A Note on Non-Negative Singular Infinity-Harmonic Functions in the Half-Space

Tilak Bhattacharya

Department of Mathematics
Purdue University
W. Lafayette, IN 47907 — USA
tbhatta@math.purdue.edu

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ABSTRACT

In this work we study non-negative singular infinity-harmonic functions in the half-space. We assume that solutions blow-up at the origin while vanishing at infinity and on a hyperplane. We show that blow-up rate is of the order $|x|^{-1/3}$.

Key words: viscosity solutions, infinity-harmonic, singular.

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1. Introduction.

Our effort in this note will be to derive growth rates for non-negative singular infinity-harmonic functions in the half-space. The functions of interest will have a singularity at the boundary while vanishing elsewhere on the hyperplane and at infinity. In particular we will show that any two such singular functions are comparable and thus have the same growth rate. This is to be viewed as a follow-up of [7] where singular infinity-harmonic functions were studied in greater generality. In the present case the precise nature of the growth rate will follow by adapting an example constructed in [2]. Our framework in this note will be that of viscosity solutions and to describe our results more precisely, we introduce the following notations. Let $x = (x_1, x_2, \ldots, x_n)$ denote a point in $\mathbb{R}^n$, $n \geq 2$, $O = (0, 0, \ldots, 0)$ be the origin and $H = \{x : x_n > 0\}$

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the half-space. We define $u = u(x)$ to be infinity-harmonic (or ∞-harmonic) in H, if u solves

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad \forall\, x \in H,$$

in the viscosity sense [3,6,9,10]. It is well known that u is locally Lipschitz continuous and obeys the comparison principle. Let $B_R(P)$ be the open ball in $\mathbb{R}^n$ with center P and radius R, $B^+(R,O) = B(R,O) \cap H$, and $S_R = \partial B(R,O) \cap H$. Also let $T = \{x : x_n = 0\}$. Set $M(R) = \sup_{S_R} u(x)$, $R > 0$. We will always assume that $u(x) > 0$, for every $x \in H$ and

(A1) $u(x)$ is continuous up to $T \setminus \{O\}$ and $u(x) = 0$, $x \in T \setminus \{O\}$,

(A2) for every $R > 0$, $0 < M(R) < \infty$, $\sup_{R>0} M(R) = \infty$, and $\lim_{R\to\infty} M(R) = 0$.

Note that in (A2) we do not specify any growth or decay rates and the blow-up occurs only at the origin O. Our main result in this work is

Theorem 1.1 (Comparison). *Let $u > 0$ and $v > 0$ be infinity-harmonic in H and satisfy the assumptions (A1) and (A2). For every $R > 0$, let*

$$S(R) = \sup_{x\in\partial B_R(O)\cap H} u(x)/v(x) \qquad \text{and} \qquad s(R) = \inf_{x\in\partial B_R(O)\cap H} u(x)/v(x).$$

Then

(i) $S(0) = \lim_{R\to 0} S(R) < \infty$ *and* $\lim_{R\to 0} s(R) = s(0) > 0$ *exist,*

(ii) *there exists a universal constant $C > 0$ such that $S(0) \le Cs(0)$,*

(iii) *for every $x \in H$, $\frac{S(0)}{C} \le \frac{u(x)}{v(x)} \le S(0)$.*

Moreover, any non-negative singular solution is axially symmetric, i.e., $u(x) = h(r,\theta)$, where $r = r(x) = |x|$ and $\theta = \theta(x) = \cos^{-1} x_n/r$.

In light of the example in [2] and the discussion in the Appendix, it follows that $u(x) \sim |x|^{-1/3}$. This result then improves the lower bound proven in [7] in this special case (see [7, Remark 3]). The main ingredients of the proof are the Harnack inequality [3,5,6,11], the boundary Harnack principle [6] and the comparison principle [3,4]. The question as to whether $S(0) = s(0)$ is true, thereby showing that $u(x) = Cv(x)$, is unclear to us. If true it would then show that such solutions are separable in r and θ. In section 2, we first prove a somewhat sharper version of the Harnack inequality which will provide a lower bound for a solution u. We then present the proofs of parts (i)–(iii) of Theorem 1.1. This will then be followed up by the proof of the fact that $u(x)$ has axial symmetry with respect to the x_n-axis. Additional properties will also be shown. In the Appendix we recall the example studied in [1,2,7]. Finally, we remark that if a boundary Harnack inequality could be proven for C^2 domains then a version of Theorem 1.1 would hold for cones.

