}$, $g^{(3)}$, $g^{(2)}$, and $g^{(1)}$ do not change.) The new position of the x-axis is the lowest dash-dotted line on fig. 2. After this change of constant one has

$$s_1 > x_2, \qquad x_3 > s_2, \qquad s_3 > x_4 \tag{8}$$

and $s_2 = l_1 = 0$.

4.$^{\text{o}}$ After this increase c_4 (without changing c_5). This increasing must be much smaller than the decreasing of c_5 because it changes the graph of g; if it is small enough, then conditions (8) are preserved and one has

$$f_2 < t_1, \qquad t_2 < f_3. \tag{9}$$

5.$^{\text{o}}$ After this increase (resp. decrease) c_3 without changing c_4 and c_5. If this change of c_3 is small enough, then conditions (8) and (9) are preserved and one has

$$l_1 < s_2 < x_3 \quad (\text{resp.} \quad s_2 < l_1 < x_3\ ). \tag{10}$$

6.$^{\text{o}}$ Change $g^{(5)}$ from $\delta(x)$ to an even C^∞-smooth function h with compact support $[-\eta, \eta]$ where η is so small that no root of $g, g^{(1)}, g^{(2)}, g^{(3)}$ belongs to $[-\eta, \eta]$, and one has $h > 0$ for $x \in (-\eta, \eta)$, $\int_{-\eta}^{\eta} h(x)dx = 1$. If η is small enough, then conditions (8), (9), and (10) still hold and g is a C^∞-smooth function. Moreover, $l_1 = 0$ is the only zero of $g^{(4)}$.

The function g is still not a PLF because $g^{(5)}$ vanishes outside $(-\eta, \eta)$. To make it a PLF one has to change $g^{(5)}$ from h to $h + b$ where $b > 0$ is so small that conditions (8), (9), and (10) still hold. This means that the function g thus constructed is a PLF and realizes one of the arrangements from matrix (7); which one exactly depends on whether one increases or decreases c_3 in 5.$^{\text{o}}$. Their symmetric arrangements are realized by the function $g(-x)$. $\square$

Remark 2.4. If one decreases c_4 in 4^0 instead of increasing it, then in the same way one can construct a PLF realizing one of the arrangements

$$(0, f, 0, s, t, f, l, s, 0, f, t, 0, s, f, 0) \quad \text{and} \quad (0, f, 0, s, t, f, s, l, 0, f, t, 0, s, f, 0)$$

and their symmetric ones; these four arrangements belong not to $\{(ffttff))\}$, but to $\{(ftfftf)\}$.

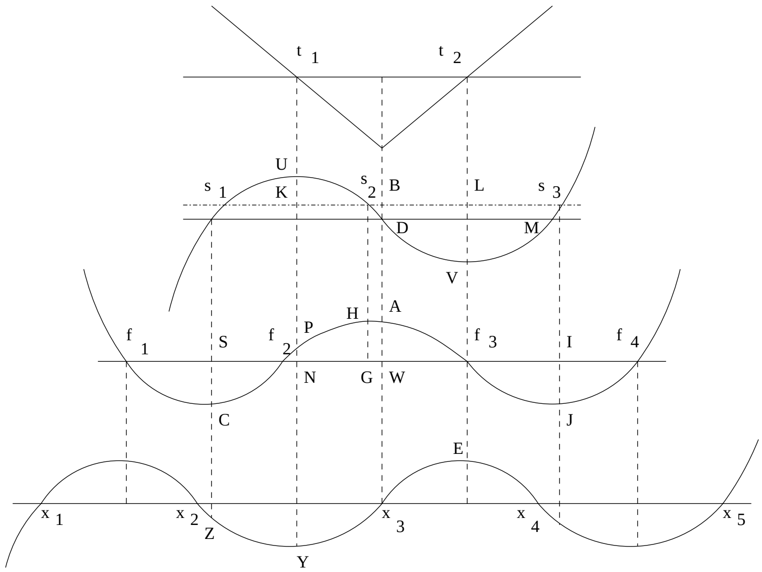

Figure 3: Construction of the PLF realizing arrangement (11).

Notation 2.5. We use the notation $S(\cdot)$ for the area of a curvilinear figure and $\bar{S}(\cdot)$ for the area of a (rectilinear) polygon. We use different notation for the same thing in order to distinguish rectilinear from curvilinear polygons. We denote the figures whose area is expressed by closed contours. Example: $S(ABCA)$ denotes the area of the curvilinear triangle ABC.

Lemma 2.6. *The non-degenerate arrangement represented by the following matrix and its symmetric one are realizable by PLFs of degree* 5*:*

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & l & & & & & & \\
 & & & & & t & & & & t & & & & & \\
 & & & s & & & s & & & & & & s & & \\
 & f & & & f & & & & & & f & & & f & \\
0 & & 0 & & & & & 0 & & & & 0 & & & 0
\end{array} \tag{11}$$

Proof. 1.$^\circ$ The proof follows the same ideas as the ones of Lemma 2.3, this is why fig. 3 which illustrates the construction is so much like fig. 2. Not completely, though.

We start by defining the function g and the constants c_j in the same way as in 1.$^\circ$–3.$^\circ$ of the proof of Lemma 2.3. So the initial graphs of the functions g, $g^{(1)}$, $g^{(2)}$, $g^{(3)}$ are given on fig. 2.

2.$^\circ$ Decrease the constant c_3. The change is presumed to be small. The new position of the x-axis is the dash-dotted line on fig. 3. The point D (and not the root s_2) is the center of symmetry of the graph of $g^{(2)}$.

3.^{o} Choose the new value of the constant c_4 such that $f_3 = t_2$. This means that it is close to the old one. Hence, the graph of the new function $g^{(1)}$ looks like on fig. 3.

Notice that one has indeed $f_2 < t_1$. This is so because when a point follows the graph of $g^{(1)}$ moving to the left or right, its initial position being the point H, then it descends to 0 which is due to the integration of $g^{(2)}$. When moving to the right, the point descends to 0 at f_3, so one has $\|HG\| = S(s_2DVLBs_2)$. One has also $S(s_2DVLBs_2) > S(s_2UKs_2)$ (remember that the point D is the center of symmetry of the graph of $g^{(2)}$). Hence, when the point is moving to the left, it will go beyond the vertical line t_1UKPNY before descending to 0.

4.^{o} Choose the new constant c_5 such that $x_3 = 0$. Hence, when c_3 is changed little enough, the new constant c_5 is close to the old one and the graph of g looks like on fig. 3.

Notice that one has $x_4 < s_3$ and $x_2 < s_1$. The first of these inequalities follows from $S(VMs_3LV) > S(VDBLV)$ (remember that the line t_2LVf_3 is the axis of symmetry of the parabola MVD; hence, f_3 is the center of symmetry of the curve Af_3J); this in turn implies $S(f_3JIf_3) > S(f_3AWf_3)$.

