

On Higher Dimensional Hirzebruch-Jung Singularities

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ABSTRACT

A germ of normal complex analytical surface is called a Hirzebruch-Jung singularity if it is analytically isomorphic to the germ at the 0-dimensional orbit of an affine toric surface. Two such germs are known to be isomorphic if and only if the toric surfaces corresponding to them are equivariantly isomorphic. We extend this result to higher-dimensional Hirzebruch-Jung singularities, which we define to be the germs analytically isomorphic to the germ at the 0-dimensional orbit of an affine toric variety determined by a lattice and a simplicial cone of maximal dimension. We deduce a normalization algorithm for quasi-ordinary hypersurface singularities.

Key words: Hirzebruch-Jung singularities, quasi-ordinary singularities, toric singularities, normalization

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Introduction

In this paper we generalize to arbitrary dimensions the notion of *Hirzebruch-Jung singularities* and we show how to classify them up to analytical isomorphism by combinatorial data. Then we give normal forms for these data and we compute these normal forms when the germ is the normalization of an irreducible quasi-ordinary hypersurface singularity.

A germ of reduced equidimensional complex analytical space is called *quasi-ordinary* if there exists a finite morphism from it to a smooth space of the same dimension, such that the discriminant locus of the morphism is contained in a divisor with normal crossings.

If the term “quasi-ordinary” seems to appear first in the '60s, in works of Zariski and Lipman, the study of quasi-ordinary germs goes back at least to the work [16] of Jung on the problem of local uniformization of surfaces. For details on it see the first chapter of [29]. The idea of Jung was to study an arbitrary germ of surface embedded in \mathbf{C}^3 by considering a finite linear projection and an embedded desingularization of the discriminant curve. By changing the base of the initial projection using this desingularization morphism, he obtained a surface which is quasi-ordinary in the neighborhood of any of its points.

This method was used by Walker [28] in order to prove the existence of a resolution of the singularities of a complex algebraic surface. This work is considered by Zariski [29] to be the first rigorous proof of this fact. Hirzebruch [15] uses again Jung's method in order to prove the existence of a desingularization for complex analytical surfaces which are locally embeddable in \mathbf{C}^3 . This last restriction was eliminated by Laufer [17].

An important step in Hirzebruch's method was to consider the normalizations of the quasi-ordinary germs he arrived at by Jung's method. He gave explicit constructions of their minimal resolutions by patching affine planes. Later on, those germs were called “Hirzebruch-Jung singularities”. After Artin's work on rational surface singularities in the '60s, they were seen to be precisely the rational surface singularities which have as dual resolution graph a segment. This is the definition used in [3]. Hirzebruch-Jung singularities are usually classified up to analytical isomorphism by an ordered pair $(n, q) \in \mathbf{N}^* \times \mathbf{N}$ of coprime numbers with $q < n$. In order to get this classification, Hirzebruch studied the exceptional divisor of the minimal resolution morphism of the singularity and introduced the numbers n, q starting from the self-intersection numbers of its components (see [3] and section 6). It is also known that this classification is topological up to orientation. For historical details, see Brieskorn [5].

After the introduction of toric geometry in the '70s, Hirzebruch-Jung surface singularities were seen to be precisely the germs analytically isomorphic to the germs of toric surfaces taken at 0-dimensional orbits (see [10, 20]). It is this view-point which we generalize here.

If \mathcal{W} is a lattice and σ is a strictly convex finite rational polyhedral cone in $\mathcal{W}_{\mathbf{R}} := \mathcal{W} \otimes \mathbf{R}$, we denote by \mathcal{M} the dual lattice of \mathcal{W} and by $\check{\sigma} \subset \mathcal{M}_{\mathbf{R}}$ the dual cone of σ . We denote by $\mathcal{Z}(\mathcal{W}, \sigma) := \text{Spec } \mathbf{C}[\check{\sigma} \cap \mathcal{M}]$ the affine normal toric variety determined by the pair (\mathcal{W}, σ) . When σ and $\mathcal{W}_{\mathbf{R}}$ have the same dimension d , we say that (\mathcal{W}, σ) is a *maximal pair of dimension d* . When σ is a simplicial cone, we say that (\mathcal{W}, σ) is a *simplicial pair*. We say that the simplicial cone σ is *regular with respect to \mathcal{W}* if it is generated by a subset of a basis of \mathcal{W} . In this case we say also

that (\mathcal{W}, σ) is a *regular pair*. Two pairs $(\mathcal{W}_1, \sigma_1)$ and $(\mathcal{W}_2, \sigma_2)$ are called *isomorphic* if there exists an isomorphism of lattices $\phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ sending σ_1 onto σ_2 .

By analogy with the bidimensional case, one can define:

A germ of irreducible normal complex analytical space of arbitrary dimension is called a *Hirzebruch-Jung singularity* if it is analytically isomorphic with the normalization of an n -dimensional irreducible quasi-ordinary germ.

In [21] (see also [23] and section 2) we showed that such a normalization is in fact analytically isomorphic to the germ at the 0-dimensional orbit of an affine toric variety defined by a maximal simplicial pair. Conversely (see Proposition 2.5), the germ at the 0-dimensional orbit of a toric variety defined by a maximal simplicial pair is quasi-ordinary. This shows that, alternatively, one can define Hirzebruch-Jung singularities by combinatorial data (see section 2):

A germ of irreducible normal complex analytical space of arbitrary dimension is called a *Hirzebruch-Jung singularity* if it is analytically isomorphic with the germ at the 0-dimensional orbit of an affine toric variety defined by a maximal simplicial pair.

It is clear that isomorphic maximal simplicial pairs give rise to analytically isomorphic Hirzebruch-Jung singularities. Our main theorem (see Theorem 3.4) shows the converse statement:

The analytical type of a Hirzebruch-Jung singularity $(\mathcal{Z}, 0) \simeq (\mathcal{Z}(\mathcal{W}, \sigma), 0)$ determines the pair (\mathcal{W}, σ) up to isomorphism.

In order to prove this result, we make the Riemann extension of the universal covering map of the smooth part of $(\mathcal{Z}, 0)$ over all of $(\mathcal{Z}, 0)$. We call this map $\mu : (\tilde{\mathcal{Z}}, 0) \rightarrow (\mathcal{Z}, 0)$ the *orbifold map* of \mathcal{Z} (see section 3). Then we look at the action $\rho(\mathcal{Z})$ of the local fundamental group of $(\mathcal{Z}, 0)$ on the Zariski cotangent space of $(\tilde{\mathcal{Z}}, 0)$ and we construct from it a pair $(W(\rho(\mathcal{Z})), \sigma_0)$ determined by the analytical type of \mathcal{Z} . Theorem 3.4 says that the pairs (\mathcal{W}, σ) and $(W(\rho(\mathcal{Z})), \sigma_0)$ are isomorphic.

We used for the first time orbifold maps in [23] in order to get analytical invariants of quasi-ordinary singularities. When we began to study the problems solved in [23] and in the present paper, we tried to use some desingularization morphism of $(\mathcal{Z}, 0)$. We could not manage their high non-canonicity, and so the idea to use instead the orbifold map came as a relief.