2. Proofs of Theorem 1.1 and other results.

We first state a more refined version of the Harnack inequality. See [3, 6, 11].

Lemma 2.1 (Harnack's inequality). *Let $\Omega \subset \mathbb{R}^n$, be an open and connected set. Let $u > 0$ be infinity-harmonic in Ω. Suppose A and B are two points in Ω and $\sigma(t)$, $0 \le t \le 1$, be a smooth curve from A to B with $\sigma(0) = A$ and $\sigma(1) = B$. Let $\delta(t) = \operatorname{dist}(\sigma(t), \partial\Omega) \ne 0$, then*

$$u(A) \le u(B) \exp\biggl(\int_0^1 \frac{|\sigma'(t)|}{\delta(t)}\, dt\biggr).$$

Proof. First recall that if P_1 and P_2 are any two points in Ω that are joined by a straight segment that is at least η away from $\partial\Omega$, then $u(P_1) \le u(P_2)e^{|P_1 - P_2|/\eta}$ [3, 5–7]. Now partition $[0, 1]$ and approximate the curve by finitely many chords. Since the Harnack inequality is multiplicative, the finite sum may then be replaced by a Riemann integral by successive refinement of the partition. □

We now prove a comparison result which will be applied in what follows.

Lemma 2.2 (Comparison). *Let $u_1 > 0$ and $u_2 > 0$ be two infinity-harmonic functions in $H \setminus B_R(O)$, $R > 0$. For $i = 1, 2$, assume that*

(i) $\lim_{r\to\infty} \sup_{\{|x|=r\}\cap H} u_i(x) = 0$,

(ii) *$u_i(x)$ is continuous up to $(T \setminus B_R(O)) \cup (\partial B_R(O) \cap H)$, and*

(iii) $u_i(x) = 0$, $x \in T \setminus B_R(O)$.

If $u_1(x) \le u_2(x)$, $x \in \partial B_R(O) \cap H$, then $u_1(x) \le u_2(x)$, $x \in H \setminus B_R(O)$.

Proof. Let $\varepsilon > 0$ be arbitrary and r_ε be such that $u_1(x) \le \varepsilon$, for every $x \in H \setminus B_{r_\varepsilon}(O)$. For $r > r_\varepsilon$, consider the region $P_{r,\varepsilon} = \{x \in H : R < |x| < r\}$. Let $u_{2,\varepsilon}(x) = u_2(x) + \varepsilon$. Then

(i) $u_{2,\varepsilon}(x)$ is ∞-harmonic in $P_{r,\varepsilon}$,

(ii) $u_{2,\varepsilon}(x) \ge u_1(x)$, $x \in (\partial B_r(O) \cup \partial B_R(O)) \cap H$, and

(iii) $u_{2,\varepsilon}(x) \ge u_1(x)$, $x \in T \cap (B_r(O) \setminus B_R(O))$.

By comparison $u_1(x) \le u_{2,\varepsilon}(x)$, $x \in P_{r,\varepsilon}$. Since r and ε are arbitrary $u_1(x) \le u_2(x)$, $x \in H \setminus B_R(O)$. □

Corollary 2.3. *For $i = 1, 2$, let $u_i > 0$, be infinity-harmonic in $H \setminus B_R(O)$ and satisfy the assumptions* (i)–(iii) *in Lemma 2.2. For every $r \ge R$, define $S(r) = \sup_{\{|x|=r\}\cap H} u_1(x)/u_2(x)$ (possibly infinite) and $s(r) = \inf_{\{|x|=r\}\cap H} u_1(x)/u_2(x)$ (possibly zero). Suppose that $S(\rho) < \infty$ for some $\rho \ge R$, then $S(r)$ is decreasing in r for $r \ge \rho$. Analogously if $s(\rho) > 0$ then $s(r)$ is increasing in r when $r \ge \rho$.*