Hence, the area $S(f_3AWf_3) = \left|\int_{f_3}^{x_3} g^{(1)}(x)\,dx\right|$ is sufficient to make the point E from the graph of g go down to 0 at x_3 when moving to the left, and the area $S(f_3JIf_3) = \left|\int_{f_3}^{s_3} g^{(1)}(x)\,dx\right|$ makes this point descend below 0 when moving to the right.

The inequality $x_2 < s_1$ follows from $S(s_1KUs_1) = S(s_2KUs_2)$ (remember that the line t_1UK is the axis of symmetry of the parabola DUs_1), hence, $S(f_2SCf_2) < S(NWAPN)$ and the point Y from the graph of g will go faster to 0 when moving to the right than when moving to the left. In fact, when moving to the left it will first descend (till f_2) and only then go up.

5.^{o} After this increase the constant c_5 so that x_3 become < 0. The increasing can be chosen so small that all strict inequalities between roots of g and its derivatives be preserved. Then increase c_4 so that f_3 become greater than t_2 and all strict inequalities between roots of g and its derivatives be preserved.

Finally, repeat the reasoning from 6.^{o} of the proof of Lemma 2.3 (i.e., change $g^{(5)}$ from $\delta(x)$ to $h(x)$ and then to $h(x) + b$). The function $g(x)$ thus obtained is a PLF and realizes the arrangement from matrix (11). Its symmetric arrangement is realized by the function $g(-x)$. □

Remark 2.7. If in 5^0 of the proof one decreases c_5 instead of increasing it, and if after this one decreases c_4, then one similarly realizes the arrangement

$$(0, f, 0, s, f, t, s, l, 0, f, t, 0, s, f, 0) \in \{(fftftf)\}$$

by a PLF $g(x)$, and by $g(-x)$ its symmetric one.

Lemma 2.8. (i) *The two non-degenerate arrangements defined by matrix* (12) *(hence, their symmetric ones as well) are realizable by PLFs of degree* 5*:*

$$\begin{array}{cccccccccccccccc}
 & & & & & & & l & & & & & & & & \\
 & & & & & t & & & t & & & & & & & \\
 & & & s & & & s & & & & & \vdots & s & \vdots & & \\
 & f & & & f & & & & & & f & \vdots & & \vdots & f & \\
0 & & 0 & & & & & & & 0 & & \vdots & 0 & \vdots & & 0
\end{array} \qquad (12)$$

(ii) *The two arrangements from matrix* (12) *differ from the two arrangements from matrix* (17) *(not realizable by PLFs) only by the sign of the inequality between* l_1 *and* s_2 *which is an inequality between roots of derivatives of highest possible orders.*

Proof. 1.$^{\mathrm{o}}$ Part (ii) of the lemma is self-evident, so we prove only part (i) of it. For each of the arrangements $([0f0], s, [ft], l, s, [t0], f, \mathcal{P}, f, 0)$ where $\mathcal{P} = (0, s)$ or $\mathcal{P} = (s, 0)$, we construct C^3-smooth functions g which realize them. (In principle, one should not put the letter l in the CV given that the functions are only C^3-smooth; however, in the subsequent approximation of g by a PLF the root l_1 will be in the same place as indicated in the CV.)

The function $g^{(4)}$ is piecewise-constant, it equals -24 on some interval $(-\infty, \tau)$ and $24a$, $a > 0$ on (τ, ∞). (In the PLF approximating g the root l_1 will be close to τ). On $(-\infty, \tau)$ one has $g(x) = P(x) := -x^4 + x^2 - \frac{1}{4}$. Notice that the polynomial $P(x)$ realizes the arrangement $([0f0], s, [ft], s, [0f0])$. We choose τ to be bigger than $s_2 = \frac{1}{\sqrt{6}}$. On (τ, ∞) one has

$$g(x) = a(x-\tau)^4 + \frac{P^{(3)}(\tau)}{6}(x-\tau)^3 + \frac{P^{(2)}(\tau)}{2}(x-\tau)^2 + P^{(1)}(\tau)(x-\tau) + P(\tau) \quad (13)$$

Observe first that the function g possesses the following properties:

(a) If one sets $\tau = \frac{1}{2}$ and $g(x) = P_1(x) := 3x^4 - 8x^3 + 7x^2 - 2x$ for $x \geq \tau$, then such a function g realizes the arrangement $([0f0], s, [ft], s, [0t], f, s, [0f0])$. This can be checked directly — the roots of P_1 (resp. $P_1^{(3)}$) equal 0, $\frac{2}{3}$, 1, 1 (resp. $\frac{2}{3}$); one has $P^{(i)}(\frac{1}{2}) = P_1^{(i)}(\frac{1}{2})$, $i = 0, 1, 2, 3$.

(b) The functions g which we construct below are obtained by increasing τ ($\tau \in (\frac{1}{2}, \frac{2}{3})$) and by keeping the same root $t_2 = \frac{2}{3}$. The restriction of g to (τ, ∞) is a polynomial P_k, $k = 2$ or 3, of degree 4, whose roots we denote by x_2, x_3, x_4, x_5 (and not by x_1, x_2, x_3, x_4) in order to have the same notation for the roots of g or of P_k; we make a similar shift by 1 of the indices of the roots of the derivatives of P_k. We list all the roots of P_k and its derivatives, but the ones smaller than τ are of no importance for the arrangement realized by the roots of g and its derivatives because for $x \leq \tau$ one has $g(x) = P(x)$.

(c) The condition $t_2 = \frac{2}{3}$ implies that one has

$$a = -\frac{P^{(3)}(\tau)}{24(\frac{2}{3} - \tau)} = \frac{\tau}{\frac{2}{3} - \tau} \tag{14}$$

When τ is close to $\frac{1}{2}$ one has $s_3 < x_4$ (see the polynomial P_2), when it becomes big enough, then one has $s_3 > x_4$ (see the polynomial P_3).