We see that, in order to classify up to analytical isomorphism n -dimensional Hirzebruch-Jung singularities, one needs only to classify up to isomorphism the pairs (\mathcal{W}, σ) , which is a combinatorial problem. We give normal forms for such pairs once an ordering of the edges of σ is chosen (Proposition 3.7). We define *the type* of a

Hirzebruch-Jung singularity $(\mathcal{Z}, 0)$ to be one of the normal forms associated to the pair $(W(\rho(\mathcal{Z})), \sigma_0)$ (Definition 3.8).

In section 4 we give an *algorithm of normalization* of an irreducible quasi-ordinary hypersurface singularity (Proposition 4.5). More precisely, we compute the type of the normalization, the ordering being the one determined by the choice of the ambient coordinates of the starting quasi-ordinary singularity. The algorithm starts from the characteristic exponents and constitutes a generalization of the normalization algorithm for surfaces that we published in [21, 22] (see Proposition 6.5). Incidentally, if (\mathcal{W}, σ) is a maximal regular pair, we compute the normal forms for the pairs (\mathcal{W}', σ) , where \mathcal{W}' is a sublattice of finite index of \mathcal{W} defined by a congruence (Lemma 4.3).

Section 5 contains a tridimensional example of application of the algorithm. In section 6 we restrict our attention to the bidimensional case and we compare our definition of the type with the classical one. We conclude by stating in section 7 some questions about the topological types of Hirzebruch-Jung singularities and of the germs at the 0-dimensional orbits of general affine toric varieties.

Since the first version of this work (see [24]), another proof of the analytical invariance of the pair (\mathcal{W}, σ) was obtained by González Pérez and González-Sprinberg in [13]. Their result is more general, as it applies in particular to any maximal pair (\mathcal{W}, σ) , not necessarily a simplicial one. They use as an essential tool results of Gubeladze [14], which show that two affine toric varieties are isomorphic if and only if the defining pairs are isomorphic. Their method gives no canonical way to associate to a germ $(\mathcal{Z}, 0) \simeq (\mathcal{Z}(\mathcal{W}, \sigma))$ a pair $(\mathcal{W}', \sigma') \simeq (\mathcal{W}, \sigma)$. It would be interesting to find one.

1. Generalities on quasi-ordinary germs

For any point P on a complex analytical space \mathcal{V} , we denote by $\mathcal{O}_{\mathcal{V}, P}$ the local algebra of \mathcal{V} at P . In the sequel we will denote with the same letter a germ and a sufficiently small representative of it. It will be deduced from the context if one deals with one or the other notion. We denote by $\text{Sing}(\mathcal{V})$ the singular locus of \mathcal{V} .

Let $d \geq 1$ be an integer. Define the *algebra of fractional series*

$$\widetilde{\mathbf{C}\{X\}} := \varinjlim_{N \geq 0} \mathbf{C}\{X_1^{\frac{1}{N}}, \dots, X_d^{\frac{1}{N}}\},$$

where $X := (X_1, \dots, X_d)$. If $m = (m_1, \dots, m_d) \in \mathbf{Q}_+^d$, we denote $X^m := X_1^{m_1} \cdots X_d^{m_d}$. If $\eta \in \widetilde{\mathbf{C}\{X\}}$ can be written $\eta = X^m u(X)$, with $m \in \mathbf{Q}_+^d$ and $u \in \widetilde{\mathbf{C}\{X\}}$, $u(0, \dots, 0) \neq 0$, we say that η has a *dominating exponent*.

Definition 1.1. Let $(\mathcal{S}, 0)$ be a germ of reduced equidimensional complex space. The germ $(\mathcal{S}, 0)$ is called *quasi-ordinary* if there exists a finite morphism ψ from $(\mathcal{S}, 0)$ to a smooth space of the same dimension, whose discriminant locus is contained in a hypersurface with normal crossings. Such a morphism ψ is also called *quasi-ordinary*.

For instance, all reduced germs of curves are quasi-ordinary with respect to any finite morphism whose target is a smooth curve.

In the special case in which \mathcal{S} is a d -dimensional hypersurface germ, one can find local coordinates X on the target space of ψ such that the discriminant locus of ψ is contained in $\{X_1 \cdots X_d = 0\}$ and an element Y in the maximal ideal of $\mathcal{O}_{\mathcal{S},0}$ such that (ψ, Y) embeds $(\mathcal{S}, 0)$ in $\mathbf{C}^d \times \mathbf{C}$. So ψ appears as a map

$$\psi : \mathcal{S} \rightarrow \mathbf{C}^d,$$

which is *unramified over* $(\mathbf{C}^*)^d$. By the Weierstrass preparation Theorem, the image of \mathcal{S} by (ψ, Y) , identified in the sequel with \mathcal{S} , is defined by a unitary polynomial $f \in \mathbf{C}\{X\}[Y]$. The discriminant locus of ψ is defined by the discriminant $\Delta_Y(f)$ of f , which has therefore a dominating exponent.

Definition 1.2. Let $f \in \mathbf{C}\{X\}[Y]$ be unitary. If $\Delta_Y(f)$ has a dominating exponent, we say that f is *quasi-ordinary*.

The following Theorem (see [1, 19]), generalizes the Theorem of Newton-Puiseux for plane curves:

Theorem 1.3 (Jung-Abhyankar). *If $f \in \mathbf{C}\{X\}[Y]$ is quasi-ordinary, then the set $R(f)$ of roots of f embeds canonically in the algebra $\widetilde{\mathbf{C}\{X\}}$.*

In the sequel, we consider $R(f)$ as a subset of $\widetilde{\mathbf{C}\{X\}}$. Moreover, we suppose that f is *irreducible*. Then all the differences of roots of f have dominating exponents, which are totally ordered for the componentwise order (see [18, 19]). If G is their number, denote them by $A_1 < \cdots < A_G$, $A_i = (A_i^1, \dots, A_i^d)$, $\forall i \in \{1, \dots, G\}$.

Definition 1.4. We call the vectors $A_1, \dots, A_G \in \mathbf{Q}_+^d$ the *characteristic exponents* and the monomials X^{A_1}, \dots, X^{A_G} the *characteristic monomials* of f or of ψ .

Following Proposition 4.5, we explain a way to *normalize* the characteristic exponents.

2. Generalized Hirzebruch-Jung singularities

In this section we recall some results about the normalization of quasi-ordinary singularities and we define Hirzebruch-Jung singularities in any dimension.

For details about toric geometry, see Oda [20] and Fulton [10].

We denote by $W_0 = \mathbf{Z}^d$ the canonical d -dimensional lattice, by $M_0 = \mathbf{Z}^d$ its canonical dual and by σ_0 the canonical regular cone of maximal dimension in W_0 .