Proof. We use comparison. Let $w(x) = S(\rho)u_2(x)$. Clearly $w(x) \geq u_1(x)$, for every $x \in \partial B_\rho(O) \cap H$. Clearly, the assumptions (i)–(iii) in Lemma 2.2 are then met and $S(\rho)u_2(x) \geq u_1(x)$, for every $x \in H \setminus B_\rho(O)$. Thus for every $r \geq \rho$ it follows that $S(r) = \sup_{\{|x|=r\}\cap H} u_1(x)/u_2(x) \leq S(\rho)$. Replacing ρ by r and applying the above argument the conclusion follows. To obtain the conclusion for $s(r)$ we work with $w(x) = s(\rho)u_2(x)$. □

The basic argument used in Lemma 2.2 will be often used in the rest of this work. Our proof of Theorem 1.1 will involve the application of the boundary Harnack inequality and Corollary 2.3. For notational ease we will often write $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$. For $Q = (Q', 0) \in T$, let (Q, d) denote the point (Q', d) and $(0, d)$ the point on x_n-axis with $x_n = d$. Also $|x|^2 = \sum_{i=1}^n x_i^2$ and $|x'|^2 = \sum_{i=1}^{n-1} x_i^2$.

Proofs of parts (i)–(iii) *of Theorem 1.1.* Let $u(x) > 0$ and $v(x) > 0$ be two singular solutions in H satisfying the assumptions (A1) and (A2). We will use the boundary Harnack principle [6] near the flat boundary T. For every $R > 0$, let $J(R) = \partial B_R(O) \cap T$, and for any $P \in J(R)$ set $D(P, R) = \{x \in T : |x - P| < R\}$. Let $C(P, R) = D(P, R) \times (0, 2R) = \{x \in H : (x', 0) \in D(P, R),\ 0 < x_n < 2R\}$ denote the cylinder of height $2R$, radius R, with axis parallel to x_n-axis and P the center of the flat face on T. Fix $R > 0$. By the boundary Harnack principle, there exist absolute constants C_1 and C_2 such that

$$C_1 \frac{u(P, R/4)}{v(P, R/4)} \leq \frac{u(x)}{v(x)} \leq C_2 \frac{u(P, R/4)}{v(P, R/4)}, \quad \forall\, x \in C(P, R/8). \tag{1}$$

We now relate $u(P, R/4)/v(P, R/4)$ to $u(0, R)/v(0, R)$ by using the regular Harnack inequality in Lemma 2.1. Let L be the segment joining $(0, R)$ to $(P, R/4)$, then $\operatorname{dist}(L, H) \geq R/4$. Thus $e^{-5}u(0, R) \leq u(P, R/4) \leq e^5 u(0, R)$. With new absolute constants C_3 and C_4, (1) yields

$$C_3 \frac{u(0, R)}{v(0, R)} \leq \frac{u(x)}{v(x)} \leq C_4 \frac{u(0, R)}{v(0, R)}, \quad \forall\, x \in C(P, R/8). \tag{2}$$

By taking the union over all $P \in J(R)$, (2) holds for $x \in \partial B_R(O) \cap \{x : 0 < x_n \leq |x'|/4\}$. Using the Harnack inequality, we now show that an analogous estimate holds for $x \in \partial B_R(O) \cap \{x : x_n \geq |x'|/4\}$. If $y \in \partial B_R(O) \cap \{x : x_n \geq |x'|/4\}$ then $y_n \geq R/\sqrt{17}$ and $|y'| \leq 4R/\sqrt{17}$. The Harnack inequality then implies that $e^{-7}u(y) \leq u(0, R) \leq e^7 u(y)$. This together with (2) implies that there are absolute constants C_5 and C_6 such that for every $R > 0$ and $x \in \partial B_R(O)$,

$$C_5 \frac{u(x)}{v(x)} \leq \frac{u(0, R)}{v(0, R)} \leq C_6 \frac{u(x)}{v(x)}, \quad \forall\, x \in \partial B_R(O) \cap H. \tag{3}$$

For every $R > 0$, it is clear that $0 < s(R) \leq S(R) < \infty$, and there is an absolute constant C_7 such that $S(R) \leq C_7 s(R)$. By Corollary 2.3, $S(R)$ is decreasing in R and

$s(R)$ is increasing in R. Taking limits we have that $S(0) \le C_7 s(0)$. By Lemma 2.2, for every $R > 0$, $s(0) \le s(R) \le u(x)/v(x) \le S(R) \le S(0)$, $x \in H \setminus B_R(O)$. Thus parts (i)–(iii) of Theorem 1.1 are now proven. □

The rest of this section is devoted to deriving additional properties of non-negative singular solutions including the last conclusion in Theorem 1.1 (see Lemma 2.6). For $i = 1, 2, \dots, n$, let $\vec{e_i}$ be the unit vector along the x_i-axis.