2.° For $\tau = \dfrac{51}{100}$ and for $x \geq \dfrac{51}{100}$ one has $a = \dfrac{153}{47}$ and

$$\begin{aligned} g(x) = P_2(x) &:= \frac{153}{47}\Big(x - \frac{51}{100}\Big)^4 - \frac{51}{25}\Big(x - \frac{51}{100}\Big)^3 \\ &\quad - \frac{2803}{5000}\Big(x - \frac{51}{100}\Big)^2 + \frac{122349}{250000}\Big(x - \frac{51}{100}\Big) - \frac{5755201}{10000000} \\ &= \frac{153}{47}x^4 - \frac{408}{47}x^3 + \frac{8978}{1175}x^2 - \frac{132651}{58750}x + \frac{890201}{23500000}. \end{aligned}$$

The roots of P_2 equal

$$x_2 = 0.01783147674, \qquad x_3 = 0.6691445423,$$
$$x_4 = 0.9224950606, \qquad x_5 = 1.057195587,$$

the ones of $P_2^{(2)}$ equal $s_2 = 0.4359153127$, $s_3 = 0.8974180206$ (and one has $t_2 = \frac{2}{3} < x_3$). Thus the function g defined for $\tau = \frac{51}{100}$ realizes the arrangement

$$([0f0], s, [ft], s, l, t, 0, f, s, 0, f, 0).$$

3.° For $\tau = \frac{6}{10}$ and for $x \geq \frac{6}{10}$ one has $a = 9$ and

$$\begin{aligned} g(x) = P_3(x) &:= 9\Big(x - \frac{6}{10}\Big)^4 - \frac{12}{5}\Big(x - \frac{6}{10}\Big)^3 - \frac{29}{25}\Big(x - \frac{6}{10}\Big)^2 \\ &\quad + \frac{42}{125}\Big(x - \frac{6}{10}\Big) - \frac{49}{2500} \\ &= 9x^4 - 24x^3 + \frac{113}{5}x^2 - \frac{216}{25}x + \frac{523}{500}. \end{aligned}$$

The roots of P_3 equal

$$x_2 = 0.2225775312, \qquad x_3 = 0.6891187808,$$
$$x_4 = 0.7667987632, \qquad x_5 = 0.9881715914,$$

the ones of $P_3^{(2)}$ equal $s_2 = 0.5056513695$, $s_3 = 0.8276819638$. One has again $t_2 = \frac{2}{3} < x_3$. The function g defined for $\tau = \frac{6}{10}$ realizes the arrangement

$$([0f0], s, [ft], s, l, t, 0, f, 0, s, f, 0).$$

4.º Add small positive constants to the functions g constructed for $\tau = \frac{51}{100}$ and $\tau = \frac{6}{10}$ so that the double root $x_1 = x_2$ split into two simple roots while preserving all strict inequalities between roots. After this add to the thus modified functions g affine functions $\varepsilon(x - f_2)$ where $\varepsilon > 0$ is small enough (to preserve all existing strict inequalities between roots). After this last modification the function g (denoted from now on by $\tilde{g}$) with $\tau = \frac{51}{100}$ (resp. $\tau = \frac{6}{10}$) realizes the arrangement

$$(\Omega)\ :\ (0, f, 0, s, f, t, s, l, t, 0, f, s, 0, f, 0)$$
$$\text{(resp.}\quad (\Xi)\ :\ (0, f, 0, s, f, t, s, l, t, 0, f, 0, s, f, 0)).$$

5.º One can smoothen the functions $\tilde{g}^{(3)}$ in a neighborhood of τ to make them C^∞-smooth and so that $\tilde{g}^{(4)}$ be increasing. The new functions are denoted by g^*. The smoothening can make the functions g^* as close to the functions $\tilde{g}$ as to preserve all strict inequalities between roots. Hence, the functions g^* realize the same arrangements as the functions $\tilde{g}$ and the root l_1 of ${g^*}^{(4)}$ appears in the CV indeed between s_2 and t_2.

There remains to change g^* into a PLF by adding a small positive constant to ${g^*}^{(5)}$. (Remember that ${g^*}^{(5)}$ is with compact support; it is greater than 0 because ${g^*}^{(4)}$ is increasing.) When this constant is small enough the functions g^* are PLFs and realize arrangements (Ω) and (Ξ). $\square$

2.2. Proof of the non-existence part

It follows from the results in [5] that all non-degenerate arrangements with $s_1 < x_2$, $f_2 < t_1$, $s_2 < x_3$, or with $x_4 < s_3$, $t_2 < f_3$, $x_3 < s_2$, are not realizable by PLFs of degree 5. This makes 46 arrangements. Out of them exactly 20 belong to the case $(ffttff)$ (the reader will easily check this oneself). Up to the symmetry induced by the change $x \mapsto -x$ (see Remarks 1.13) these are the arrangements defined by the following two matrices:

$$\begin{array}{cccccccccccccccccc}
 & & & & & & \vdots & l & \vdots & & & & & & & & & \\
 & & & & & t & \vdots & & \vdots & \vdots & t & \vdots & & & & & & \\
 & & s & & & & \vdots & s & \vdots & \vdots & & \vdots & & \vdots & s & \vdots & & \\
 & f & & & f & & \vdots & & \vdots & \vdots & & \vdots & f & \vdots & & \vdots & f & \\
0 & & & 0 & & & \vdots & & \vdots & \vdots & 0 & \vdots & & \vdots & 0 & \vdots & & 0
\end{array} \tag{15}$$

This matrix defines eight arrangements (there are three couples where one can choose each of the two permutations). The following matrix defines two more arrangements:

$$\begin{array}{cccccccccccccccc}
 & & & & & & & l & & & & & & & & \\
 & & & & & & t & & & & t & & & & & \\
 & & \vdots & s & \vdots & & & & & s & & & & s & & \\
 & f & \vdots & & \vdots & f & & & & & & f & & & f & \\
0 & & \vdots & 0 & \vdots & & & & 0 & & & & 0 & & & 0
\end{array} \tag{16}$$

Remark 2.9. Matrix (15) implies the non-realizability of the following partial arrangements of the roots of a PLF of degree 5 and its first three (and not four) derivatives:

$$(0, f, s, 0, f, t, s, t, 0, f, \mathcal{P}, f, 0) \quad \text{where} \quad \mathcal{P} = (s, 0) \quad \text{or} \quad (0, s).$$

Indeed, in the absence of equalities between roots these partial arrangements allow only two possibilities for l_1: $s_2 < l_1 < t_2$ and $t_1 < l_1 < s_2$. For each of them the corresponding (complete) arrangements are defined by matrix (15), hence, are not realizable by PLFs.

On the other hand, all non-degenerate partial arrangements of the roots of a PLF (even better — of a hyperbolic polynomial) and its first two derivatives are realizable. This follows from [7, Theorem 2] — it suffices to realize the partial arrangement of the roots of the polynomial and its second derivative which defines the relative positions of the roots of its first derivative.

Lemma 2.10. *The two non-degenerate arrangements represented by matrix* (17) *and their symmetric ones are not realizable by PLFs of degree* 5.

$$\begin{array}{cccccccccccccccc}
 & & & & & & l & & & & & & & & & \\
 & & & & & t & & & t & & & & & & & \\
 & & & s & & & & s & & & & \vdots & s & \vdots & & \\
 & f & & & f & & & & & & f & \vdots & & \vdots & f & \\
0 & & 0 & & & & & & & 0 & & \vdots & 0 & \vdots & & 0
\end{array} \tag{17}$$

Proof. I) Plan of the proof

Definition 2.11. We define the set W as the one consisting of the two non-degenerate arrangements defined by matrix (17) and of the degenerate arrangement obtained from anyone of them by replacing the inequality $s_3 < x_4$ or $s_3 > x_4$ by $s_3 = x_4$.