Let $(\mathcal{S}, 0)$ be an irreducible d -dimensional quasi-ordinary germ and let $\psi : (\mathcal{S}, 0) \rightarrow (\mathbf{C}^d, 0)$ be a finite morphism unramified over $(\mathbf{C}^*)^d$. We look at \mathbf{C}^d as the affine toric variety $\mathcal{Z}(W_0, \sigma_0)$. Then the fundamental group $\pi_1((\mathbf{C}^*)^d)$ can be canonically identified with W_0 (see [10]).

Define

$$W(\psi) := \psi_* \pi_1(\psi^{-1}((\mathbf{C}^*)^d)).$$

It is a subgroup of $\pi_1((\mathbf{C}^*)^d) = W_0$. Moreover, $W(\psi)$ is of finite index in W_0 , as ψ is a finite morphism. Consider the affine toric variety $\mathcal{Z}(W(\psi), \sigma_0)$ obtained by changing the lattice from W_0 to $W(\psi)$. Denote by

$$\gamma_{W_0:W(\psi)} : \mathcal{Z}(W(\psi), \sigma_0) \rightarrow \mathcal{Z}(W_0, \sigma_0) = \mathbf{C}^d$$

the canonical morphism associated to this change of lattice. We proved topologically the following Theorem in [21, 23]. A more algebraic proof was given later by Aroca and Snoussi in [2].

Theorem 2.1. *One has the following commutative diagram, in which ν is a normalization morphism:*

$$\begin{array}{ccc} (\mathcal{Z}(W(\psi), \sigma_0), 0) & \xrightarrow{\nu} & (\mathcal{S}, 0) \\ & \searrow \gamma_{W_0:W(\psi)} & \swarrow \psi \\ & & (\mathbf{C}^d, 0) \end{array}$$

In the special case in which \mathcal{S} is a hypersurface germ, we can express the lattice $W(\psi)$ using the characteristic exponents of ψ . In order to do this let us introduce, following Lipman [19], the abelian groups $M_0 := \mathbf{Z}^d, M_i := M_{i-1} + \mathbf{Z}A_i, \forall i \in \{1, \dots, G\}$ and the successive indices $N_i := (M_i : M_{i-1}), \forall i \in \{1, \dots, G\}$. Following González Pérez [12] we consider also the dual lattices W_k of the lattices M_k :

$$W_k := \text{Hom}(M_k, \mathbf{Z}), \quad \forall k \in \{1, \dots, G\}.$$

One has the inclusions: $M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_G, W_0 \supsetneq W_1 \supsetneq \dots \supsetneq W_G$. The following Proposition was proved in [21, 23]:

Proposition 2.2. *Let $f \in \mathbf{C}\{X\}[Y]$ be an irreducible quasi-ordinary polynomial and ψ be the associated quasi-ordinary projection. Then $W(\psi) = W_G$.*

Using this identification, Theorem 2.1 becomes:

Corollary 2.3 (González Pérez). *If f is an irreducible quasi-ordinary polynomial defining the germ \mathcal{S} , then one has the following commutative diagram, in which ν is a normalization morphism:*

$$\begin{array}{ccc} (\mathcal{Z}(W_G, \sigma_0), 0) & \xrightarrow{\nu} & (\mathcal{S}, 0) \\ & \searrow \gamma_{W_0:W_G} & \swarrow \psi \\ & & (\mathbf{C}^d, 0) \end{array}$$

This theorem was first proved algebraically by González Pérez in [12], without passing through Proposition 2.2. It inspired our Theorem 2.1.

Theorem 2.1 and the fact that in dimension 2 Hirzebruch-Jung singularities are precisely the normalizations of quasi-ordinary ones, motivates us to introduce the following definition in arbitrary dimension:

Definition 2.4. The irreducible germ $(\mathcal{Z}, 0)$ of complex analytical space is called a *Hirzebruch-Jung singularity* if it is analytically isomorphic with the germ at the 0-dimensional orbit of an affine toric variety defined by a maximal simplicial pair.

One can give another definition of Hirzebruch-Jung singularities:

Proposition 2.5. *Hirzebruch-Jung singularities are precisely the quasi-ordinary singularities which are normal.*

Proof. By Theorem 2.1, each normal quasi-ordinary singularity is a Hirzebruch-Jung one.

Conversely, let $(\mathcal{Z}, 0)$ be a Hirzebruch-Jung singularity, according to Definition 2.4. Then $(\mathcal{Z}, 0) \simeq (\mathcal{Z}(\mathcal{W}, \sigma), 0)$, where (\mathcal{W}, σ) is a maximal simplicial pair. Let v_1, \dots, v_d be the primitive elements of \mathcal{W} situated on the edges of σ and let w_1, \dots, w_d be a basis of the lattice \mathcal{W} . The matrix transforming (v_1, \dots, v_d) in (w_1, \dots, w_d) has rational coefficients. So, there is a number $q \in \mathbf{N}^*$ such that the matrix transforming $(\frac{1}{q}v_1, \dots, \frac{1}{q}v_d)$ into (w_1, \dots, w_d) has integral coefficients. If $\mathcal{W}_0 := \sum_{i=1}^d \mathbf{Z} \frac{1}{q} v_i \subset \mathcal{W}_{\mathbf{Q}}$, then \mathcal{W} is a sublattice of finite index of \mathcal{W}_0 and (\mathcal{W}_0, σ) is a maximal regular pair. Denote also by 0 the 0-dimensional orbit of $\mathcal{Z}(\mathcal{W}_0, \sigma)$. Consider the toric morphism obtained by changing the lattice:

$$\eta : \mathcal{Z}(\mathcal{W}, \sigma) \rightarrow \mathcal{Z}(\mathcal{W}_0, \sigma) \simeq \mathbf{C}^d.$$

It is finite and unramified over the torus $\mathcal{Z}(\mathcal{W}_0, \{0\})$. So, its germ

$$\eta : (\mathcal{Z}(\mathcal{W}, \sigma), 0) \rightarrow (\mathcal{Z}(\mathcal{W}_0, \sigma), 0)$$

is quasi-ordinary. But $\mathcal{Z}(\mathcal{W}, \sigma)$ is normal, and the Proposition is proved. \square

3. The analytical classification of Hirzebruch-Jung singularities

In this section we prove our main result (Theorem 3.4) which classifies Hirzebruch-Jung singularities up to analytical isomorphism by combinatorial data. It states that a maximal simplicial pair (\mathcal{W}, σ) can be reconstructed from the analytical type of the germ $(\mathcal{Z}(\mathcal{W}, \sigma), 0)$. Our essential tool is *the orbifold map* μ associated to \mathcal{Z} (Definition 3.3). Then we give a normal form for maximal simplicial pairs (\mathcal{W}, σ) , once an ordering of the edges of σ was fixed (Proposition 3.7). This allows us to define *the type* of a Hirzebruch-Jung singularity (Definition 3.8).

Let V be a complex finite dimensional vector space. An element of $\mathrm{GL}(V)$ is called a *complex reflection* if its fixed-point set is a hyperplane of V . A finite group $\Gamma \subset \mathrm{GL}(V)$ is called *small* (see Prill [25]) if it contains no complex reflections.