Remark 2.4. Recall that $M(r) = \sup_{\partial B_r(O) \cap H} u(x)$. A comparison argument involving the function $w(x) = M(a) + \frac{M(b)-M(a)}{b-a}(|x| - a)$, $x \in H$, $a \le |x| \le b$, shows that $M(r)$ is convex in r. Since $M(r) > 0$ and u satisfies (A2), it follows that $M(r)$ is decreasing in r.

Lemma 2.5. *Let u be as in Theorem 1.1, and for $r > 0$ let $M(r)$ be as defined above. Let $D_r(O)$ be a fixed great circle centered at O. For every $x \in D_r(O)$ define $\theta = \theta(x) = \cos^{-1} x_n/r$. Then $u(x)$ is decreasing in θ and $u(0, r) = M(r)$.*

Proof. We give a proof based on reflection and comparison. Fix $r > 0$, and let $A(r)$ be the point $(0, r)$ on the x_n-axis. Let $\vec{\eta} = (\eta_1, \eta_2, \dots, \eta_{n-1}, 0)$ be a unit vector orthogonal to the x_n-axis, and $\Pi(\vec{\eta})$ be the 2-dimensional plane containing the x_n-axis and $\vec{\eta}$. Select $\vec{\eta}$ such that $D_r(O) = \partial B_r(O) \cap \Pi(\vec{\eta})$. Let $P_1, P_2 \in \partial D_r(O)$ (on one side of x_n-axis) be such that $0 < \cos\theta_1 = \frac{\langle P_1, e_n\rangle}{r} < \cos\theta_2 = \frac{\langle P_2, e_n\rangle}{r} \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product, and θ_1 and θ_2 ($\theta_1 > \theta_2$) are the angles made with the x_n axis. We show that $u(P_1) \le u(P_2)$. This would imply that $u(x)$ increases along a great circle as $x \to A(r)$ (or as $\theta \to 0$). Let $\vec{e} = (P_1 - P_2)/|P_1 - P_2|$, then $-1 < \langle \vec{e}, \vec{e_n}\rangle < 0$. Set $P_3 = (P_1 + P_2)/2$, and note $\vec{e} \perp \vec{P_3}$. Let

(i) $Z(\vec{e}) = \{x : \langle x - P_3, \vec{e}\rangle = 0\}$ be the hyperplane containing O and P_3,

(ii) $Z(\vec{e})^+ = \{x : \langle x - P_3, \vec{e}\rangle > 0\}$ and $Z(\vec{e})^- = \{x : \langle x - P_3, \vec{e}\rangle < 0\}$, the half-spaces,

(iii) $T(\vec{e}) = T \cap Z(\vec{e})^+$, and

(iv) the infinite wedge $W(\vec{e}) = H \cap Z(\vec{e})^+$.

For $x \in W(\vec{e})$, define the reflection about $Z(\vec{e})$ by $x^f = x - 2\langle x, \vec{e}\rangle e$. Clearly $(x^f)^f = x$. Also set

(v) $W(\vec{e})^f = \{x : x^f \in W(\vec{e})\}$, the reflection of $W(\vec{e})$ about $Z(\vec{e})$, and

(vi) $T^f(\vec{e})$ the reflection of $T(\vec{e})$ about $Z(\vec{e})$.