We first change a PLF g supposed to realize an arrangement from the set W to another function g_2 with simpler graph of $g_2^{(3)}$ which also realizes such an arrangement, see **II)**. The function g_2 is not a PLF but can be approximated by PLFs which also realize arrangements from the set W. (In fact, one can perturb them so that they realize non-degenerate arrangements from W.) After this in **III)** we prove that it is impossible to realize an arrangement from W by the function g_2.

The change of g into g_2 is done in two steps, see **IIA)**, **IIB)**. At these steps we modify the graph of g respectively on $(-\infty, t_1]$ and $[t_2, \infty)$. The functions thus obtained are denoted by g_1 and g_2. We include them into a homotopy g_τ, $\tau \in [0, 2]$; we set $g_0 = g$. (When g_j and g_{j-1} are defined, one sets $g_\tau = (\tau - j + 1)g_j + (j - \tau)g_{j-1}$ for $\tau \in [j-1, j]$.)

II) Replacing the PLF realizing a given arrangement by another function

IIA) Modification of the graph on $(-\infty, t_1]$

1.$^{\mathrm{o}}$ Suppose that at least one of the arrangements of the set W is realizable by a PLF g. The graphs of g, $g^{(1)}$, $g^{(2)}$ and $g^{(3)}$ are shown on fig. 4, one above the other.

Definition 2.12. An *almost PLF (APLF)* of degree n is a function $g : \mathbf{R} \to \mathbf{R}$ for which $g^{(n)} \geq 0$ and $g^{(n)}$ is piecewise-smooth, with a finite number of discontinuities at which there exist the left and right limits. If an APLF g has n real zeros (counted with the multiplicities), then it is called *hyperbolic*. In what follows all APLFs are presumed hyperbolic.

Remark 2.13. It is clear that if a non-degenerate arrangement is realizable by an APLF, then it is realizable by a PLF as well (one has to approximate the n-th derivative by a smooth positive-valued function; if the approximation is good enough, then all strict inequalities between roots are preserved). We use APLFs in situations when they yield easier estimations.

2.$^{\mathrm{o}}$ Consider a PLF g realizing an arrangement from the set W as a five-fold integral of $g^{(5)}$. Integration is performed from a fixed point from the interval (t_1, t_2).

Replace g by an APLF g_1 (with the same constants of integration) for which $g_1^{(5)} \equiv g^{(5)}$ for $x \geq t_1$ and $g_1^{(5)} \equiv 0$ for $x < t_1$. This means that $g_1^{(4)}$ is constant for $x < t_1$; to obtain the graph of $g_1^{(3)}$ for $x < t_1$ one has to replace the one of $g^{(3)}$ by the tangent to this graph at $(t_1, 0)$, see fig. 4.

Statement 2.14. *The APLF g_1 and the PLF g realize the same arrangement.*

Proof of Statement 2.14. Consider the homotopy $g_\tau := (1-\tau)g + \tau g_1$ defined after $(1-\tau)g^{(5)} + \tau g_1^{(5)}$ with the same constants of integration for all $\tau \in [0, 1]$. For each τ this is an APLF (and a PLF for $\tau = 0$). For all $\tau \in [0, 1]$ the function $g_\tau^{(2)}$ has exactly one zero in $(-\infty, t_1)$ because one is integrating the positive-valued function $g_\tau^{(3)}$ from t_1 to x, $x < t_1$, and $g_\tau^{(2)}(t_1) > 0$ is fixed.

When τ increases from 0 to 1, then the root s_1 moves to the left, and for each $a \in (-\infty, t_1)$ fixed the value of $g_\tau^{(2)}$ increases with τ. Hence, the root f_2 moves to

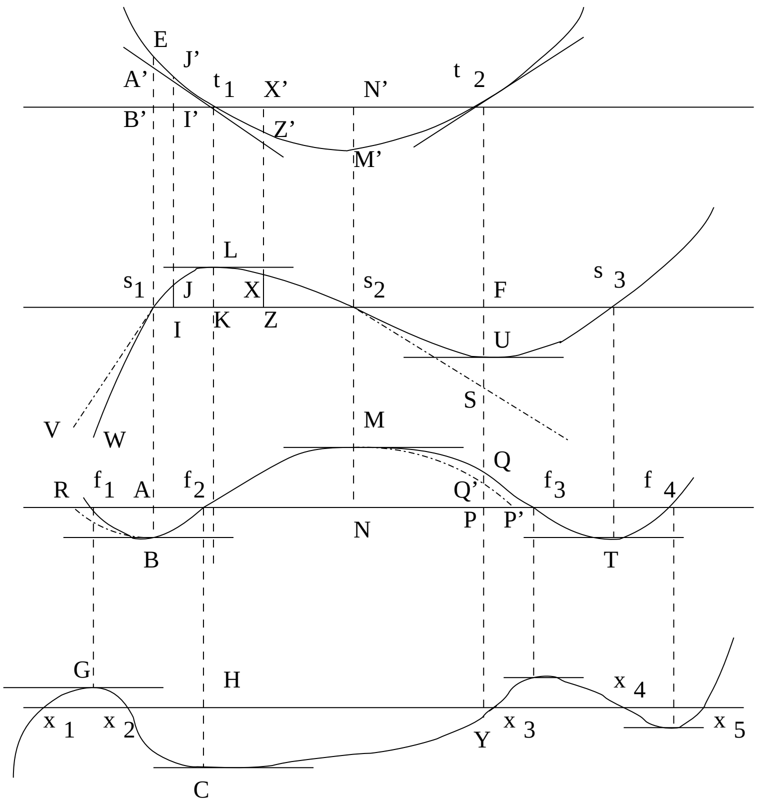

Figure 4: The graphs of g, $g^{(1)}$, $g^{(2)}$, and $g^{(3)}$.

the right (without reaching t_1 because $g_\tau^{(1)}(t_1) > 0$ does not depend on τ). For each $a \in (-\infty, t_1)$ fixed the value of $g_\tau^{(1)}(a)$ decreases. For each τ fixed one has $g_\tau^{(1)}(x) \to \infty$ when $x \to -\infty$. Hence, $g_\tau^{(1)}$ has a root $f_1 < f_2$, and it is clear that it can have no roots other than f_1, f_2, f_3, f_4.

In the same way one shows that the root x_2 of g_τ moves to the right when τ increases (without reaching t_1 because $g_\tau(t_1) < 0$ is fixed). As $g_\tau(x) \to -\infty$ when $x \to -\infty$, the function g_τ has a root $x_1 < x_2$. Hence, it is a hyperbolic APLF.

The arrangement realized by g_τ may change only if for some τ one has $x_2 = s_1$ (because t_1 and all roots to the right of t_1 do not change their positions). The following statement implies that this doesn't happen.

