Cartan Subalgebras, Weight Spaces, and Criterion of Solvability of Finite Dimensional Leibniz Algebras

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ABSTRACT

In this work the properties of Cartan subalgebras and weight spaces of finite dimensional Lie algebras are extended to the case of Leibniz algebras. Namely, the relation between Cartan subalgebras and regular elements are described, also an analogue of Cartan's criterion of solvability is proved.

Key words: Leibniz algebra, Cartan subalgebra, weight space.

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Introduction

The present work is devoted to the investigation of the Leibniz algebras, which were introduced by J.-L. Loday in [10] and considered further in works [5–8, 11].

In studying the properties of the homology of Lie algebras Loday noted that if in the definition of an n-th chain the exterior product is changed by the tensor product then in order to prove the derived property defined on chains it is sufficient

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that the algebra satisfies the Leibniz identity instead of antisymmetricity and Jacobi identities. This motivated the introduction of Leibniz algebras, which are a "non skew-symmetric" generalization of Lie algebras. For Leibniz algebras a natural problem arises — to prove analogues of theorems from the theory of Lie algebras.

In the structure theory of finite dimensional Lie algebras it is known that an arbitrary Lie algebra is decomposed into the direct sum of solvable and semisimple parts. In Malcev's work [12] it was shown that the description of solvable Lie algebras is reduced to the description of nilpotent algebras.

Investigations of nilpotent Leibniz algebras [1–3] show that many nilpotent properties of Lie algebras can be extended to the case of nilpotent Leibniz algebras.

In the structure theory of Lie algebras the crucial role is played by Cartan subalgebras and the decomposition of algebras into weight (root) spaces with respect to Cartan (or nilpotent) subalgebras.

In non-Lie Leibniz algebras the ideal generated by squares of elements of the algebra is very important. It is easy to see that this ideal for such algebras is abelian and non trivial.

In the present paper we consider a similar approach to the investigation of Cartan subalgebras and weight spaces of Leibniz algebras.

1. Preliminaries

Definition 1.1. An algebra L over a field F is called *Leibniz algebra* if for any $x, y, z \in L$ the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, where $[\cdot,\cdot]$ is a bilinear multiplication in L.

Note that if in L the identity $[x, x] = 0$ holds, then the Leibniz and Jacobi identities coincide. Thus, Leibniz algebras are a "noncommutative" analogue of Lie algebras.

For an arbitrary algebra we define the sequences:

(i) $L^{[1]} = L$, $L^{[n+1]} = [L^{[n]}, L^{[n]}]$;

(ii) $L^1 = L$, $L^{n+1} = [L^1, L^n] + [L^2, L^{n-1}] + \cdots + [L^{n-1}, L^2] + [L^n, L^1]$.

Definition 1.2. An algebra L is called *solvable* if there exists $m \in N$ such that $L^{[m]} = 0$.

An algebra L is called *nilpotent* if there exists $s \in N$ such that $L^s = 0$.

For an arbitrary element x of L we consider the operator of right multiplication $R_x : L \to L$, where $R_x(z) = [z, x]$. The set $R(L) = \{ R_x : x \in L \}$ forms a Lie algebra with respect to the operation of commutating, and the following identity holds:

$$R_x R_y - R_y R_x = R_{[y,x]}.$$

From this identity it is easy to see that the solvability of the Lie algebra $R(L)$ is equivalent to the solvability of the Leibniz algebra L.

Further in this paper we will assume that all algebras, vector spaces and modules are finite dimensional.

The following lemma gives the decomposition of a vector space into the direct sum of invariant subspaces with respect to a linear transformation.

Lemma 1.3 (Fitting's lemma). *Let V be a vector space and $A : V \to V$ be a linear transformation. Then $V = V_{0A} \oplus V_{1A}$, where $A(V_{0A}) \subseteq V_{0A}$, $A(V_{1A}) \subseteq V_{1A}$ and $V_{0A} = \{ v \in V \mid A^i(v) = 0$ for some $i\}$, $V_{1A} = \bigcap_{i=1}^{\infty} A^i(V)$. Moreover, $A_{|V_{0A}}$ is a nilpotent transformation and $A_{|V_{1A}}$ is an automorphism.*

Proof. See [9, chapter II, §4]. □

Definition 1.4. The spaces V_{0A} and V_{1A} are called respectively *Fitting null component* and *Fitting one component* of the space V with respect to the transformation A.

Lemma 1.5. *Let V be a vector space. And let A, B be linear transformations of V such that*

$$[\cdots[[B, \underbrace{A], A], \dots, A}_{k \text{ times}}] = 0.$$

Then the Fitting components V_{0A}, V_{1A} of the space V with respect to A are also invariant with respect to the transformation B.