Define $u^f(x) = u(x^f)$, $x \in W^f(\vec{e})$, then u^f is ∞-harmonic in $W^f(\vec{e})$. Noting that $u(x) > 0$ in $W^f(\vec{e})$, comparison would then yield $u(x) \ge u^f(x)$ in $W^f(\vec{e})$. However, due to the singularity at O, we modify the geometry. For $\delta > 0$, small, let $Z(\vec{e})_\delta = \{x + \delta\vec{e} : x \in Z(\vec{e})\} = \{x : \langle x - P_3, \vec{e}\rangle = \delta\}$ be the translated plane and the half-spaces $Z(\vec{e})^{\pm}_\delta$ similarly defined. Set $W(\vec{e})_\delta = H \cap Z(\vec{e})^+_\delta$ and let $W^f(\vec{e})_\delta$ be the reflection of

$W(\vec{e})_\delta$ about $Z(\vec{e})_\delta$. Also for $x \in W(\vec{e})_\delta$, let $x^f_\delta = x + 2(\delta - \langle x, \vec{e}\rangle)\vec{e}$. Also set $T^f_\delta(\vec{e})$ to be the reflection of $T \cap Z(\vec{e})^+_\delta$. Define $u^f_\delta(x) = u(x^f_\delta)$ in $W^f(\vec{e})_\delta$. Then

(a) $u^f_\delta(x)$ is ∞-harmonic in $W^f(\vec{e})_\delta$,

(b) $u^f_\delta(x) = u(x)$, on $Z(\vec{e})_\delta \cap H$, and

(c) $u(x) \geq u^f_\delta(x) = 0$ in $T^f_\delta(\vec{e})$.

Adapting the argument of Lemma 2.2 we conclude that $u^f_\delta(x) \leq u(x)$, $x \in W^f(\vec{e})_\delta$. Define $P_{2,\delta} \in W^f(\vec{e})_\delta$ the reflection of P_1. Clearly, $u(P_1) = u^f_\delta(P_{2,\delta}) \leq u(P_{2,\delta})$. Since this holds for all small $\delta > 0$, continuity of u implies that $u(P_1) = u^f(P_2) \leq u(P_2)$. Clearly, the statement of the Lemma holds and $M(r) = u(0,r)$. Also by Remark 2.4, $u(0,r)$ is decreasing in r. □

For $\theta \in [0, \pi/2)$, let $C_\theta = \left\{x \in H : \cos^{-1}\left(\frac{\langle x, e_n\rangle}{|x|}\right) = \theta\right\}$ be the cone of opening θ with apex at O and symmetric about the x_n-axis. For $r > 0$, let $I(\theta, r) = C_\theta \cap \partial B_r(O)$.

Lemma 2.6 (Symmetry). *Let u be as in Theorem 1.1. Then for each $r > 0$ and every $\theta \in [0, \pi/2]$, we have $u(x) = u(y)$, for every $x, y \in I(\theta, r)$. Thus u is axially symmetric in H and $u(x) = h(r, \theta)$, where $r = |x|$ and $\theta = \cos^{-1} x_n/r$.*

Proof. We again use reflection and comparison. The arguments are similar to those in Lemmas 2.2 and 2.5. Let $P_1, P_2 \in I(\theta, r)$. Both P_1 and P_2 lie in the plane $\{x : x_n = r\cos\theta\}$. Set $\vec{e} = (P_1 - P_2)/|P_1 - P_2|$ and $P_3 = (P_1 + P_2)/2$, then $\vec{e} \perp \vec{e_n}$ and $\vec{e} \perp P_3$. Define the hyperplane $Z(\vec{e}) = \{x : \langle x - P_3, \vec{e}\rangle = \langle x, \vec{e}\rangle = 0\}$. Note that $Z(\vec{e}) \ni O$. We tilt and translate $Z(\vec{e})$ as follows. Define $Z_\varepsilon(\vec{e}) = \{x : \langle x, \vec{e} - \varepsilon\vec{e_n}\rangle = 0\}$, where $\varepsilon > 0$ is so small that $P_1 \in Z_\varepsilon(\vec{e})^+ = \{x : \langle x, \vec{e} - \varepsilon\vec{e_n}\rangle > 0\}$. For $\delta > 0$, small, define the translated plane $Z_{\varepsilon,\delta}(\vec{e}) = \{x + \delta(\vec{e} - \varepsilon\vec{e_n}) : x \in Z_\varepsilon(\vec{e})\}$ and the half-space $Z_{\varepsilon,\delta}(\vec{e})^+$ accordingly. For small $\delta > 0$, $P_1 \in Z_{\varepsilon,\delta}(\vec{e})^+$. We reflect the wedge $H \cap Z_{\varepsilon,\delta}(\vec{e})^+$ about $Z_{\varepsilon,\delta}(\vec{e})$. We now apply comparison as was done in Lemma 2.5. For $x \in H \cap Z_{\varepsilon,\delta}(\vec{e})^+$, let $x^f(\varepsilon, \delta)$ be the reflection of x about $Z_{\varepsilon,\delta}(\vec{e})$, and $u^f(x) = u(x^f(\varepsilon, \delta))$. Comparison yields that $u^f(x) \leq u(x)$ in the wedge obtained by reflecting $H \cap Z_{\varepsilon,\delta}(\vec{e})^+$. Clearly, $u(P_1) \leq u(P^f_1(\varepsilon, \delta))$. Letting $\varepsilon, \delta \to 0$, we see that $u(P_1) \leq u(P_2)$. Now replacing P_1 by P_2 and repeating the above argument we see $u(P_1) \geq u(P_2)$ and equality holds. □