Let us recall a generalization of the Riemann existence theorem (see [4]):

Theorem 3.1 (Grauert-Remmert). *Let S be a connected normal complex space and $T \subset S$ a proper closed analytical subset. Let $Y := S - T$, let X be a normal complex space and $\phi : X \rightarrow Y$ be a ramified covering. Then ϕ extends to a ramified covering $\tilde{\phi} : \tilde{X} \rightarrow S$ with \tilde{X} normal if and only if the closure \bar{B} in S of the branch locus $B \subset Y$ of ϕ is an analytical subset in S . In this case, the extension is unique.*

In the case it exists, we say that $\tilde{\phi}$ is obtained by *Riemann extension* of ϕ .

Let (\mathcal{W}, σ) be a maximal simplicial pair of dimension $d \geq 2$. Denote by v_i , $i \in \{1, \dots, d\}$ the primitive elements of \mathcal{W} situated on the edges of σ . Denote by $\tilde{\mathcal{W}}$ the sublattice of \mathcal{W} generated by v_1, \dots, v_d . Then $(\tilde{\mathcal{W}}, \sigma)$ is a maximal regular pair. Consider the toric morphism

$$\mu : \mathcal{Z}(\tilde{\mathcal{W}}, \sigma) \rightarrow \mathcal{Z}(\mathcal{W}, \sigma),$$

obtained by keeping the same cone σ and by replacing the lattice \mathcal{W} by $\tilde{\mathcal{W}}$. In what follows, we will denote by $(\mathcal{Z}, 0)$ and $(\tilde{\mathcal{Z}}, 0)$ the complex analytic germs $(\mathcal{Z}(\mathcal{W}, \sigma), 0)$ and $(\mathcal{Z}(\tilde{\mathcal{W}}, \sigma), 0)$, or sufficiently small representatives of them. Notice that $0 = \mu^{-1}(0)$.

Proposition 3.2. *The map μ is obtained by Riemann extension of the universal covering map of the smooth part of $\mathcal{Z}(\mathcal{W}, \sigma)$. In particular, the restriction of μ over the germ $(\mathcal{Z}, 0)$ depends only on the analytical structure of $(\mathcal{Z}, 0)$.*

Proof. The proof of this Proposition is also contained in the section 6 of [23].

By general results of toric geometry (see [20, Corollary 1.16]), μ is the quotient map of $\mathcal{Z}(\tilde{\mathcal{W}}, \sigma)$ by the natural action of the finite group $\mathcal{W}/\tilde{\mathcal{W}}$. Moreover, in toric coordinates, this action is linear, faithful, and does not contain complex reflections. So, as a linear group $\mathcal{W}/\tilde{\mathcal{W}}$ is small (see the Remark which follows Theorem 3.4).

This shows that the locus $\mathrm{Fix}(\mu)$ of the fixed points of the elements of $\mathcal{W}/\tilde{\mathcal{W}}$ distinct from the identity has codimension at least 2 in $\mathcal{Z}(\tilde{\mathcal{W}}, \sigma)$. Moreover,

$$\mu^{-1}(\mathrm{Sing}(\mathcal{Z}(\mathcal{W}, \sigma))) \subset \mathrm{Fix}(\mu).$$

As $\mathcal{Z}(\tilde{\mathcal{W}}, \sigma)$ is smooth, the complement $\mathcal{Z}(\tilde{\mathcal{W}}, \sigma) - \mu^{-1}(\mathrm{Sing}(\mathcal{Z}(\mathcal{W}, \sigma)))$ is simply connected, and so the restriction of μ over the smooth part of $\mathcal{Z}(\mathcal{W}, \sigma)$ is a universal covering map. The uniqueness in Theorem 3.1 implies then the Proposition. \square

Following a terminology used in [6], we define:

Definition 3.3. The morphism μ obtained by Riemann extension of the universal covering map of the smooth part of $(\mathcal{Z}, 0)$ is called *the orbifold map* associated to $(\mathcal{Z}, 0)$.

Denote by $\Gamma(\mathcal{Z})$ the *group of covering transformations* of μ (in the terminology of [8]), formed by those analytical automorphisms $\phi : (\tilde{\mathcal{Z}}, 0) \rightarrow (\tilde{\mathcal{Z}}, 0)$ which verify $\mu = \mu \circ \phi$. Consider its action

$$\Gamma(\mathcal{Z}) \xrightarrow{\rho(\mathcal{Z})} \text{GL}(\tilde{m}/\tilde{m}^2) \tag{1}$$

on the Zariski cotangent space of $\tilde{\mathcal{Z}}$ at 0. Here \tilde{m} denotes the maximal ideal of $\tilde{\mathcal{Z}}$ at 0. Being abelian, the group $\Gamma(\mathcal{Z})$ is canonically isomorphic with the local fundamental group of $(\mathcal{Z}, 0)$. As an abstract representation, $\rho(\mathcal{Z})$ is clearly determined by the analytical type of the germ $(\mathcal{Z}, 0)$. The proof of Proposition 3.2 shows that the map (1) is a faithful \mathbf{C} -linear representation of $\Gamma(\mathcal{Z})$, whose image is small.

More generally, consider a faithful finite-dimensional \mathbf{C} -linear representation

$$\Gamma \xrightarrow{\rho} \text{GL}(V)$$

of a finite abelian group Γ , such that its image is small. Denote $d = \dim V$. Choose a decomposition $V = E_1 \oplus E_2 \oplus \dots \oplus E_d$ of ρ as a sum of irreducible (1-dimensional) representations. This is possible, since Γ is abelian (see [27]). Denote by \mathcal{E} this decomposition.

For any $g \in \Gamma$ and any $k \in \{1, \dots, d\}$, g acts on E_k by multiplication by a root of unity $e^{2i\pi w_k(g)}$. Here $w_k(g) \in \mathbf{Q}$ is well-defined modulo \mathbf{Z} . Define then

$$w_{\mathcal{E}}(g) := (w_1(g), \dots, w_d(g)) \in \mathbf{Q}^d.$$

This vector is well-defined modulo \mathbf{Z}^d . Define the following over-lattice of $W_0 = \mathbf{Z}^d$:

$$W_{\mathcal{E}}(\rho) := \mathbf{Z}^d + \sum_{g \in \Gamma} \mathbf{Z}w_{\mathcal{E}}(g). \tag{2}$$

As the vectors $w_{\mathcal{E}}(g)$ are well-defined modulo \mathbf{Z}^d , it is clear that $W_{\mathcal{E}}(\rho)$ does not depend on their choices. Moreover, as the decomposition of a representation of a finite group as direct sum of irreducible ones is unique up to the order of the summands (see [27]), the pair $(W_{\mathcal{E}}(\rho), \sigma_0)$ is independent up to isomorphism of the choice of decomposition \mathcal{E} . That is why we denote it shortly

$$(W(\rho), \sigma_0).$$

Our main theorem is:

Theorem 3.4. *The pairs (\mathcal{W}, σ) and $(W(\rho(\mathcal{Z})), \sigma_0)$ are isomorphic.*