Proof. See [9, chapter II, §4]. □

Let L be a nilpotent Leibniz algebra, then it is evident that the Lie algebra $R(L)$ is also nilpotent. Further we will use the following results:

Theorem 1.6. *Let L be a nilpotent Lie algebra of linear transformations of a vector space V and $V_0 = \bigcap_{A\in L} V_{0A}$, $V_1 = \bigcap_{i=1}^{\infty} L^i(V)$. Then the subspaces V_0 and V_1 are invariant with respect to L (i.e., V_0 and V_1 are invariant with respect to every transformation B from L) and $V = V_0 \oplus V_1$. Moreover, $V_1 = \sum_{A\in L} V_{1A}$.*

Proof. See [9, chapter II, §4]. □

Theorem 1.7. *Let G be a split nilpotent Lie algebra of linear transformations of a vector space M. Then G has a finite number of different weights, weight subspaces are submodules of M, and M is decomposed into the direct sum of these modules. Moreover, if $M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$ is an arbitrary decomposition of M into the sum of subspaces M_i ($\neq 0$), which are invariant with respect to G such that the following conditions hold:*

(i) *for each i the restriction of $A \in G$ on M_i has only one characteristic root $\alpha_i(A)$ (of some multiplicity);*

(ii) *if $i \neq j$, then there exists $A \in G$ such that $\alpha_i(A) \neq \alpha_j(A)$;*

then the maps $A \to \alpha_i(A)$ are weights and M_i are weight subspaces.

Proof. See [9, theorem 7, page 43]. □

Proposition 1.8. *Let $\aleph$ be a Lie algebra of linear transformations of a vector space over a field of zero characteristic, G be the radical (solvable) of the algebra $\aleph$ and $\mathfrak{R}$ be the radical of the associative algebra $\aleph^*$. Then $\aleph \cap \mathfrak{R}$ consists of all nilpotent elements of the radical G and $[G, \aleph] \subseteq \mathfrak{R}$. ($\mathfrak{R}$ is considered as a nilpotent radical of the associative algebra $\aleph^*$.)*

Proof. See [9, chapter II]. □

2. Cartan subalgebras of finite dimensional Leibniz algebras

Let $\mathfrak{I}$ be a nilpotent subalgebra of a Leibniz algebra L and $L = L_0 \oplus L_1$ be the Fitting decomposition of the algebra L with respect to the nilpotent Lie algebra $R(\mathfrak{I}) = \{ R_x \mid x \in \mathfrak{I} \}$ of transformations of the vector space L according to theorem 1.6.

The set $l(\mathfrak{I}) = \{ x \in L \mid [x, \mathfrak{I}] \subseteq \mathfrak{I} \}$ is called *left normalizator* of the subalgebra $\mathfrak{I}$ in the algebra L.

The set $r(\mathfrak{I}) = \{ x \in L \mid [\mathfrak{I}, x] \subseteq \mathfrak{I} \}$ is called *right normalizator* of the subalgebra $\mathfrak{I}$ in the algebra L.

Definition 2.1. A subalgebra $\mathfrak{I}$ of a Leibniz algebra L is called *Cartan subalgebra* if the following two conditions are satisfied:

(i) $\mathfrak{I}$ is nilpotent;

(ii) $\mathfrak{I}$ coincides with the left normalizator of $\mathfrak{I}$ in the algebra L.

Since in the Lie algebras case we have antisymmetricity, the sets $l(\mathfrak{I})$ and $r(\mathfrak{I})$ obviously coincide. For a Cartan subalgebra of the Leibniz algebra we have only $l(\mathfrak{I}) \subseteq r(\mathfrak{I})$. It is easy to see that if $\mathfrak{I}$ contains the ideal generated by squares of elements of the algebra L, then we have $l(\mathfrak{I}) = r(\mathfrak{I})$. In non-Lie Leibniz algebras the non coincidence of these sets in general follows from the following example.

Example 2.2. Let L be the Leibniz algebra defined by the following multiplication:

$$[x, z] = x, \quad [z, y] = y, \quad [y, z] = -y, \quad [z, z] = x,$$

where $\{ x, y, z \}$ is the basis of L and omitted products are equal to zero. Then $\mathfrak{I} = \{x - z\}$ is the Cartan subalgebra of the algebra L, but $r(\mathfrak{I}) = \{x, z\}$.

Proposition 2.3. *A nilpotent subalgebra $\mathfrak{I}$ of a Leibniz algebra L is a Cartan subalgebra if and only if $\mathfrak{I}$ coincides with L_0 in the Fitting decomposition of the algebra L with respect to $R(\mathfrak{I})$.*

Proof. Firstly, we note that $l(\mathfrak{I}) \subseteq L_0$. In fact, if $x \in l(\mathfrak{I})$ then $[x, h] \in \mathfrak{I}$ for any $h \in \mathfrak{I}$. Since the subalgebra $\mathfrak{I}$ is nilpotent, there exists $k \in N$ such that

$$[\cdots[[x, \underbrace{h], h], \ldots, h}_{k \text{ times}}] = R_h^k(x) = 0,$$

this implies that $x \in L_0$, i.e., $l(\mathfrak{I}) \subseteq L_0$. Since $\mathfrak{I} \subseteq l(\mathfrak{I})$ we have $\mathfrak{I} \subseteq L_0$.