Remark 2.7. Using Lemma 2.5 and Lemma 2.6, it is clear that $u(x) = h(r, \theta)$ is decreasing in θ. If $\rho = |x'|$ then $u(x) = h(r, \theta) = g(\rho, x_n)$.

Remark 2.8. Adapting the arguments of Lemmas 2.5 and 2.6, it can be shown that $u(x) = u(\rho, x_n)$ is decreasing in ρ.

Lemma 2.9. *Let u be as in Theorem 1.1, $P \in H$ be a point on the x_n-axis and $\vec{e}$ be such that $\langle \vec{e}, \vec{e_n}\rangle > 0$. Then for $t \geq 0$, $u(P + t\vec{e})$ is a decreasing function of t. In particular, $u = h(r, \theta)$ is decreasing in r.*

Proof. Select $P \in H$, then $P = (0, \eta)$ for some $\eta > 0$. Set $H_\eta = \{x : x_n = \eta\}$ and $H_\eta^+ = \{x : x_n \geq \eta\}$. For $x \in H_\eta^+$ and $a > 1$, let $x_a = a(x - P) + P$. Under this scaling

(i) H_η^+ stays invariant,

(ii) x and x_a are collinear with P, and

(iii) if $x \in H_\eta$ then $x_a \in H_\eta$ and $|x'| = |x - P| < |x'_a| = a|x - P|$.

Now define $u_a(x) = u(x_a)$ in H_η^+. Then

(a) u_a is ∞-harmonic in H_η^+,

(b) for $x \in H_\eta$, $u_a(x) = u(x_a) \leq u(x)$ (by Remarks 2.7 and 2.8), and

(c) by (A1) and (A2),

$$\lim_{r\to\infty} \sup_{\{|x-P|=r\}\cap H_\eta^+} u(x) = \lim_{r\to\infty} \sup_{\{|x-P|=r\}\cap H_\eta^+} u_a(x) = 0.$$

We may now adapt Lemma 2.2 to conclude that $u(x_a) = u_a(x) \leq u(x)$, $x \in H_\eta^+$. This shows that $u(P + t\vec{e})$ is decreasing in t.

Now let $z \in H$, $t > 1$ and $\vec{e} = z/|z|$. Let $L(z)$ be the ray $\{sz : s > 0\}$. For small $\varepsilon > 0$, set $P_\varepsilon = (0, \varepsilon)$, $y = P_\varepsilon + |z|\vec{e}$, and $y_t = P_\varepsilon + t|z|\vec{e}$. Clearly, $|y - z| = |y_t - tz| = |P_\varepsilon| \to 0$ as $\varepsilon \to 0$. From the previous argument it is clear that for every $\varepsilon > 0$, $u(y_t) \leq u(y)$. Since u is continuous in H^+, it follows that $u(tz) \leq u(z)$. □

Remark 2.10. We may derive a lower bound for u using Lemma 2.1 on $\partial B_r(O) \cap H$, $r > 0$. Let $P = (0, r)$ and $Q = (r, \theta)$, $0 < \theta < \pi/2$. Fix Q and let L denote the circular arc joining P to Q. Parameterizing L by θ, we see that $M(r) \leq (\sec\theta + \tan\theta)u(r, \theta)$. Working with the chord instead $[\sigma(t) = r(t\sin\theta, t\cos\theta + (1 - t))$, $\delta(t) = r(1 - t(1 - \cos\theta))$, $0 \leq t \leq 1]$ we get $M(r) \leq u(r, \theta)(\sec\theta)^{\nu(\theta)}$, where $\nu(\theta) = \sqrt{2/(1 - \cos\theta)}$. Also if $P = (r_1, \theta)$ and $Q = (r_2, \theta)$, $0 < r_1 < r_2$, we may show by using Lemma 2.1 that $u(r_1, \theta)r_1^{\sec\theta} \leq u(r_2, \theta)r_2^{sec\theta}$.