Proof. As a \mathbf{C} -vector space, $\mathbf{C}[\tilde{\mathcal{M}} \cap \check{\sigma}]$ is generated by the monomials $X^{\tilde{m}}$, with $\tilde{m} \in \tilde{\mathcal{M}} \cap \check{\sigma}$. The canonical action of \mathcal{W} on these monomials is given by

$$(w, X^{\tilde{m}}) \rightarrow e^{2i\pi(w, \tilde{m})} X^{\tilde{m}} \tag{3}$$

Let $(\check{v}_1, \dots, \check{v}_d)$ be the basis of $\widetilde{\mathcal{M}}$ dual to the basis (v_1, \dots, v_d) of $\widetilde{\mathcal{W}}$. Then the images of $X^{\check{v}_1}, \dots, X^{\check{v}_d}$ constitute a basis of the \mathbf{C} -vector space $\widetilde{m}/\widetilde{m}^2$. For any $k \in \{1, \dots, d\}$, denote by F_k the subspace of $\widetilde{m}/\widetilde{m}^2$ generated by the image of $X^{\check{v}_k}$. Denote by \mathcal{F} the decomposition $\widetilde{m}/\widetilde{m}^2 = F_1 \oplus F_2 \oplus \dots \oplus F_d$. Then, by formula (3), for any $w \in \mathcal{W}$ and any $k \in \{1, \dots, d\}$, w acts on F_k by multiplication with $e^{2i\pi(w, \check{v}_k)}$. If $g(w)$ denotes the image of w in the group $\Gamma(\mathcal{Z}) \simeq \mathcal{W}/\widetilde{\mathcal{W}}$, this shows that

$$w_{\mathcal{F}}(g(w)) = ((w, \check{v}_1), \dots, (w, \check{v}_d)),$$

and so, by formula (2),

$$(W(\rho(\mathcal{Z})), \sigma_0) \simeq (\mathbf{Z}^d + \sum_{w \in \mathcal{W}} \mathbf{Z}((w, \check{v}_1), \dots, (w, \check{v}_d)), \sigma_0).$$

If we express the pair (\mathcal{W}, σ) using the basis (v_1, \dots, v_d) of the associated \mathbf{Q} -vector space, we get the isomorphism

$$\begin{aligned} (\mathcal{W}, \sigma) &= \left(\sum_{k=1}^d \mathbf{Z}v_k + \sum_{w \in \mathcal{W}} \mathbf{Z}((w, \check{v}_1)v_1 + \dots + (w, \check{v}_d)v_d), \sum_{k=1}^d \mathbf{R}_+v_k \right) \\ &\simeq \left(\mathbf{Z}^d + \sum_{w \in \mathcal{W}} \mathbf{Z}((w, \check{v}_1), \dots, (w, \check{v}_d)), \sigma_0 \right) \end{aligned}$$

which proves the Theorem. □

Remark. The constructions done in the previous proof show easily that the image of the group $\Gamma(\mathcal{Z}) \simeq \mathcal{W}/\widetilde{\mathcal{W}}$ by the representation $\rho(\mathcal{Z})$ is small. Suppose this is false and consider $w \in \mathcal{W}$ such that $g(w)$ acts on $\widetilde{m}/\widetilde{m}^2$ as a complex reflection. Consider again the basis of $\widetilde{m}/\widetilde{m}^2$ formed by the images of $X^{\check{v}_1}, \dots, X^{\check{v}_d}$. Possibly after reordering it, we can suppose that $(w, \check{v}_i) \in \mathbf{Z}, \forall i \in \{1, \dots, d-1\}$ and $(w, \check{v}_d) \notin \mathbf{Z}$. As $w = \sum_{i=1}^d (w, \check{v}_i)v_i$, this implies that $(w, \check{v}_d)v_d \in \mathcal{W}$. As $(w, \check{v}_d) \notin \mathbf{Z}$, this contradicts the fact that v_d is a primitive element of \mathcal{W} .

The following proposition shows that a representation ρ and the pair $(W(\rho), \sigma_0)$ it determines contain equivalent information.

Proposition 3.5. *Let $\Gamma \xrightarrow{\rho} GL(V)$ be faithful finite-dimensional \mathbf{C} -linear representation of a finite abelian group Γ whose image is small. If $(\mathcal{Z}, 0)$ denotes the Hirzebruch-Jung singularity defined by $(W(\rho), \sigma_0)$, then the representations ρ and $\rho(\mathcal{Z})$ are isomorphic.*

Proof. Choose \mathcal{E} , an arbitrary decomposition of ρ as a sum of irreducible representations. As $\rho(\Gamma)$ is small, we see that no d -tuple $w_{\mathcal{E}}(g)$, with $g \neq 1$, is contained on a line defined by an edge of σ_0 . This shows that the d -tuples modulo \mathbf{Z}^d can be recovered from $(W(\rho), \sigma_0)$, simply by expressing the elements of $W(\rho)$ in terms of the primitive

with $\tilde{\alpha}_{i,k} \in \mathbf{Z}, \forall i \in \{1, \dots, k\}$.

If e'_k is such that $(e_1, \dots, e_{k-1}, e'_k)$ is also a basis of $\mathcal{W} \cap (\sum_{i=1}^k \mathbf{Q}v_i)$, then

$$\tilde{e}_k = \epsilon e'_k + \lambda_{k-1}e_{k-1} + \dots + \lambda_1e_1,$$

where $\epsilon \in \{+1, -1\}$ and $\lambda_i \in \mathbf{Z}, \forall i \in \{1, \dots, k-1\}$. So,

$$v_k = -(\tilde{\alpha}_{1,k} - \lambda_1\tilde{\alpha}_{k,k})e_1 - \dots - (\tilde{\alpha}_{k-1,k} - \lambda_{k-1}\tilde{\alpha}_{k,k})e_{k-1} + \epsilon\tilde{\alpha}_{k,k}e'_k.$$

The number ϵ is uniquely determined by the condition $\epsilon\tilde{\alpha}_{k,k} > 0$. Then, $\forall i \in \{1, \dots, k-1\}$, the integer λ_i is clearly uniquely determined by the condition $0 \leq \tilde{\alpha}_{i,k} - \lambda_i\tilde{\alpha}_{k,k} < \epsilon\tilde{\alpha}_{k,k}$.

So, there exists a unique e_k such that $(e_1, \dots, e_{k-1}, e_k)$ verify the first k relations of (4). This proves the Proposition by induction. \square

We denote by $\mathcal{B}(\mathcal{W}, \sigma, \prec)$ the basis (e_1, \dots, e_d) of W and by $\mathbf{m}(\mathcal{W}, \sigma, \prec)$ the matrix

$$\begin{pmatrix} 1 & -\alpha_{1,2} & \dots & -\alpha_{1,d} \\ 0 & \alpha_{2,2} & \dots & -\alpha_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{d,d} \end{pmatrix}$$

Definition 3.8. Let $(\mathcal{Z}, 0)$ be a Hirzebruch-Jung singularity isomorphic with $(\mathcal{Z}(\mathcal{W}, \sigma), 0)$, where (\mathcal{W}, σ) is a maximal simplicial pair. If \prec is an ordering of the edges of σ , we say that $(\mathcal{Z}, 0)$ is of type $\mathbf{m}(\mathcal{W}, \sigma, \prec)$.