Statement 2.15. *For each* $\tau \in [0,1]$ *one has*

$$\left| \int_{s_1}^{f_2} g_\tau^{(1)}(x)\,dx \right| = S(f_2ABf_2) < S(f_2MNf_2) = \left| \int_{f_2}^{s_2} g_\tau^{(1)}(x)\,dx \right| \tag{18}$$

and

$$\|AB\| < \|MN\| \tag{19}$$

The statement implies that one cannot have $g_\tau(s_1) > g_\tau(t_2)$ because

$$\left| \int_{s_1}^{f_2} g_\tau^{(1)}(x)\,dx \right| < \left| \int_{f_2}^{s_2} g_\tau^{(1)}(x)\,dx \right| < \left| \int_{f_2}^{t_2} g_\tau^{(1)}(x)\,dx \right|,$$
$$\text{hence,} \quad g_\tau(s_1) < g_\tau(s_2) < g_\tau(t_2) \tag{20}$$

However, if one has $x_2 = s_1$ for some τ, then one must have $g_\tau(s_1) = 0$ and $g_\tau(t_2) < 0$ — a contradiction with (20). Hence, the arrangement realized by g_τ does not change throughout the homotopy. □

Proof of Statement 2.15. Consider two points I, Z such that $I \in (s_1, K)$, $Z \in (K, s_2)$, $\|IK\| = \|KZ\|$ (see the graph of $g^{(2)}$ on fig. 4). One has $\|IJ\| < \|ZX\|$. Indeed, consider $g^{(2)}$ as a primitive of $g^{(3)}$. The graph of $g^{(3)}$ is convex and lies above its tangent at $(t_1, 0)$. Hence, if a point follows the graph of $g^{(2)}$ starting at L, then it descends faster when it is moving to the left than to the right because one has

$$\begin{aligned} \|KL\| - \|IJ\| &= \left| \int_{I'}^{t_1} g^{(3)}(x)\,dx \right| = S(t_1J'I't_1) > S(t_1X'Z't_1) \\ &= \left| \int_{t_1}^{X'} g^{(3)}(x)\,dx \right| = \|KL\| - \|ZX\|. \end{aligned}$$

This implies that $\|s_1K\| < \|Ks_2\|$. In the same way one shows that when a point follows the graph of $g^{(1)}$ starting at f_2, then it climbs faster to the right than it descends to the left. Moreover, one has $\|f_2N\| > \|f_2A\|$. This proves condition (18) and (having in mind that $\|s_1f_2\| < \|s_1t_1\| = \|s_1K\| < \|Ks_2\| = \|t_1s_2\| < \|f_2s_2\|$) also condition (19). □

Remark 2.16. One can prove in a similar way (using the concavity of $g^{(2)}$ on $(-\infty, t_1]$) that one has $s_1 - f_1 < f_2 - s_1$, and, hence, $s_1 - f_1 < t_1 - s_1 < s_2 - t_1$.

IIB) Modification of the graph on $[t_2, \infty)$

3.° Replace the APLF g_1 by another APLF g_2 this time modifying the graph of g_1 to the right of t_2. Namely, we set $g_2^{(5)}(x) = 0$ for $x > t_2$, $g_2^{(5)}(x) = g_1^{(5)}(x)$ for $x \leq t_2$, and we keep the same constants of integration when defining $g_2^{(j)}$, $j = 0, \dots, 4$. Hence, $g_2^{(4)}$ is constant for $x \geq t_2$ and to obtain the graph of $g_2^{(3)}$ from the one of $g_1^{(3)}$ one has to replace this graph to the right of t_2 by the tangent line at $(t_2, 0)$ to the graph of $g_1^{(3)}$, see fig. 4.

Statement 2.17. *The function g_2 is a hyperbolic APLF.*

Proof of Statement 2.17. We follow the same ideas as in the proof of Statement 2.14. First of all, include g_2 in the homotopy g_τ (see **I)**), i.e., set $g_\tau = (2-\tau)g_1 + (\tau - 1)g_2$, $\tau \in [1, 2]$. For τ close to 1 the APLFs g_1 and g_τ define the same arrangement.

The following property is evident:

Property 2.18. *When τ increases from 1 to 2, then for each $a > t_2$ fixed, the values of $g_\tau^{(j)}(a)$ $(j = 0, \dots, 4)$ decrease.*

Hence, the roots s_3, f_4, and x_3 move to the right while the root f_3 moves to the left (without attaining t_2 because $g_\tau^{(1)}(t_2) > 0$ remains fixed). For $x > t_2$, $g_\tau^{(2)}$ (resp. $g_\tau^{(1)}$) is a polynomial of x of degree 2 (resp. 3) with positive leading coefficient, and one has $g_\tau^{(3)}(x) > 0$ for $x > t_2$. This implies that $g_\tau^{(2)}$ has exactly one root (namely, s_3) for $x > t_2$, and $g_\tau^{(1)}$ has exactly two roots (namely, f_3 and f_4) there.

To prove that g_2 is a hyperbolic APLF it suffices to prove the following

Statement 2.19. *For no $\tau \in [1, 2]$ does one have $g_\tau(f_3) \leq g_\tau(f_1)$.*

Indeed, it follows from Property 2.18 that $g_\tau(f_3)$ decreases when τ increases — to obtain $g_\tau(f_3)$ one integrates the function $g_\tau^{(1)}(x)$ (positive-valued on (t_2, f_3) and whose value as a function of τ is decreasing for each $x > t_2$ fixed) along an interval whose length decreases with τ. On the other hand, the difference $g_\tau(f_3) - g_\tau(f_4)$ increases with τ due to Property 2.18. Therefore one also has $g_\tau(f_4) < 0$ which means that g_τ has a root $x_4 \in (f_3, f_4)$. As g_τ has at least four roots $x_1, \dots, x_4$, then it has also a fifth one $x_5 > f_4$, (For $x \to \infty$ it behaves like bx^4, $b > 0$.) hence, g_τ is a hyperbolic APLF. □

Remarks 2.20. (i) In Statement 2.17 we do not claim that the APLFs g_2 and g_1 (hence, the PLF g as well) realize the same arrangement. This is so because we do not take into account the relative position of s_3 and x_4. However, if this position changes (from $x_4 < s_3$ to $x_4 > s_3$ or vice versa), then the arrangement changes to another arrangement from the set W.

(ii) One need not consider separately the particular situation when $x_4 = s_3$ (all other strict inequalities between roots being preserved) because one can approximate the APLF g_2 by a PLF so that this equality changes to $x_4 < s_3$ or to $x_4 > s_3$.

(iii) To prove that g_2 realizes one of the arrangements from the set W it is sufficient to show that g_2 is a hyperbolic APLF because the relative position of t_2 and of the roots to the left of t_2 are the same for g_1 and g_2.