Suppose that $\mathfrak{I} \subset L_0$ ($\mathfrak{I} \neq L_0$). By theorem 1.6, the space L_0 is invariant with respect to $R(\mathfrak{I})$ and the restriction of the operator R_h on L_0, where $h \in \mathfrak{I}$, is nilpotent. Moreover, $\mathfrak{I}$ is an invariant subspace of the space L_0 with respect to $R(\mathfrak{I})$. Thus, we obtain the induced Lie algebra $\bar{\mathfrak{I}}$ of linear transformations which acts on the non null space $L_0/\mathfrak{I}$. Since these transformations are nilpotent then by a version of Engel's theorem we have that $\bar{\mathfrak{I}}(x + \mathfrak{I}) = \bar{0}$, where $x + \mathfrak{I}$ a non zero vector. It means that the condition $[x, h] \in \mathfrak{I}$ is verified for any $h \in \mathfrak{I}$; therefore $x \in l(\mathfrak{I})$ and $x \notin \mathfrak{I}$, so $\mathfrak{I} \neq l(\mathfrak{I})$. Thus, $\mathfrak{I} \subset l(\mathfrak{I})$ if and only if $\Im \subset L_0$, and the assertion is proved. □

Proposition 2.4. *Let $\mathfrak{I}$ be a nilpotent subalgebra of a Leibniz algebra L, and let $L = L_0 \oplus L_1$ be the Fitting decomposition of the algebra L with respect to $R(\mathfrak{I})$. Then L_0 is a subalgebra and $[L_1, L_0] \subseteq L_1$.*

Proof. Let $h \in \mathfrak{I}$, $a \in L_0$. Then there exists $k \in N$ such that

$$[\cdots[a, \underbrace{h], h], \ldots, h}_{k \text{ times}}] = 0.$$

From this we have $[\cdots[[R_a, R_h], R_h], \ldots, R_h] = (-1)^k R_{[\cdots[a,h],h],\ldots,h]} = 0$. From this relation and lemma 1.5 it follow that the Fitting subspaces L_{0R_h} and L_{1R_h} of the algebra $\mathfrak{I}$ which correspond to the endomorphism R_h are invariant subspaces with respect to R_a. Since $L_0 = \bigcap_{h\in\mathfrak{I}} L_{0R_h}$ and $L_1 = \bigcap_{h\in\mathfrak{I}} L_{1R_h}$, then $R_a(L_0) \subseteq L_0$ and $R_a(L_1) \subseteq L_1$. And since a is an arbitrary element in L_0, we obtain $[L_0, L_0] \subseteq L_0$ and $[L_1, L_0] \subseteq L_1$. □

Definition 2.5. An element h of L is called *regular* if the dimension of the Fitting null component of the Leibniz algebra L with respect to R_h is minimal. This dimension is called *the rank* of the algebra L.

The set $Z(L) = \{ x \in L \mid [L, x] = 0 \}$ will be called *right annihilator* of the algebra L. It is easy to see that the dimension of the Fitting null component of a linear transformation A is equal to the order of the zero root of the characteristic polynomial of its transformation.

Therefore an element h is regular if and only if the order of the zero characteristic root for R_h is minimal.

Note that in the Lie algebras case the linear transformation R_h is degenerated since $[h, h] = 0$ for any h, and therefore the rank of the Lie algebra is greater than zero.

The following lemma shows that in the Leibniz algebras case the rank is also greater than zero.

Lemma 2.6. *Let L be a Leibniz algebra. Then operator R_x is degenerated for any x in L.*

Proof. Suppose that the algebra L is a non Lie algebra. And suppose that there exists a non zero x in L such that R_x is not degenerated. Then $[L,x] = L$ and therefore, there exists a non null element y of algebra L such that $[y,x] = x$. Since $I^{\text{ann}} = \text{ideal}\langle\, [a,a] \mid a \in L\,\rangle \subseteq Z(L)$, then it is easy to see that $\{\, x, y, [x,y]\,\} \not\subset I^{\text{ann}}$.

Let the dimension of the algebra L be equal to n. Put $x_1 := y$, $x_2 := x$, $x_3 := [x_2, x_2]$, $\dots$, $x_{n+1} := [x_n, x_2]$. Since the operator R_x is not degenerated we have $x_i \neq 0$ for any $i = 3, \dots, n+1$ and every x_i is contained in I^{ann}. Let us show that the system $\{\, x_1, x_2, \dots, x_{n+1}\,\}$ is linearly independent.

The elements x_1 and x_2 are linearly independent. In fact, otherwise $y = \beta x$, where β is different from zero. Then $x = [y,x] = \beta[x,x] = \beta[x,[y,x]] = \beta^2[x,[x,x]] = 0$, and we obtain contradiction with a condition $x \neq 0$.

Suppose that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{n+1} x_{n+1} = 0$ for appropriate α_i, $i = 1, \dots, n+1$. Multiplying this equation from the left hand side by the element x_2, we obtain $\alpha_1[x_2, x_1] + \alpha_2 x_3 = 0$. Since $[x_2, x_1]$ does not belong to the ideal I^{ann}, unlike x_3, then $\alpha_1 = \alpha_2 = 0$. Consider the equation $\alpha_3 x_3 + \alpha_4 x_4 + \dots + \alpha_{n+1} x_{n+1} = 0$, which we can rewrite in the following form: $\alpha_3[x_2, x_2] + \alpha_4[x_3, x_2] + \dots + \alpha_{n+1}[x_n, x_2] = R_x(\alpha_3 x_2 + \alpha_4 x_3 + \dots + \alpha_{n+1} x_n) = 0$. And again using the fact that R_x is non degenerated we have that $\alpha_3 x_2 + \alpha_4 x_3 + \dots + \alpha_{n+1} x_n = 0$. Since x_2 does not belong to the ideal I^{ann}, unlike $\alpha_4 x_3 + \dots + \alpha_{n+1} x_n$, we have $\alpha_3 = 0$. Continuing similarly we obtain that $\alpha_4 = \alpha_5 = \dots = \alpha_{n+1} = 0$, i.e., the system $\{\, x_1, x_2, \dots, x_{n+1}\,\}$ is linearly independent. This is in contradiction with the dimension of the algebra L. $\square$

The following theorem establishes relations between regular elements of Leibniz algebra and Cartan subalgebras.