We see that there is a finite ambiguity in the definition of the type of $(\mathcal{Z}, 0)$. Indeed, there are $d!$ possible orderings, and so $d!$ possible matrices $\mathbf{m}(\mathcal{W}, \sigma, \prec)$.

Remark. It would be interesting to find a method to decide if two matrices correspond to the same pair (\mathcal{W}, σ) but to distinct choices of the ordering of the edges of σ . Such a method is known classically in dimension 2 (see Proposition 6.4).

4. A normalization algorithm for quasi-ordinary hypersurface singularities

We proved in [21] (see also [22]) an algorithm for computing the Hirzebruch-Jung type of the normalization of a quasi-ordinary singularity of hypersurface in \mathbf{C}^3 . In this section we generalize it to arbitrary dimensions (Proposition 4.5). In order to do it, we need to give a normal form for sublattices $\overline{\mathcal{W}}$ of finite index of a lattice \mathcal{W} , once a basis of \mathcal{W} is fixed (Proposition 4.1). As an important intermediate result, we give an algorithm of computation of this normal form when $\overline{\mathcal{W}}$ is defined by a congruence (Lemma 4.3).

By Corollary 2.3, the normalization of the germ $(\mathcal{S}, 0)$ defined by an irreducible quasi-ordinary polynomial is a Hirzebruch-Jung singularity of type $\mathbf{m}(W_G, \sigma_0, \prec_0)$.

of Proposition 3.7 gives an algorithm of computation of $n_a(\mathbf{m})$ starting from the knowledge of \mathbf{m} .

If $t \in \mathbf{Q}$, one can write in a unique way $t = \frac{p}{q}$ with $\gcd(p, q) = 1$ and $q > 0$. Define the *numerator* and the *denominator* of t by

$$\begin{aligned} \text{num}(t) &:= p, \\ \text{den}(t) &:= q. \end{aligned}$$

The following lemma relates the normal forms of the Propositions 3.7 and 4.1:

Lemma 4.2. *If (\mathcal{W}, σ) is a maximal simplicial pair, that \prec is an ordering of the edges of σ and that $\overline{\mathcal{W}}$ is a sublattice of finite index of \mathcal{W} , then the matrix $\mathbf{m}(\mathcal{W}, \sigma, \prec; \overline{\mathcal{W}})$ determines the matrix $\mathbf{m}(\overline{\mathcal{W}}, \sigma, \prec)$.*

Proof. As $\mathbf{m}(\mathcal{W}, \sigma, \prec; \overline{\mathcal{W}})$ is upper triangular, so is its inverse. But unlike the entries of $\mathbf{m}(\mathcal{W}, \sigma, \prec; \overline{\mathcal{W}})$, the entries $t_{i,j}$ of $\mathbf{m}(\mathcal{W}, \sigma, \prec; \overline{\mathcal{W}})^{-1}$ are not in general integers. As $w_j = \sum_{i=1}^j t_{i,j} \overline{w}_i, \forall j \in \{1, \dots, d\}$, one sees that $(d_j w_j)_{1 \leq j \leq d}$ are the primitive elements of $\overline{\mathcal{W}}$ situated on the edges of σ , their ordering \prec being the same as before. We have denoted

$$d_j := \text{lcm}(\text{den}(t_{1,j}), \dots, \text{den}(t_{j,j})), \forall j \in \{1, \dots, d\}.$$

This shows that

$$\mathbf{m}(\overline{\mathcal{W}}, \sigma, \prec) = n_a((d_j t_{i,j})_{i,j}). \tag{6}$$

□

Let \mathcal{M}, \mathcal{W} be two dual rank d lattices endowed with dual basis (u_1, \dots, u_d) , respectively (w_1, \dots, w_d) . Denote by σ the cone spanned by (w_1, \dots, w_d) . Consider $a \in \mathcal{M}_{\mathbf{Q}}$ and let $\mathcal{W}(a)$ be the sublattice of \mathcal{W} dual to $\mathcal{M}(a) := \mathcal{M} + \mathbf{Z}a$, i.e. $\mathcal{W}(a) := \text{Hom}(\mathcal{M} + \mathbf{Z}a, \mathbf{Z})$.

If we write $a = \sum_{i=1}^d a^i u_i$, with $a^1, \dots, a^d \in \mathbf{Q}$, then

$$\mathcal{W}(a) = \{ w \in \mathcal{W} \mid (w, a) \in \mathbf{Z} \} = \left\{ \sum_{i=1}^d c_i w_i \mid \sum_{i=1}^d c_i a^i \in \mathbf{Z} \right\}. \tag{7}$$

Remark. The relation $\sum_{i=1}^d c_i a^i \in \mathbf{Z}$ can also be written $\sum_{i=1}^d (l_d a^i) c_i \equiv 0 \pmod{l_d}$, where $l_d := \text{lcm}(\text{den}(a^1), \dots, \text{den}(a^d))$. So, $\mathcal{W}(a)$ can be seen as a sublattice of finite index of \mathcal{W} defined by a congruence.

In the sequel we will also denote

$$\mathbf{m}(a^1, \dots, a^d) := \mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}(a)).$$

The following lemma gives an algorithm which computes the matrix $\mathbf{m}(a^1, \dots, a^d)$ starting from the values of a^1, \dots, a^d .

Lemma 4.3. Consider the matrix $\mathbf{m}(a^1, \dots, a^d) = (r_{i,j})_{i,j}$ and introduce the numbers $l_k := \text{lcm}(\text{den}(a^1), \dots, \text{den}(a^k)), \forall k \in \{1, \dots, d\}$. Then

$$r_{k,k} = \frac{l_k}{l_{k-1}}, \forall k \in \{1, \dots, d\}.$$

Moreover, for any $k \in \{1, \dots, d\}$ and any $j \in \{1, \dots, k-1\}$, one has the equivalent relations:

$$\sum_{i=j}^k l_j a^i r_{i,k} \equiv 0 \pmod{r_{j,j}}$$

$$r_{j,k} = \begin{cases} -(\sum_{i=j+1}^k l_j a^i r_{i,k})(l_j a^j)^{-1} \text{ in } \mathbf{Z}/r_{j,j}\mathbf{Z}, & \text{if } r_{j,j} \neq 1 \\ 0, & \text{if } r_{j,j} = 1 \end{cases}$$

Proof. Denote by $t_k(a) := \sum_{j=1}^k a^j u_j$ the k -truncation of a , for all $k \in \{1, \dots, d\}$.