Proof of Statement 2.19. One has

$$\begin{aligned} g_\tau(f_1) - g_\tau(f_3) &= (g_\tau(f_1) - g_\tau(f_2)) - (g_\tau(f_3) - g_\tau(f_2)) \\ &= \left|\int_{f_1}^{f_2} g_\tau^{(1)}(x)\,dx\right| - \left|\int_{f_2}^{f_3} g_\tau^{(1)}(x)\,dx\right| \\ &= S(f_1Bf_2Af_1) - S(f_2NPf_3QMf_2). \end{aligned}$$

We show that $S(f_1Bf_2Af_1) < S(f_2NPf_3QMf_2)$ which implies the statement. The last inequality follows from inequality (18) and from

$$S(f_1BAf_1) - S(NPf_3QMN) = \left|\int_{f_1}^{s_1} g_\tau^{(1)}(x)\,dx\right| - \left|\int_{s_2}^{f_3} g_\tau^{(1)}(x)\,dx\right| < 0 \tag{21}$$

which we prove now.

One has

$$|g_\tau^{(3)}(s_2)| = \|N'M'\| \leq \|B'A'\| = |g_\tau^{(3)}(s_1)| \tag{22}$$

because

$$\bar{S}(A'B't_1A') = \frac{\|B'A'\|\,\|s_1K\|}{2} \geq S(t_1N'M't_1) \geq \frac{\|N'M'\|\,\|Ks_2\|}{2}$$

and $\|s_1K\| \leq \|Ks_2\|$ (see Remark 2.16). The absolute value of $g_\tau^{(3)}$ grows to the left of s_1 not slowlier than to the right of s_2 on (s_2, t_2). (This follows from the convexity of the graph of $g_\tau^{(3)}$.) Therefore the same statement holds for the absolute value of $g_\tau^{(2)}$. This in turn means that when a point is following the graph of $g_\tau^{(1)}$, then it climbs faster on (s_1f_1) than it descends on (s_2, f_3). As one has also (19), this implies inequality (21). □

III) End of the proof

4.º Denote by s_1V the tangent line at $(s_1, 0)$ to the graph of $g_2^{(2)}$ and by s_2S the one at $(s_2, 0)$ (in dash-dotted lines on fig. 4). Denote by BR and $MQ'P'$ the graphs of $g_2^{(1)}$ when its derivative $g_2^{(2)}$ is replaced by the affine function whose graph is one of these tangent lines. We keep the constant of integration the same; integration starts at s_1 or s_2.

Remark 2.21. We use the fact that $l_1 \leq s_2$ as follows: the curve s_2U, see fig. 4, lies above the tangent line s_2S for $x \in (s_2, t_2]$. (For $l_1 > s_2$ this is not true.) This implies that the curve $MQ'P'$ (which is a parabola) lies below the curve MQf_3.

Statement 2.22. *If there exists an APLF g_2 realizing an arrangement from the set W, then there exists an APLF realizing an arrangement obtained from one of the set W in which the inequalities $f_2 < t_1$, $t_2 < x_3$, $x_4 < x_5$ are replaced by the corresponding equalities.*

Proof of Statement 2.22. Consider the one-parameter deformation $g_2(x) + s(x - t_1)$, $s \leq 0$. By abuse of notation we denote it again by g_2. It defines (for all $s \leq 0$ such that $f_2 \leq t_1$) an APLF. Indeed, $g_2(f_1)$ increases and $g_2(f_3)$ decreases when s decreases. (The positions of f_1 and f_3 also change.) Hence, g_2 has two real roots which are $\leq t_1$. (It behaves like $-bx^4$, $b > 0$, for $x \to -\infty$.) For all $s \leq 0$ one has $g_2(f_3) > g_2(f_1) > 0$. (This can be proved by analogy with Statement 2.19.) As $g_2(f_4)$ decreases, g_2 must have three real roots which are $\geq t_2$. As for $x \to \infty$ g_2 behaves like bx^4, $b > 0$, g_2 has exactly five real roots. As $g_2^{(5)}$ does not depend on s, g_2 is an APLF for all $s \leq 0$ until one has $f_2 = t_1$. So assume that $f_2 = t_1$.

Case 1). Suppose that one has $g_2(t_2) \geq g_2(f_4)$.

Set $g_2(x) \mapsto g_2(x) - g_2(t_2)$. Hence, the new function $g_2(x)$ is also an APLF and one has $g_2(t_2) = 0$, $t_2 = x_3$.

If one has $x_4 = x_5$, then there is nothing to do. If $x_4 < x_5$, then observe first that one has $t_2 - t_1 < t_1 - f_1$. Indeed, define $r \in \mathbf{R}$ by the condition $r - t_1 = t_1 - s_1$. One proves by analogy with Statement 2.15 that one has

$$\left| \int_{t_1}^{r} g_2^{(1)}(x)\, dx \right| \geq \left| \int_{s_1}^{t_1} g_2^{(1)}(x)\, dx \right| \quad \text{and} \quad |g_2^{(1)}(r)| \geq |g_2^{(1)}(s_1)| = \|AB\|. \tag{23}$$

The value of the function $|g_2^{(1)}(x)|$ decreases faster when x decreases from s_1 to f_1 than when x grows from r to t_2. (In fact, $|g_2^{(1)}(x)|$ increases for $x \in [r, s_2]$, and then decreases for $x \in [s_2, t_2]$.) This can be proved by analogy with the proof of Statement 2.15. Hence, if one has $t_2 - t_1 \geq t_1 - f_1$, then one must have

$$g_2(t_2) - g_2(t_1) = \left| \int_{t_1}^{t_2} g_2^{(1)}(x)\, dx \right| > \left| \int_{f_1}^{t_1} g_2^{(1)}(x)\, dx \right| = g_2(f_1) - g_2(t_1),$$

i.e., $g_2(t_2) > g_2(f_1)$ which is impossible; the inequality in the middle follows from (23) and from the lines above.

Thus one has $t_2 - t_1 < t_1 - f_1$. Set $g_2(x) \mapsto g_2(x) + u((x - t_1)^2 - (t_2 - t_1)^2)$, $u \geq 0$. Hence, $g_2(t_1)$ decreases when u increases, $g_2(t_2)$ does not change while $g_2(f_1)$, $g_2(f_3)$, and $g_2(f_4)$ increase. This means that there exists $u > 0$ for which one has $x_4 = x_5$ (and one has already $f_2 = t_1$, $x_3 = t_2$).

Case 2). Suppose that one has $g_2(t_2) < g_2(f_4)$.

Set $g_2(x) \mapsto g_2(x) - vq(x)$, $q(x) := (x - t_1)^2 - (f_3 - t_1)^2$, $v \geq 0$. Notice that $q^{(1)}(t_1) = 0$. When v increases, then f_3 and $g_2(f_3)$ remain the same, x_3 and x_4 decrease, x_5, $g(t_2)$, and $-g_2(f_4)$ increase.