Theorem 2.7. *Let L be a Leibniz algebra over a infinite field F and a be a regular element of L. Then the Fitting null component $\mathfrak{I}$ of algebra L with respect to R_a is a Cartan subalgebra.*

Proof. Let $L = \mathfrak{I} \oplus \mathfrak{R}$ be a Fitting decomposition of L with respect to R_a. As it will shown below one may assume without loss of generality that the one-generated subalgebra $\langle a \rangle$ is nilpotent. Then by proposition 2.4 we have that $\mathfrak{I}$ is a subalgebra and $[\mathfrak{R}, \mathfrak{I}] \subseteq \mathfrak{I}$. Now we prove that any transformation $R_{b|\mathfrak{I}}$, where $b \in \mathfrak{I}$, is nilpotent. Otherwise, let $b \in \mathfrak{I}$ be an element such that $R_{b|\mathfrak{I}}$ is not nilpotent. Choose a basis in L consisting of bases of $\mathfrak{I}$ and $\mathfrak{R}$. The matrix of R_h, $h \in \mathfrak{I}$, in this basis has the form $\begin{pmatrix} (\rho_1) & 0 \\ 0 & (\rho_2) \end{pmatrix}$, where (ρ_1) is the matrix of $R_{h|\mathfrak{I}}$ and (ρ_2) is the matrix of $R_{h|\mathfrak{R}}$.

Let $A = \begin{pmatrix} (\alpha_1) & 0 \\ 0 & (\alpha_2) \end{pmatrix}$ and $B = \begin{pmatrix} (\beta_1) & 0 \\ 0 & (\beta_2) \end{pmatrix}$ be the matrices of R_a and R_b, respectively. Since (α_2) is not degenerated we have $\det(\alpha_2) \neq 0$. By hypothesis, the matrix (β_1) is not nilpotent. Therefore if $l = \operatorname{rank} L$ then $\dim \mathfrak{I} = l$ and the characteristic polynomial of the matrix (β_1) is not divisible by λ^l. Let λ, μ, ν be algebraic independent variables and let $P(\lambda, \mu, \nu)$ be the characteristic polynomial, i.e., $P(\lambda, \mu, \nu) = \det(\lambda 1 - \mu A - \nu B) = \det(\lambda 1 - (\mu A + \nu B))$. Then the equality $P(\lambda, \mu, \nu) = P_1(\lambda, \mu, \nu)P_2(\lambda, \mu, \nu)$, where $P_i(\lambda, \mu, \nu) = \det(\lambda_1 - \mu(\alpha_i) - \nu(\beta_i)) = \det(\lambda 1 - (\mu(\alpha_i) + \nu(\beta_i)))$ holds. As it was noted above the polynomial $P_2(\lambda, 1, 0) = \det(\lambda 1 - (\alpha_2))$ is not divisible by λ and the polynomial $P_1(\lambda, 0, 1) = \det(\lambda 1 - (\beta_1))$ is not divisible by λ^l. Therefore the greatest degree of λ on which the polynomial $P(\lambda, \mu, \nu)$ can be divided is $\lambda^{l'}$, where $l' < l$. Since the field F is infinite, we can choose μ_0 and ν_0 such that $P(\lambda, \mu_0, \nu_0)$ is not divisible by $\lambda^{l'+1}$. Put $c := \mu_0 a + \nu_0 b$, then the characteristic polynomial $\det(\lambda 1 - R_c) = \det(\lambda 1 - \mu_0 A - \nu_0 B) = P(\lambda, \mu_0, \nu_0)$ is not divisible by $\lambda^{l'+1}$. Therefore the order of zero root for R_c will be equal to $l' < l$. But a is a regular element. Thus, for any $b \in \mathfrak{I}$ the operator $R_{b|\mathfrak{I}}$ is nilpotent. Using now Engel's theorem [4] we obtain that $\mathfrak{I}$ is a nilpotent Leibniz algebra. Let L_0 be a Fitting null-component of the algebra L with respect to $R(\mathfrak{I})$. Since $\mathfrak{I}$ is the Fitting null component of the transformation R_a, then $L_0 \subseteq \mathfrak{I}$. Indeed, when $a \in \mathfrak{I}$ we have that $L_0 = \bigcap_{b \in \mathfrak{I}} L_{0R_b} \subseteq L_{0R_a}$.

Let $a \notin \mathfrak{I}$. Then $a^k \neq 0$ for any $k \in N$. Consider the following elements:

$$a_1 := a, \quad a_2 := [a, a], \quad \ldots, \quad a_{n+1} := \underbrace{[[[a, a], a], \ldots, a]}_{n+1 \text{ times}} = 0$$

where $n = \dim L$. These elements are not equal to zero but linearly dependent, i.e., there exists a non trivial linear combination

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{n+1} a_{n+1} = 0.$$

Let $\alpha_1 \neq 0$, then using the fact that $a_i \in I^{\mathrm{ann}} = \mathrm{ideal}\langle\, [x, x] \mid x \in L\,\rangle$ for any $2 \leq i \leq n+1$, we obtain that $a_1 \in I^{\mathrm{ann}} \subseteq Z(L)$ and consequently $a_2 = 0$, i.e., we have a contradiction with the condition $a_2 \neq 0$. Thus, $\alpha_1 = 0$.