One knows (see the proof of Lemma 4.1) that $(\bar{w}_1, \dots, \bar{w}_k)$ is a basis of $\mathcal{W}(a) \cap (\sum_{j=1}^k \mathbf{Z}w_j) = \mathcal{W}(t_k(a))$. This shows that $\prod_{j=1}^k r_{j,j} = (\mathcal{W} : \mathcal{W}(t_k(a)))$. But, as the pairs of lattices \mathcal{M}, \mathcal{W} and $\mathcal{M}(t_k(a)), \mathcal{W}(t_k(a))$ are in duality, one has the equality of indices: $(\mathcal{W} : \mathcal{W}(t_k(a))) = (\mathcal{M}(t_k(a)) : \mathcal{M})$. This last index is equal to the order of $t_k(a)$ in $\mathcal{M}(t_k(a))/\mathcal{M}$, which is obviously equal to l_k . This implies

$$\prod_{j=1}^k r_{j,j} = l_k$$

which proves the first equalities.

Let us fix now $k \in \{2, \dots, d\}$ and $j \in \{1, \dots, k\}$. The relation (7) implies $\sum_{i=1}^k a^i r_{i,k} \in \mathbf{Z}$. Multiplying this relation by l_j , we get

$$\left(\sum_{i=1}^{j-1} l_j a^i r_{i,k}\right) + \left(\sum_{i=j}^k l_j a^i r_{i,k}\right) \in l_j \mathbf{Z} \subset r_{j,j} \mathbf{Z}. \tag{8}$$

But $\forall i \in \{1, \dots, j-1\}, l_j a^i = r_{j,j}(l_{j-1} a^i) \in r_{j,j} \mathbf{Z}$, as $l_{j-1} a^i \in \mathbf{Z}$ by the definition of l_{j-1} . This shows that $\sum_{i=1}^{j-1} l_j a^i r_{i,k} \in r_{j,j} \mathbf{Z}$, and (8) implies

$$\sum_{i=j}^k l_j a^i r_{i,k} \in r_{j,j} \mathbf{Z}. \tag{9}$$

This is one of the forms in which were written the second relations of the Lemma.

Formula (9) can also be written

$$l_j a^j r_{j,k} + \sum_{i=j+1}^k l_j a^i r_{i,k} \in r_{j,j} \mathbf{Z}. \tag{10}$$

As

$$\sum_{i=j+1}^k l_j a^i r_{i,k} = \frac{\sum_{i=j+1}^k l_{j+1} a^i r_{i,k}}{r_{j+1,j+1}},$$

relation (9) at the order $j + 1$ implies that $\sum_{i=j+1}^k l_j a^i r_{i,k} \in \mathbf{Z}$. Moreover, $\gcd(l_j a^j, r_{j,j}) = 1$. Indeed, if p is a prime number dividing $r_{j,j} = \frac{l_j}{l_{j-1}}$, then $p \mid \text{den}(a^j)$ and $p \nmid \frac{l_j}{\text{den}(a^j)}$. As $\gcd(\text{den}(a^j), \text{num}(a^j)) = 1$, we also have $p \nmid \text{num}(a^j)$, and so $p \nmid (l_j a^j)$. This shows that $l_j a^j$ is invertible in the ring $\mathbf{Z}/r_{j,j}\mathbf{Z}$ if $r_{j,j} \neq 1$, and from relation (10) we get the last formulae of the Lemma. If $r_{j,j} = 1$, as $0 \leq r_{j,k} < r_{j,j}$ we get $r_{j,k} = 0$. \square

Remark. The previous Lemma shows that once $r_{1,1}, \dots, r_{d,d}$ are computed, one has to compute the entries of the k -th column in the order: $r_{k-1,k}, r_{k-2,k}, \dots, r_{1,k}$. As $0 \leq r_{i,j} < r_{i,i}, \forall 1 \leq i < j \leq d$, the entries $r_{i,j}$ of the matrix $\mathbf{m}(a^1, \dots, a^d)$ are completely determined by the congruences of the Lemma.

Suppose now that $g \geq 1$ and a_1, \dots, a_g is a sequence of vectors of $\mathcal{M}_{\mathbf{Q}}$. Define for all $k \in \{1, \dots, g\}$:

$$\begin{aligned} \mathcal{M}_k &:= \mathcal{M} + \mathbf{Z}a_1 + \dots + \mathbf{Z}a_k \\ \mathcal{W}_k &:= \text{Hom}(\mathcal{M}_k, \mathbf{Z}). \end{aligned}$$

We denote $(r_{i,j}^k)_{i,j} = \mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}_k)$. Write $a_k = a_k^1 u_1 + \dots + a_k^d u_d$, with $a_k^1, \dots, a_k^d \in \mathbf{Q}$. Introduce also the basis $\mathfrak{B}_k = (w_1^k, \dots, w_d^k) := \mathfrak{B}(\mathcal{W}, \sigma, \prec; \mathcal{W}_k)$. Denote by \prec_k its ordering, deduced canonically from \prec , and by σ_k the cone generated by \mathfrak{B}_k . Then

$$\begin{aligned} \mathcal{W}_k &= \{ w \in \mathcal{W}_{k-1} \mid (w, a_k) \in \mathbf{Z} \} = \\ &= \left\{ w = \sum_{i=1}^d c_i w_i^{k-1} \mid \left(\sum_{i=1}^d \sum_{j=1}^d c_i r_{j,i}^{k-1} w_j, \sum_{j=1}^d a_k^j w_j \right) \in \mathbf{Z} \right\} = \\ &= \left\{ w = \sum_{i=1}^d c_i w_i^{k-1} \mid \sum_{i=1}^d c_i \left(\sum_{j=1}^d a_k^j r_{j,i}^{k-1} \right) \in \mathbf{Z} \right\}. \end{aligned}$$

We get:

Lemma 4.4. *One has the equality of matrices*

$$\mathbf{m}(\mathcal{W}_{k-1}, \sigma_{k-1}, \prec_{k-1}; \mathcal{W}_k) = \mathbf{m}\left(\sum_{j=1}^d a_k^j r_{j,1}^{k-1}, \dots, \sum_{j=1}^d a_k^j r_{j,d}^{k-1}\right).$$

The Lemmas 4.3 and 4.4 allow to compute recursively the matrices $\mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}_k)$ for $k \in \{1, \dots, g\}$ from the knowledge of the components of a_1, \dots, a_k in the basis (u_1, \dots, u_d) . Indeed:

$$\mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}_k) = n_s(\mathbf{m}(\mathcal{W}_{k-1}, \sigma_{k-1}, \prec_{k-1}; \mathcal{W}_k)\mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}_{k-1})). \tag{11}$$

Once the matrix $\mathbf{m}(\mathcal{W}, \sigma, \prec; \mathcal{W}_k)$ is known, Lemma 4.2 shows that $\mathbf{m}(\mathcal{W}_k, \sigma, \prec)$ is also known.