Appendix

We recall the example of a planar singular solution in [2,7]. In [2] this singular solution is expressed in terms of r and θ which refer to the polar coordinates. This example is written as $u(x) = r^{-1/3}f(\theta)$ where

$$\begin{aligned} f(\theta) &= \frac{\cos t}{(1 + 2\cos^2 t)^{2/3}}, \quad \theta = t - 2\arctan[(\tan t)/2], \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ \Delta_\infty u &= u_r^2 u_{rr} + \frac{2u_r u_\theta u_{r\theta}}{r^2} + \frac{u_\theta^2 u_{\theta\theta}}{r^4} - \frac{u_r u_\theta^2}{r^3} = 0. \end{aligned} \tag{4}$$

Also $f(0) = 1$, $f(\pm\pi/2) = 0$, $f'(0) = 0$, and $f(-\theta) = f(\theta)$. This solution is C^∞ except at $\theta = 0$ and it was verified in [7] (see the appendix of this work) that this is also a viscosity solution of (4) in the half-plane. We extend the planar example to $n > 2$ as follows. Recall that in this work $\theta \in [0, \pi/2]$, and refers to the opening of the cone with apex at O, and that every non-negative singular solution is axially symmetric. For $n > 2$, define $r^2 = \sum_{i=1}^n x_i^2$, $\cos\theta = x_n/r$, and set for $r > 0$, $0 \le \theta \le \pi/2$, $u(x) = h(r,\theta) = r^{-1/3} f(\theta)$. Now u is defined in all of H. Routine calculations show that (4) holds in $0 < \theta < \pi/2$. Showing that u is a viscosity solution in H will largely be a repetition of the work in [7]. We provide details where slight differences occur. We take $\theta = 0$. Let $\psi(x) \in C^2$ be such that $u(x) - \psi(x)$ has a maximum at $x^1 = (0, r_1)$ on x_n-axis. Then as $x \to x^1$,

$$\begin{aligned}\frac{f(\theta)}{r^{1/3}} - \frac{1}{r_1^{1/3}} &\le \psi(x) - \psi(x^1)\\ &= \sum_{i=1}^{n-1} \psi_{x_i}(x^1) x_i + \psi_{x_n}(x^1)(x_n - r_1) + O(|x - x^1|^2).\end{aligned} \tag{5}$$

Taking $\theta = 0$ and $x = (0, r)$, we see that $\psi_{x_n}(x^1) = -r_1^{-4/3}/3$. For a fixed $i = 1, 2, \dots, n-1$, set $x = s\vec{e_i} + x^1$. It is clear that for any $s \neq 0$, $s\psi_{x_i} + O(s^2) \ge r^{-1/3} f(\theta) - r_1^{-1/3}$. It is easily seen that $\psi_{x_i}(x^1) = 0$. Taking $x = (0, r_1 \pm \delta)$, $\delta > 0$ and working with quadratic approximations, $\psi_{x_n x_n}(x^1) \ge 0$. Thus $\Delta_\infty \psi(x^1) = \psi_{x_n}^2(x^1)\psi_{x_n x_n}(x^1) \ge 0$. Now let $u - \psi$ have a minimum at $x^1 = (0, r_1)$. Then (5) holds with the inequality reversed. Once again $\psi_{x_i}(x^1) = 0$, $i = 1, 2, \dots, n-1$, and $\psi_{x_n}(x^1) = -r_1^{-4/3}/3$. Using quadratic approximation with $x = s\vec{e_i} + x^1$ yields $\psi_{x_i x_i}(x^1) s^2 \le r^{-1/3} f(\theta) - r_1^{-1/3}$. Recalling that $f''(\theta) \to -\infty$ as $s \to 0$, we see that $\psi_{x_i x_i}(x^1) \le -\infty$, contradicting that ψ is C^2. Thus minimum occurs only when $\theta \neq 0$.

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