Let k be the minimal number, such that $\alpha_k \neq 0$. Then $\alpha_k a_k + \cdots + \alpha_{n+1} a_{n+1} = 0$ and therefore for the element $t = \alpha_k a_1 + \cdots + \alpha_{n+1} a_{n+1-k}$ we have that

$$\underbrace{[[[t, t], t], \ldots, t]}_{k \text{ times}} = 0$$

and $R_t = \alpha_k R_a$, i.e., t is a regular element and $L_{0R_a} = L_{0R_t} = \mathfrak{I}$. On the other hand, $L_0 \supseteq \mathfrak{I}$ for any nilpotent subalgebra $\mathfrak{I}$. In fact, $L_0 = \bigcap_{b \in \mathfrak{I}} L_{0R_b}$ and if $h \in \mathfrak{I}$ then

$$\underbrace{[[[h, b], b], \ldots, b]}_{s \text{ times}} = 0$$

for any $b \in \mathfrak{I}$ (here s is the index of nilpotence of the algebra $\mathfrak{I}$) thus $h \in L_0$ and $L_0 = \mathfrak{I}$. Using proposition 2.3 we obtain that $\mathfrak{I}$ is a Cartan subalgebra, which completes the proof of the theorem. □

Another useful remark about regular elements and Cartan subalgebras is the following: if a Cartan subalgebra contains a regular element a, then $\mathfrak{I}$ is uniquely defined by the element a as Fitting null component of the algebra L with respect to R_a, i.e., $\mathfrak{I} = L_{0R_a}$.

In fact, if we denote L_{0R_a} by $\mathfrak{R}$ then it is evident that $\mathfrak{I} \subseteq \mathfrak{R}$, since $\mathfrak{I}$ is nilpotent. (If $h \in \mathfrak{I}$ then using the nilpotence of $\mathfrak{I}$ and from $a \in \mathfrak{I}$ we have $[[[h, a], a], \dots, a] = 0$ and then $h \in L_{0R_a} = \mathfrak{R}$.)

On the other hand, from theorem 1.6 we have that $\mathfrak{R}$ is nilpotent. And if $\mathfrak{I} \neq \mathfrak{R}$ then there exists $z \in \mathfrak{R} \setminus \mathfrak{I}$. If $[z, \mathfrak{I}] \subseteq \mathfrak{I}$, since $\mathfrak{I}$ is a Cartan subalgebra, we have $z \in \mathfrak{I}$, a contradiction.

Therefore for any $z \in \mathfrak{R} \setminus \mathfrak{I}$ we have $[z, \mathfrak{I}] \not\subset \mathfrak{I}$. Then there exist $l_1, l_2, \dots, l_k \in \mathfrak{I}$ such that $[[[z, l_1], l_2], \dots, l_k] \in \mathfrak{R} \setminus \mathfrak{I}$ and $[[[z, l_1], l_2], \dots, l_k] \neq 0$, i.e., $\mathfrak{R}$ is not nilpotent, also a contradiction. Therefore $\mathfrak{I} = \mathfrak{R}$.

From this it follows that two Cartan subalgebras having the same regular element coincide.

For the Leibniz algebra L, we consider the natural homomorphism φ into the factor algebra L_{Lie}, where $L_{\text{Lie}} = L/I^{\text{ann}}$

Proposition 2.8. *Let L be a complex finite dimensional Leibniz algebra. Then the image of a regular element of the algebra L by a homomorphism φ is a regular element of the Lie algebra L_{Lie}.*

Proof. Let a be a regular element of the algebra L. We shall prove that the element $\bar{a} = a + I^{\text{ann}}$ will be a regular element of the Lie algebra L_{Lie}. Suppose the opposite, i.e., $\bar{a} = a + I^{\text{ann}}$ is not a regular element. Let $\bar{b} = b + I^{\text{ann}}$ be any regular element of the Lie algebra L_{Lie} and $a - b \notin I^{\text{ann}}$.

Since I^{ann} is an ideal, then for any $x \in L$ we have $R_x(I^{\text{ann}}) \subseteq I^{\text{ann}}$. It means that the matrix of the transformation R_x has the following block form

$$R_x = \begin{pmatrix} X, & 0 \\ Z_x, & I_x \end{pmatrix}$$

in the basis $\{ e_1, e_2, \dots, e_m, i_1, i_2, \dots, i_n \}$ of L, where $\{ i_1, i_2, \dots, i_n \}$ is the basis of I^{ann}. Here X is the matrix of the transformation $R_x|_{\{ e_1, e_2, \dots, e_m \}}$ and I_X is the matrix of the transformation $R_x|_{I^{\text{ann}}}$.

Let

$$R_a = \begin{pmatrix} A, & 0 \\ Z_a, & I_a \end{pmatrix}, \quad R_b = \begin{pmatrix} B, & 0 \\ Z_b, & I_b \end{pmatrix}$$

be the matrices of the transformations R_a and R_b respectively.

Let k (respectively k') be the order of the characteristic zero root of the matrix A (respectively B) and s and s' be the orders of the characteristic zero root of the matrices I_a and I_b, respectively. Then we have $k' < k$, $s < s'$.

