Global Gronwall Estimates for Integral Curves on Riemannian Manifolds

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ABSTRACT

We prove Gronwall-type estimates for the distance of integral curves of smooth vector fields on a Riemannian manifold. Such estimates are of central importance for all methods of solving ODEs in a verified way, i.e., with full control of roundoff errors. Our results may therefore be seen as a prerequisite for the generalization of such methods to the setting of Riemannian manifolds.

Key words: Riemannian geometry, ordinary differential equations, Gronwall estimate.

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Introduction

Suppose that X is a complete smooth vector field on $\mathbb{R}^n$, let p_0, $q_0 \in \mathbb{R}^n$ and denote by $p(t)$, $q(t)$ the integral curves of X with initial values p_0 resp. q_0. In the theory of

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ordinary differential equations it is a well known consequence of Gronwall's inequality that in this situation we have

$$|p(t) - q(t)| \leq |p_0 - q_0| e^{C_T t} \qquad (t \in [0, T)) \tag{1}$$

with $C_T = \|DX\|_{L^\infty(K_T)}$ (K_T some compact convex set containing the integral curves $t \mapsto p(t)$ and $t \mapsto q(t)$) and DX the Jacobian of X (cf., e.g., [1, 10.5]).

The aim of this paper is to derive estimates analogous to (1) for integral curves of vector fields on Riemannian manifolds. Apart from a purely analytical interest in this generalization, we note that Gronwall-type estimates play an essential role in the convergence analysis of numerical methods for solving ordinary differential equations (cf. [5]). Concerning notation and terminology from Riemannian geometry our basic references are [2–4].

1. Estimates

The following proposition provides the main technical ingredient for the proofs of our Gronwall estimates. Here and in what follows, for $X \in \mathfrak{X}(M)$ (the space of smooth vector fields on M) we denote by ∇X its covariant differential and by $\|\nabla X(p)\|_g$ the mapping norm of $\nabla X(p) : (T_pM, \|\cdot\|_g) \to (T_pM, \|\cdot\|_g)$, $Y_p \mapsto \nabla_{Y_p} X$.

Proposition 1.1. *Let $[a, b] \ni \tau \mapsto c_0(\tau) =: c(0, \tau)$ be a smooth regular curve in a Riemannian manifold (M, g), let $X \in \mathfrak{X}(M)$ and set $c(t, \tau) := \mathrm{Fl}^X_t c(0, \tau)$ where Fl^X is the flow of X. Choose $T > 0$ such that Fl^X is defined on $[0, T] \times c_0([a, b])$. Then denoting by $l(t)$ the length of $\tau \mapsto c(t, \tau)$, we have*

$$l(t) \leq l(0) e^{C_T t} \qquad (t \in [0, T]) \tag{2}$$

where $C_T = \sup\{ \|\nabla X(p)\|_g : p \in c([0, T] \times [a, b]) \}$.

Proof. Let $\tau \mapsto c(0, \tau)$ be parameterized by arc length, $\tau \in [0, l(0)]$. Since Fl^X_t is a local diffeomorphism, $g(\partial_\tau c, \partial_\tau c) > 0$ on $[0, T] \times [a, b]$. Furthermore, since the Levi-Cività connection ∇ is torsion free, we have $\nabla_{\partial_t} c_\tau = \nabla_{\partial_\tau} c_t$, where $c_t = \partial_t c$, $c_\tau = \partial_\tau c$, see [3, 1.8.14]. Then