Prove that $g_2(f_1)$ increases. One has $s_2 - t_1 > t_1 - s_1$ (see Remark 2.16) and $f_3 - s_2 > \|P'N\| > \|AR\| \geq \|Af_1\| = s_1 - f_1$. (This follows from the fact that the parabolas MP' and BR have horizontal tangents at M and B, from (19) and from (22).) Hence,

$$f_3 - t_1 = (f_3 - s_2) + (s_2 - t_1) > (s_1 - f_1) + (t_1 - s_1) = t_1 - f_1$$

which means that $q(f_1) < 0$ and that $g_2(f_1)$ increases with v.

When increasing v, if one has $g_2(t_2) < g_2(f_4) < 0$, then one cannot have $x_2 = t_1$ for any $v > 0$. Indeed, as one has $g_2^{(1)}(t_1) = 0$ for all v, the equality $x_2 = t_1$ would imply that x_2 is at least a double root of g_2 which together with x_1, x_3, x_4 and x_5 makes at least 6 real roots (counted with the multiplicities) — a contradiction.

Hence, one can choose v such that $g_2(t_2) = g_2(f_4)$. After this set $g_2(x) \mapsto g_2(x) - g_2(t_2)$. Thus one has all three equalities $x_4 = x_5$, $f_2 = t_1$, $x_3 = t_2$. □

Assumption 2.23. *We assume till the end of the proof of Lemma 2.10 that the APLF g_2 realizes an arrangement satisfying the conclusion of Statement 2.22.*

Statement 2.24. *One has*

$$S(BRf_1AB) = \frac{4}{3}\sqrt{\frac{2}{3}}S(f_1BAf_1)$$

Proof of Statement 2.24. Recall that the graph of $g_2^{(2)}$ restricted to $(-\infty, s_1]$ is an arc of a parabola. To ease the computation assume (after a change of the scopes of the axes) that its equation is $y = 1 - x^2$.

Hence, $t_1 = 0$, $s_1 = -1$ and $f_1 = -\sqrt{3}$. The last equality follows from the condition $\left|\int_{f_1}^{s_1} g_2^{(2)}(x)\,dx\right| = \left|\int_{s_1}^{f_2} g_2^{(2)}(x)\,dx\right|$. After this one finds that $S(f_2ABf_2) = \frac{5}{12}$, $S(f_1BAf_1) = \frac{1}{3}$. The equation of the tangent line s_1V is $y = 2x + 2$, the one of the parabola RB is $y = x^2 + 2x + \frac{1}{3}$, the x-coordinate of the point R equals $-1 - \sqrt{\frac{2}{3}}$, and one finds that $S(BRf_1AB) = \frac{4}{9}\sqrt{\frac{2}{3}}$ which implies the equality from the statement. □

Notation 2.25. On fig. 5 we show parts of the graphs of $g_2^{(3)}$, $g_2^{(2)}$, $g_2^{(1)}$, and g_2 under Assumption 2.23. The curve UU' is the continuation of the arc of parabola s_3U. As the arc $N''t_2$ lies above the tangent line to the graph of $g_2^{(3)}$ at $(t_2, 0)$, the arc UU' lies

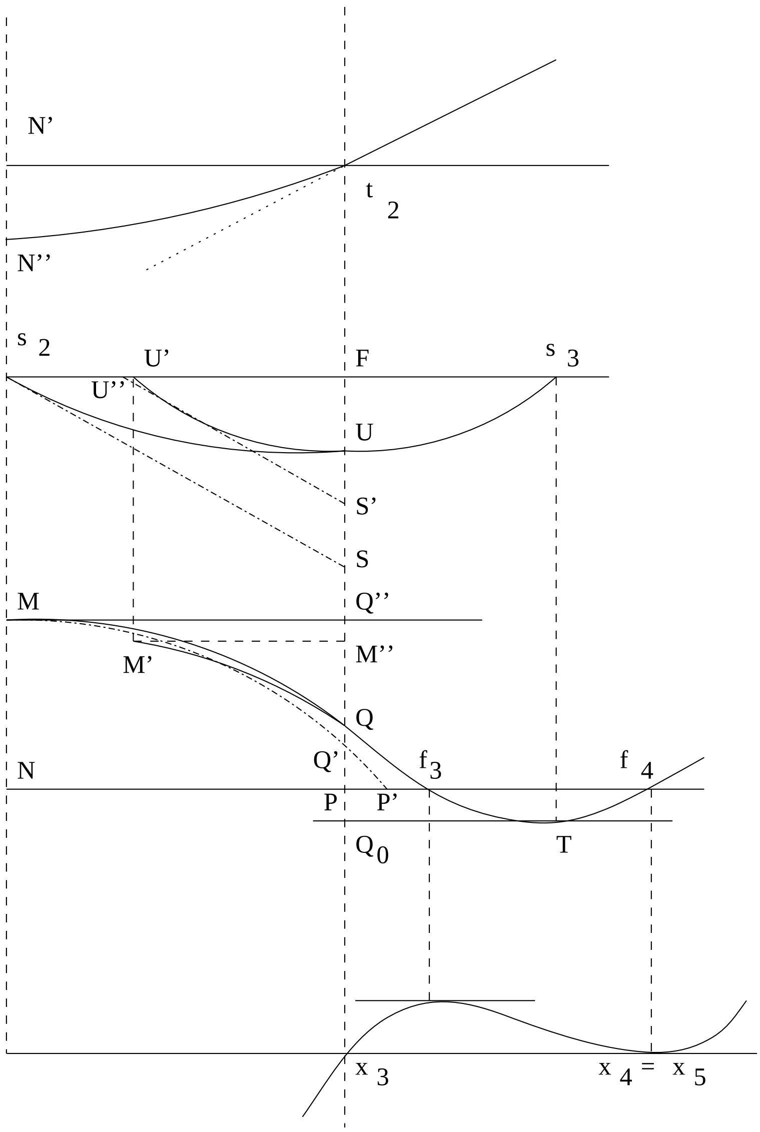

Figure 5: A detail of the graphs of $g_2^{(3)}$, $g_2^{(2)}$, $g_2^{(1)}$, and g_2.

above the arc Us_2; these two arcs are (horizontally) tangent at U. Denote by $S'U''$ a tangent line to the arc of parabola UU'.

Statement 2.26. *One has* $\bar{S}(S'U''FS') \geq \frac{2}{\sqrt{3}}S(UU'FU)$.

Proof of Statement 2.26. By changing the scopes of the axes one can assume that the equations of the parabola $U'Us_3$ and of its tangent line $S'U''$ are respectively $y = x^2 - 1$ and $y = 2x_0x - x_0^2 - 1$ where $(x_0, x_0^2 - 1)$ is the point of tangency, $x_0 < 0$. Hence, $\|S'F\| = x_0^2 + 1$, $\|U''F\| = (x_0^2 + 1)/2|x_0|$, and $\bar{S}(S'U''FS') = \phi(s) := \frac{(s^2+1)^2}{4s}$, $s = -x_0$. When $s > 0$, the function ϕ attains its minimum for $s = \frac{1}{\sqrt{3}}$ and this minimum equals $\frac{4}{3\sqrt{3}}$. One has $S(UU'FU) = \int_{-1}^{0}(x^2 - 1)\,dx = \frac{2}{3}$ from where the statement follows. $\square$

5.º Consider $g_2^{(1)}$ as a primitive of $g_2^{(2)}$ when integration starts at t_2. Construct the arc QM' when one defines the graph of $g_2^{(2)}$ to be not the arc s_2U but the arc $U'U$. Hence, the arc QM' lies below the arc QM and is tangent to it at Q.