Put

$$U = \Big\{ y \in L \setminus I^{\text{ann}} \,\Big|\, R_y = \left(\begin{smallmatrix} Y, & 0 \\ Z_y, & I_y \end{smallmatrix}\right) \text{ and } Y \text{ has the order of the characteristic zero root less than } k \Big\},$$

$$V = \Big\{ y \in L \setminus I^{\text{ann}} \,\Big|\, R_y = \left(\begin{smallmatrix} Y, & 0 \\ Z_y, & I_y \end{smallmatrix}\right) \text{ and } I_y \text{ has the order of the characteristic zero root less than } s+1 \Big\}.$$

Since $b \in U$ and $a \in V$ these sets are non empty.

Let us show that the set U is an open subset of the set $L \setminus I^{\text{ann}}$ in the Zariski topology.

Let Y have the order of the characteristic zero root less than k. Then Y^k has the rank greater than $n-k$. It means that there exists a non-zero minor of the order $n-k+1$. In other words, there exists a non-zero polynomial of structural constants of the algebra L, hence the set U is open in the Zariski topology in the subset of the set $L \setminus I^{\text{ann}}$.

One can similarly prove that the set V is open in $L \setminus I^{\text{ann}}$. It is not difficult to check that the sets U and V are dense in $L \setminus I^{\text{ann}}$. Therefore, there exists an element $y \in U \cap V$ such that Y has the order of characteristic zero root less than k and I_y has the order of the characteristic zero root less than $s+1$. Thus, for this element y the order of characteristic zero root is not greater than $k+s-1$, i.e., the rank of the algebra L is less than $k+s$ and we obtain a contradiction to the assumption that $\bar{a}$ is not a regular element of the Lie algebra L_{Lie}. □

Let L be a Leibniz algebra with a basis $\{e_1, e_2, \dots, e_n\}$ over a field F. Let $\xi_1, \xi_2, \dots, \xi_n$ be independent variables and let $P = F(\xi_1, \xi_2, \dots, \xi_n)$ be the field of rational functions of ξ_i. We construct an extension of P putting $L_P = Pe_1 + Pe_2 + \dots + Pe_n$.

The following definition and its comments are step by step modifications of the Lie algebras case and they are included for the sake of completeness.

Definition 2.9. An element $x = \sum_{i=1}^{n} \xi_i e_i$ of the algebra L_P is called a *generic element* of the algebra L and the characteristic polynomial $f_x(\lambda)$ of the transformation R_x in L_P is called *characteristic polynomial* of the Leibniz algebra L.

If we take the basis $\{e_1, e_2, \dots, e_n\}$ of L_P then $[e_i, x] = \sum_{j=1}^{n} \rho_{ij} e_j$, where $i = 1, \dots, n$ and ρ_{ij} are homogenous functions of degree 1 with respect to ξ_k. Then

$$f_x(\lambda) = \det(\lambda 1 - R_x) = \lambda^n - \tau_1(\xi)\lambda^{n-1} + \tau_2(\xi)\lambda^{n-2} - \dots + (-1)^l \tau_{n-l}(\xi)\lambda^l, \quad (*)$$

where $\tau_i(\xi)$ are homogenous polynomials of degree i in the variables ξ_i, $\xi = \{\xi_1, \dots, \xi_n\}$ and $\tau_{n-l}(\xi) \neq 0$, where $\tau_{n-l+k}(\xi) = 0$ for $k > 0$. Since $x \neq 0$ and R_x is a degenerated operator, it follows that $l > 0$ and $detR_x = 0$.

The value of the characteristic polynomial on an arbitrary element $a = \sum_{i=1}^{n} \alpha_i e_i$ of the algebra L is obtained by specialization of $\xi_i = \alpha_i$, $i = 1, \dots, n$ in equation (*). Therefore it is evident that the order of zero root of the characteristic transformation R_a is not less than l. On the other hand, if F is an infinite field then, since the polynomial $\tau_{n-l}(x)$ is different from zero in the polynomial algebra $\Phi[\xi_1, \xi_2, \dots, \xi_n]$, we can choose $\xi_i = \alpha_i$ such that $\tau_{n-l}(\alpha) \neq 0$. Then the transformation R_a for the element $a = \sum_{i=1}^{n} \alpha_i e_i$ has exactly l characteristic roots which are equal to zero and therefore a is regular. Thus in the case of an infinite field the element a is regular if and only if $\tau_{n-l}(\alpha) \neq 0$. In this sense "almost all" elements of the algebra L are regular (i.e., they form an open set in Zariski topology).

The above statement depends on the choice of the basis (e). However, it is easy to observe what happens when we pass to another basis $(f_1, f_2, \dots, f_n)$, where $f_i = \sum_{j=1}^{n} \mu_{ij} e_j$. If $\eta_1, \eta_2, \dots, \eta_n$ are independent variables then $y = \sum_{i=1}^{n} \eta_i f_i = \sum_{i=1}^{n} \eta_i \mu_{ij} e_j$. Therefore the characteristic polynomial $f_y(\lambda)$ is obtained from polynomial $f_x(\lambda)$ by substitution $\xi_j \to \sum_{i=1}^{n} \eta_i \mu_{ij}$ in its coefficients.

3. Some properties of weight spaces of Leibniz algebras and Cartan's criterion of solvability

In order to define a weight module over a Leibniz algebra we need the definition of the right representation of a Leibniz algebra.