$$\begin{aligned}
l(s) - l(0) &= \int_0^s \partial_t l(t)\, dt = \int_0^s \partial_t \int_0^{l(0)} \|c_\tau(t, \tau)\|_g \, d\tau \, dt \\
&= \int_0^s \int_0^{l(0)} \frac{\partial_t g(c_\tau(t, \tau), c_\tau(t, \tau))}{2\|c_\tau(t, \tau)\|_g} \, d\tau \, dt = \int_0^s \int_0^{l(0)} \frac{g((\nabla_{\partial_t} c_\tau)(t, \tau), c_\tau(t, \tau))}{\|c_\tau(t, \tau)\|_g} \, d\tau \, dt \\
&= \int_0^s \int_0^{l(0)} \frac{g((\nabla_{\partial_\tau} c_t)(t, \tau), c_\tau(t, \tau))}{\|c_\tau(t, \tau)\|_g} \, d\tau \, dt \leq \int_0^s \int_0^{l(0)} \|(\nabla_{\partial_\tau} c_t)(t, \tau)\|_g \, d\tau \, dt
\end{aligned}$$

$$= \int\limits_0^s \int\limits_0^{l(0)} \|\nabla_{c_\tau(t,\tau)} X\|_g \, d\tau \, dt \leq C_T \int\limits_0^s \int\limits_0^{l(0)} \|c_\tau(t,\tau)\|_g \, d\tau \, dt$$
$$= C_T \int_0^s l(t) \, dt.$$

The claim now follows by applying Gronwall's inequality. □

We may utilize this proposition to prove our first main result:

Theorem 1.2. *Let (M, g) be a connected smooth Riemannian manifold, $X \in \mathfrak{X}(M)$ a complete vector field on M and let p_0, $q_0 \in M$. Let $p(t) = \mathrm{Fl}^X_t(p_0)$, $q(t) = \mathrm{Fl}^X_t(q_0)$ and suppose that $C := \sup_{p \in M} \|\nabla X(p)\|_g < \infty$. Then*

$$d(p(t), q(t)) \leq d(p_0, q_0) e^{Ct} \qquad (t \in [0, \infty)), \tag{3}$$

where $d(p, q)$ denotes Riemannian distance.

Proof. For any given $\varepsilon > 0$, choose a piecewise smooth regular curve $\tau \mapsto c_0(\tau) =: c(0, \tau) : [0, 1] \to M$ connecting p_0 and q_0 such that $d(p_0, q_0) > l(0) - \varepsilon$. Using the notation of Proposition 1.1 it follows that

$$d(p(t), q(t)) \leq l(t) \leq l(0) e^{Ct} < (d(p_0, q_0) + \varepsilon) e^{Ct}$$

for $t \in [0, \infty)$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

Example 1.3. (i) In general, when neither M nor X is complete, the conclusion of Theorem 1.2 is no longer valid:

Consider $M = \mathbb{R}^2 \setminus \{ (0, y) \mid y \geq 0 \}$, endowed with the standard Euclidean metric. Let $X \equiv (0, 1)$, $p_0 = (-x_0, -y_0)$, and $q_0 = (x_0, -y_0)$ ($x_0 > 0$, $y_0 \geq 0$) (cf. figure 1). Then $p(t) = (-x_0, -y_0 + t)$, $q(t) = (x_0, -y_0 + t)$ and

$$d(p(t), q(t)) = \begin{cases} 2x_0, & t \leq y_0, \\ 2\sqrt{x_0^2 + (t - y_0)^2}, & t > y_0. \end{cases}$$

On the other hand, $\nabla X = 0$, so (3) is violated for $t > y_0$, i.e., as soon as the two trajectories are separated by the "gap" $\{ (0, y) \mid y \geq 0 \}$.

(ii) Replace X in (i) by the complete vector field $(0, e^{-1/x^2+1})$ and set $x_0 = 1$, $y_0 = 0$. Then $C := \|\nabla X\|_{L^\infty(\mathbb{R}^2)} = 3\sqrt{3/(2e)}$ and

$$d(p(t), q(t)) = 2\sqrt{1 + t^2} \leq d(p_0, q_0) e^{Ct} = 2e^{Ct}$$

for all $t \in [0, \infty)$, in accordance with Theorem 1.2.