One has $\|QQ''\| \geq \|QM''\| = \|Q_0Q\|$. The last equality follows from the fact that the arc (of parabola) $U'Us_3$ is symmetric w.r.t. the vertical line FU (which implies that the arc $M'QT$ is symmetric w.r.t. the point Q).

Suppose that $t_2 = 0$ and that the equation of the parabola $U'Us_3$ is $y = x^2 - 1$. Hence, the one of the arc $M'QT$ is of the form $y = \frac{x^3}{3} - x + a$, $a \in \mathbf{R}$.

Statement 2.27. *One has* $S(QPf_3Q) = S(f_3Tf_4f_3)$, $a = g_2^{(1)}(0) = \|PQ\| = \frac{\sqrt{2}}{3}$ *and* $\|PQ_0\| = \frac{2-\sqrt{2}}{3}$.

Proof of Statement 2.27. One has

$$S(QPf_3Q) = \int_0^{f_3} g_2^{(1)}(x)\,dx = \left|\int_{f_3}^{f_4} g_2^{(1)}(x)\,dx\right| = S(f_3Tf_4f_3),$$

and if $g_2 = \frac{x^4}{12} - \frac{x^2}{2} + ax + b$, $b \in \mathbf{R}$, then one has the following system of equalities:

$$\frac{(f_3)^3}{3} - f_3 + a = \frac{(f_4)^3}{3} - f_4 + a = 0,$$
$$g_2(f_3) - g_2(0) = g_2(f_3) - g_2(f_4),$$

i.e.,

$$\frac{(f_4)^4}{12} - \frac{(f_4)^2}{2} + af_4 = 0$$

with unknown variables f_3, f_4 and a. The last equation combined with the second of the first two yields $f_4 = \sqrt{2}$. Hence, $a = \frac{\sqrt{2}}{3}$ and $f_3 = \frac{\sqrt{6}-\sqrt{2}}{2}$. One has $\|PQ_0\| = -g_2^{(1)}(s_3) = -g_2^{(1)}(1) = \frac{2-\sqrt{2}}{3}$. $\square$

Remark 2.28. One has $\|Q''Q'\| \geq \frac{4}{3\sqrt{3}}$. Indeed, suppose that on fig. 5 the tangent line $U''S'$ to the parabola $U'Us_3$ is parallel to the tangent line s_2S to the curve s_2U at s_2. Then one has

$$\|Q''Q'\| = \left|\int_{s_2}^{0} g_2^{(2)}(x)\,dx\right| = \bar{S}(s_2SFs_2) \geq \bar{S}(S'U''FS') \geq \frac{2}{\sqrt{3}}S(UU'FU) = \frac{4}{3\sqrt{3}}.$$

The last inequality follows from Statement 2.26.

Statement 2.29. *One has $S(MNPQM) > S(f_1BAf_1)$.*

The statement and condition (18) together imply that one has

$$g_2(t_2) - g_2(t_1) = S(f_2NPQMf_2) > S(f_2Bf_1Af_2) = g_2(f_1) - g_2(t_1)$$

i.e.,

$$g_2(t_2) > g_2(f_1)$$

which is a contradiction. This contradiction proves the lemma. $\square$

Proof of Statement 2.29. One has $S(MNPQM) \geq S(MNPQ'M)$ and we show that $S(MNPQ'M) \geq S(f_1BAf_1)$ which implies the statement.

One has $S(MNP'Q'M) \geq S(BRf_1AB)$; this follows from BR and MP' being parabolas with horizontal tangents at B and M, and from conditions (19) and (22).

On the other hand, one has

$$\tau := \frac{S(MNPQ'M)}{S(MNP'Q'M)} = 0.9370807865 > \frac{3}{4}\sqrt{\frac{3}{2}} = \frac{S(f_1BAf_1)}{S(RBAR)}.$$

Indeed, the last equality follows from Statement 2.24. The inequality is to be checked directly. So there remains to prove only the second equality.

One has

$$\frac{\|PQ'\|}{\|PQ''\|} = \frac{\|PQ'\|}{\|PQ'\| + \|Q''Q'\|} < \frac{\|PQ\|}{\|PQ\| + \|Q''Q'\|} \leq \sigma := \frac{\frac{\sqrt{2}}{3}}{\frac{\sqrt{2}}{3} + \frac{4}{3\sqrt{3}}},$$

see Statement 2.27 and Remark 2.28. Hence,

$$\tau = \int_0^b (1 - x^2)\,dx \Big/ \int_0^1 (1 - x^2)\,dx = \frac{3b - b^3}{2} \quad \text{where} \quad 1 - b^2 = \sigma.$$

A numeric computation yields

$$\sigma = 0.3797958970, \quad b = 0.7875303823, \quad \tau = 0.9370807865. \qquad \square$$

3. Conclusions

The present paper gives the answer to the question which non-degenerate arrangements are realizable by the roots of a PLF of degree 5 and of its derivatives in the case $(ffttff)$. The author intends to publish two more papers (one treating the cases $(ftftff)$ and $(fftftf)$, and one treating the case $(ftfftf)$), after which the answer to this question will be known for all non-degenerate arrangements of degree 5.

Computations made by the author up to now show that in the other three cases (not covered by the present paper) there appear no non-realizable non-degenerate arrangements different from the ones mentioned in [5]. In particular, in the case $(ftfftf)$ all non-degenerate arrangements are realizable. This means that in the case of degree 5 exactly 50 out of 286 non-degenerate arrangements are not realizable by the roots of PLFs and of their derivatives. Out of these 50 arrangements, 46 are the ones described in paper [5] and 4 are the ones described by Lemma 2.10 of the present paper.

The author does not intend to consider the case of degree 6, at least not in detail, because of the enormous number of non-degenerate arrangements (namely, 33592; see Remark 1.5). The non-realizability of some of them follows from the results of [5] and of the present paper. For instance, if a non-degenerate arrangement (U) of degree 6 is such that the partial arrangement defined by the roots of the first five derivatives of the PLF is non-realizable (say, by the results of the present paper), then arrangement (U) is also non-realizable.

Instead of considering the case of degree 6 in detail it would be more interesting to see whether the ratio "non-degenerate arrangements of degree n realizable by the roots of PLFs and of their derivatives"$/N(n)$ (see Remark 1.5 for $N(n)$) tends to a finite limit or not when $n \to \infty$, and whether this limit is 0 or not. This has been suggested to the author by V. I. Arnold.

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