Definition 3.1. A vector space M is said to be a *right representation* of a Leibniz algebra L if an action: $[\cdot,\cdot] : M \times L \to M$ is defined, which satisfies the condition

$$[m, [x, y]] = [[m, x], y] - [[m, y], x]$$

for any $x, y \in L$, $m \in M$.

Note that this definition agrees with the definition of symmetric representation in [10].

Observe that M has natural right L-module structure (in the Lie sense) and below we shall think about M in that sense.

A map $a \to \alpha(a)$ from an algebra L into the field F ($\alpha : L \to F$) is called *weight* of the right module M if there exists a non zero $x \in M$ such that $(a - \alpha(a)1)^k(x) = 0$, i.e., $[x, (a - \alpha(a)1)^k] = 0$ for some $k \in N$.

The set of vectors which satisfy this condition and the zero vector form the subspace M_a which is called *weight subspace* (weight subspace with the weight α) corresponding to the weight α.

Let L be a Leibniz algebra and M be a weight subspace over the algebra L with respect to the weight α. Then for each element $x \in M$ one has $[x, (a - \alpha(a)1)^k] = 0$ if k is sufficiently large. Moreover, if $\dim M = n$, then the polynomial $(\lambda - \alpha(a))^n$ is the characteristic polynomial of the endomorphism a. Therefore $[x, (a - \alpha(a)1)^n] = 0$ for any $x \in M$.

Consider the contradredient (conjugated) right module M^* over a Leibniz algebra satisfying the condition

$$\langle [x,a], y^* \rangle + \langle x, [y^*, a^*] \rangle = 0,$$

where $x \in M$, $y^* \in M^*$, a^* from representation which corresponds to the right module M^* and $\langle x, y^* \rangle$ denote the value of linear function y^* at the vector x.

It is clear that $\langle \alpha(a)x, y^* \rangle + \langle x, \alpha(a)y^* \rangle = 0$.

Adding these equalities we obtain

$$\langle [x, (a - \alpha(a)1)], y^* \rangle + \langle x, [y^*, (a^* + \alpha(a)1)] \rangle = 0.$$

By repeating this procedure, we obtain the equality

$$\langle [x, (a - \alpha(a)1)^k], y^* \rangle + \langle x, [y^*, (a^* + \alpha(a)1)^k] \rangle = 0.$$

If $k = n$ then $[x, (a - \alpha(a)1)^n] = 0$ for any x and thus $\langle x, [y^*, (a^* + \alpha(a)1)^n] \rangle = 0$. Therefore $[y^*, (a^* + \alpha(a)1)^n] = 0$ for any $y^* \in M^*$. This shows that M^* is a weight module with the weight $-\alpha$.

Thus, we have proved the following proposition.

Proposition 3.2. *If M is a weight module over a Leibniz algebra with the weight α, then the contradredient module M^* is a weight module with weight $-\alpha$.*

Proposition 3.3. *If M and R are weight modules over a Leibniz algebra with the weight α and β respectively, then $B = M \otimes R$ is a weight module with weight $\alpha + \beta$.*

Proof. Proposition 3.3 is proved similarly as the corresponding proposition in the Lie algebras case (see proposition 4, page 63 in [9]). □

Definition 3.4. A nilpotent Leibniz algebra L of linear transformations is called *a split algebra* if the characteristic roots of each element of $A \in L$ is contained in the basic field.

Let L be a Leibniz algebra, $\mathfrak{I}$ be a nilpotent subalgebra and M be a left module over L (and also over $\mathfrak{I}$).

Suppose that $M = L$ and $R(\mathfrak{I})$ is a nilpotent split Lie algebra. From theorem 1.7 we have that $L = L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\delta$, where α, β, $\ldots$, δ are maps from the subalgebra $R(\mathfrak{I})$ into F such that if $x_\nu \in L_\nu$, then $(R_h - \nu(R_h)1)^m(x_\nu) = 0$ for some $m = m(\nu)$, where $\nu \in \{\alpha, \beta, \ldots, \delta\}$. The weights α, β, $\ldots$, δ are called *roots* of the algebra L with respect to the subalgebra $\mathfrak{I}$.

Proposition 3.5. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ *if $\alpha + \beta$ is a root of the Leibniz algebra L with respect to $R(\mathfrak{I})$; otherwise* $[L_\alpha, L_\beta] = 0$.

Proof. The elements of $[L_\alpha, L_\beta]$ have the form $\sum_i [x_\alpha^{(i)}, y_\beta^{(i)}]$ where $x_\alpha^{(i)} \in L_\alpha$, $y_\beta^{(i)} \in L_\beta$. From the characteristic property of the tensor product of two spaces it follows that there exists a linear map $\pi : L_\alpha \otimes L_\beta \to [L_\alpha, L_\beta]$ such that $\pi\big(\sum_i x_\alpha^{(i)} \otimes y_\beta^{(i)}\big) = \sum_i \big[x_\alpha^{(i)}, y_\beta^{(i)}\big]$. We show that π is actually a homomorphism of $R(\mathfrak{I})$-modules.