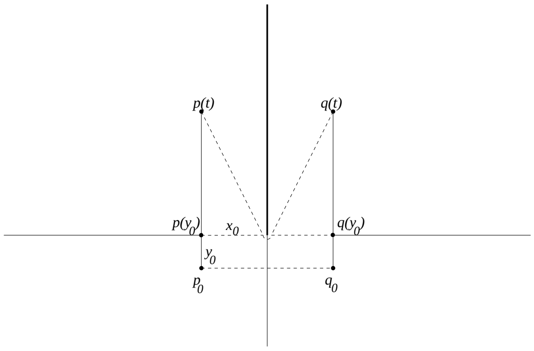

Figure 1

The following result provides a sufficient condition for the validity of a Gronwall estimate even if neither M nor X satisfies a completeness assumption.

Theorem 1.4. *Let (M, g) be a connected smooth Riemannian manifold, $X \in \mathfrak{X}(M)$ and let p_0, $q_0 \in M$. Let $p(t) = \mathrm{Fl}^X_t(p_0)$, $q(t) = \mathrm{Fl}^X_t(q_0)$ and suppose that there exists some relatively compact submanifold N of M containing p_0, q_0 such that $d(p_0, q_0) = d_N(p_0, q_0)$. Fix $T > 0$ such that Fl^X is defined on $[0, T] \times N$ and set $C_T := \sup\{\, \|\nabla X(p)\|_g : p \in \mathrm{Fl}^X([0, T] \times N)\,\}$. Then*

$$d(p(t), q(t)) \le d(p_0, q_0)e^{C_T t} \qquad (t \in [0, T]). \tag{4}$$

Proof. As in the proof of Theorem 1.2, for any given $\varepsilon > 0$ we may choose a piecewise smooth curve $\tau \mapsto c_0(\tau) : [0, 1] \to N$ from p_0 to q_0 such that $d(p_0, q_0) = d_N(p_0, q_0) > l(0) - \varepsilon$. The corresponding time evolutions $c(t, \cdot)$ of $c(0, \cdot) = c_0$ then lie in $\mathrm{Fl}^X([0, T] \times N)$, so an application of Proposition 1.1 gives the result. □

Example 1.5. Clearly such a submanifold N need not exist in general. As a simple example take $M = \mathbb{R}^2 \setminus \{(0, 0)\}$, $p_0 = (-1, 0)$, $q_0 = (1, 0)$. In Example 1.3.(i) with $y_0 > 0$ the condition is obviously satisfied with N an open neighborhood of the straight line joining p_0, q_0 and the supremum of the maximal evolution times of such N under Fl^X is $T = y_0$, coinciding with the maximal time-interval of validity of (4). On the other hand, if there is no N as in Theorem 1.4 then the conclusion in general breaks down even for arbitrarily close initial points p_0, q_0: if we set $y_0 = 0$ in Example 1.3.(i) then no matter how small x_0 (i.e., irrespective of the initial distance of the trajectories) the estimate is not valid for any $T > 0$.

Finally, we single out some important special cases of Theorem 1.4:

Corollary 1.6. *Let M be a connected geodesically complete Riemannian manifold, $X \in \mathfrak{X}(M)$, and p_0, q_0, $p(t)$, $q(t)$ as above. Let S be a minimizing geodesic segment* *connecting* p_0*,* q_0 *and choose some* $T > 0$ *such that* Fl^X *is defined on* $[0, T] \times S$*. Then* (4) *holds with* $C_T = \sup\{\, \|\nabla X(p)\|_g \mid p \in \mathrm{Fl}^X([0, T] \times S)\,\}$*. In particular, if* X *is complete then for any* $T > 0$ *we have*

$$d(p(t), q(t)) \leq d(p_0, q_0) e^{C_T t} \qquad (t \in [0, T]).$$

Proof. Choose for N in Theorem 1.4 any relatively compact open neighborhood of S. The value of C_T then follows by continuity. □

In particular, for $M = \mathbb{R}^n$ with the standard Euclidean metric, Corollary 1.6 reproduces (1).

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