Let $R_h \in R(\mathfrak{I})$ then using the Leibniz identity we obtain the following chain of equalities:

$$\begin{aligned} R_h(x_\alpha \otimes y_\beta) &= R_h(x_\alpha) \otimes y_\beta + x_\alpha \otimes R_h(y_\beta) \\ &= [x_\alpha, h] \otimes y_\beta + x_\alpha \otimes [y_\beta, h] \to [[x_\alpha, h], y_\beta] + [x_\alpha, [y_\beta, h]] \\ &= R_h([x_\alpha, y_\beta]). \end{aligned}$$

On the other hand, the image of the element $x_\alpha \otimes y_\beta$ under the homomorphism π is $[x_\alpha, y_\beta]$. So, we prove that $[L_\alpha, L_\beta]$ is a homomorphic image of the module $L_\alpha \otimes L_\beta$. Moreover, $L_\alpha \otimes L_\beta$ is a weight module with the weight $\alpha + \beta$. But from the definition it is clear that the homomorphic image of the weight module with the weight β is either 0 or a weight module with the weight β. □

Let L be a finite dimensional Leibniz algebra over an algebraically closed field F and $\mathfrak{I}$ be a nilpotent subalgebra of L. Let $L = L_\alpha \oplus L_\beta \oplus \dots \oplus L_\delta$ be a decomposition of module L into the direct sum of weight submodules with respect to $\mathfrak{I}$.

Suppose that $\mathfrak{I}$ is a Cartan subalgebra. Then it is not difficult to see that $\mathfrak{I} = L_0$, where L_0 is the root module corresponding to the root 0.

We have also the equality $[L, L] = \sum [L_\alpha, L_\beta]$, where the sum is taken over all roots α, β. From this we obtain $L_0 \cap L^2 = \mathfrak{I} \cap L^2 = \sum [L_\alpha, L_{-\alpha}]$, where summation is made over all α, such that $-\alpha$ is also a root (in particular $\alpha = 0$).

Definition 3.6. The form $f(a, b) = tr(R_a R_b)$ for $a, b \in L$ is called the *Killing form* of the Leibniz algebra L.

A bilinear form $f(a, b)$ on L satisfying the condition

$$f([a, c], b) + f(a, [b, c]) = 0$$

is called an *invariant form* on L.

The following equalities show that the Killing form is an invariant form on Leibniz algebra:

$$\begin{aligned} f([a, c], b) + f(a, [b, c]) &= \mathrm{tr}(R_{[a,c]} R_b) + \mathrm{tr}(R_a R_{[b,c]}) \\ &= \mathrm{tr}([R_c, R_a] R_b + R_a [R_c, R_b]) \\ &= \mathrm{tr}((R_c R_a - R_a R_c) R_b + R_a (R_c R_b - R_b R_c)) \\ &= \mathrm{tr}(R_c R_a R_b - R_a R_b R_c) = \mathrm{tr}[R_c, R_a R_b] = 0. \end{aligned}$$

Note that if $f(a, b)$ is the Killing form then the set

$$L^\perp = \{\, z \in L \mid f(a, z) = 0 \text{ for any } a \in L \,\}$$

is an ideal of the algebra L.

Theorem 3.7. *Let L be a Leibniz algebra over an algebraically closed field of zero characteristic. Then L is solvable if and only if* $\operatorname{tr}(R_a R_a) = 0$ *for any* $a \in L^2$.

Proof. The necessity. In proposition 1.8 put $\aleph := R(L)$, then it is clear that the Lie algebra $R(L)$ is also solvable and therefore $[R(L), R(L)] \subseteq \mathfrak{R}$. And since $[R(L), R(L)] = R(L^2)$ we have that for any $a \in L^2$ the operator R_a is nilpotent, therefore $\operatorname{tr}(R_a R_a) = 0$ for any $a \in L$.

The sufficiency. If we apply Cartan's criterion for Lie algebras [9] for algebra $R(L)$ and consider L as $R(L)$-module, we obtain the solvability of the Lie algebra $R(L)$, but as we noted in section 1 it is equivalent to the solvability of Leibniz algebra L. $\square$

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References

[1] S. Albeverio, Sh. A. Ayupov, and B. A. Omirov, *On nilpotent and simple Leibniz algebras*, Comm. Algebra **33** (2005), no. 1, 159–172.

[2] Sh. A. Ayupov and B. A. Omirov, *On some classes of nilpotent Leibniz algebras*, Sibirsk. Mat. Zh. **42** (2001), no. 1, 18–29, i (Russian).

[3] ———, *Nilpotent properties of the Leibniz algebra $M_n(\mathbb{C})_D$*, Sibirsk. Mat. Zh. **45** (2004), no. 3, 483–496 (Russian).

[4] ———, *On Leibniz algebras*, Algebra and operator theory (Tashkent, 1997), 1998, pp. 1–12.

[5] J. M. Casas and T. Pirashvili, *Ten-term exact sequence of Leibniz homology*, J. Algebra **231** (2000), no. 1, 258–264.

[6] A. Frabetti, *Leibniz homology of dialgebras of matrices*, J. Pure Appl. Algebra **129** (1998), no. 2, 123–141.

[7] Y. Gao, *The second Leibniz homology group for Kac-Moody Lie algebras*, Bull. London Math. Soc. **32** (2000), no. 1, 25–33.

[8] A. V. Gnedbaye, *Leibniz homology of extended Lie algebras*, K-Theory **13** (1998), no. 2, 169–178.

[9] N. Jacobson, *Lie algebras*, Izdat. "Mir", Moscow, 1964.

[10] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. (2) **39** (1993), no. 3-4, 269–293.

[11] J.-L. Loday and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. **296** (1993), no. 1, 139–158.

[12] A. Malcev, *On solvable Lie algebras*, Izvestia Akad. Nauk SSSR **9** (1945), 329–356 (Russian).