In the special case in which $(\mathcal{S}, 0)$ is an irreducible quasi-ordinary singularity of hypersurface having (A_1, \dots, A_G) as characteristic exponents with respect to some projection, we put $g = G$, $\mathcal{W} = W_0$, $\mathcal{M} = M_0$, and $a_k = A_k$, $\forall k \in \{1, \dots, g\}$. By combining Corollary 2.3 and Definition 3.8, we see that the normalization of $(\mathcal{S}, 0)$ is a Hirzebruch-Jung singularity of type $\mathbf{m}(W_G, \sigma_0, \prec_0)$, which can be computed by the previous method. Using the Lemmas 4.2 (more precisely the relation (6)), 4.3, 4.4 and relation (11), we get the following compact form of the algorithm:

Proposition 4.5. *Let $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ be an irreducible quasi-ordinary polynomial with characteristic exponents A_1, \dots, A_G . We look at A_k as a matrix $1 \times d$. If $R^k := \mathbf{m}(W_0, \sigma_0, \prec_0; W_k)$, $S^k := \mathbf{m}(W_{k-1}, \sigma_{k-1}, \prec_{k-1}; W_k)$, $T^k = (t_{i,j}^k)_{i,j} := (R^k)^{-1}$, $d_j^k := \text{lcm}(\text{den}(t_{1,j}^k), \dots, \text{den}(t_{j,j}^k))$, $\forall k \in \{1, \dots, G\}$, $\forall j \in \{1, \dots, d\}$, and $R^0 := I_d$, then:*

$$\begin{aligned} S^k &= \mathbf{m}(A_k R^{k-1}), \\ R^k &= n_s(S^k R^{k-1}) \\ N_k &= \det(S^k). \end{aligned}$$

The normalization of the germ defined by $f = 0$ is a Hirzebruch-Jung singularity of type

$$\mathbf{m}(W_G, \sigma_0, \prec_0) = n_a((d_j^G t_{i,j}^G)_{i,j}).$$

We recall that the numbers N_k were defined after having stated Theorem 2.1.

If $(\mathcal{S}, 0)$ is an irreducible quasi-ordinary singularity of dimension $d \geq 1$ and embedding dimension $d + 1$, Lipman [18] showed that there is always a quasi-ordinary polynomial $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ defining \mathcal{S} such that its characteristic exponents A_1, \dots, A_G verify

$$\begin{cases} (A_1^1, \dots, A_G^1) \geq_{\text{lex}} \dots \geq_{\text{lex}} (A_1^d, \dots, A_G^d) \\ A_1^2 \neq 0 \text{ or } A_1^1 > 1 \end{cases} \tag{12}$$

Lipman [19] and Gau [11] showed that a sequence A_1, \dots, A_G which verifies (12) —they called it then *normalized*— is an embedded topological invariant of $(\mathcal{S}, 0)$. In particular, it is an analytical invariant of $(\mathcal{S}, 0)$. In [23] we gave an algebraic proof of this analytical invariance. This shows that for an irreducible quasi-ordinary germ of hypersurface, there is a way to choose a well-defined matrix for the type of its normalization between the $d!$ possibilities. Indeed, one simply starts the application of the previous algorithm from normalized characteristic exponents.

5. A tridimensional example

Consider the following sequence of characteristic exponents:

$$A_1 = \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{6}\right), \quad A_2 = \left(\frac{3}{8}, \frac{5}{12}, \frac{7}{12}\right).$$

As the relations (12) are verified, it is a normalized sequence (see the definition in the last paragraph of the previous section).

Let us apply the algorithm summarized in Proposition 4.5:

$$R^1 = S^1 = \begin{pmatrix} 4 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$N_1 = \det(S^1) = 12,$$

$$T^1 = (R^1)^{-1} = \frac{1}{2^2 \cdot 3} \begin{pmatrix} 3 & -2 & -2 \\ 0 & 4 & -8 \\ 0 & 0 & 12 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix},$$

$$d_1^1 = 4, \quad d_2^1 = 6, \quad d_3^1 = 6,$$

$$(d_j^1 t_{i,j}^1)_{i,j} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 6 \end{pmatrix},$$

$$\mathfrak{m}(W_1, \sigma_0, \prec_0) = n_a((d_j^1 t_{i,j}^1)_{i,j}) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 6 \end{pmatrix},$$

$$A_2 R^1 = \left(\frac{3}{2}, 2, \frac{13}{6}\right),$$

$$S^2 = \mathfrak{m}\left(\frac{3}{2}, 2, \frac{13}{6}\right) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$N_2 = \det(S^2) = 6,$$

$$S^2 R^1 = \begin{pmatrix} 8 & 4 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix},$$

$$R^2 = n_s(S^2 R^1) = \begin{pmatrix} 8 & 4 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix},$$

$$T^2 = (R^2)^{-1} = \frac{1}{2^3 \cdot 3^2} \begin{pmatrix} 9 & -12 & -7 \\ 0 & 24 & -16 \\ 0 & 0 & 24 \end{pmatrix} = \begin{pmatrix} \frac{1}{2^3} & -\frac{1}{2 \cdot 3} & -\frac{7}{2^3 \cdot 3^2} \\ 0 & \frac{1}{3} & -\frac{2}{3^2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

$$d_1^2 = 2^3, \quad d_2^2 = 6, \quad d_3^2 = 2^3 \cdot 3^2,$$

$$(d_j^2 t_{i,j}^2)_{i,j} = \begin{pmatrix} 1 & -1 & -7 \\ 0 & 2 & -16 \\ 0 & 0 & 24 \end{pmatrix},$$

$$\mathfrak{m}(W_1, \sigma_0, \prec_0) = n_a((d_j^2 t_{i,j}^2)_{i,j}) = \begin{pmatrix} 1 & -1 & -7 \\ 0 & 2 & -16 \\ 0 & 0 & 24 \end{pmatrix}.$$

This shows that the normalization of a quasi-ordinary hypersurface singularity with one characteristic exponent $(\frac{1}{4}, \frac{1}{6}, \frac{1}{6})$ has a Hirzebruch-Jung singularity of type

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 6 \end{pmatrix}$$

and the normalization of a quasi-ordinary hypersurface singularity with two characteristic exponents $(\frac{1}{4}, \frac{1}{6}, \frac{1}{6}), (\frac{3}{8}, \frac{5}{12}, \frac{7}{12})$ has a Hirzebruch-Jung singularity of type

$$\begin{pmatrix} 1 & -1 & -7 \\ 0 & 2 & -16 \\ 0 & 0 & 24 \end{pmatrix}.$$

By the comments made at the end of the previous section, we have obtained like this well-defined normal forms for the types of the normalizations of the considered quasi-ordinary singularities.

6. The classical (2-dimensional) Hirzebruch-Jung singularities

In this section we restrict to the case of surfaces and we compare our definition of the type of a Hirzebruch-Jung singularity with the one given in Barth, Peters, Van de Ven [3]. For details on Hirzebruch's work [15], one should consult Brieskorn [5].

Let p_1, \dots, p_r be a sequence of integers, such that $p_i \leq -2$, $\forall i \in \{1, \dots, r\}$ and $r \geq 1$. Let X be a smooth complex analytical surface containing a reduced divisor with normal crossings C whose components C_1, \dots, C_r are projective lines with self-intersections $C_i^2 = p_i$, $\forall i \in \{1, \dots, r\}$, and such that

$$C_i \cdot C_j = \begin{cases} 1, & \text{if } |i - j| = 1 \\ 0, & \text{either.} \end{cases}$$

Such a couple (X, C) always exists. The curve C is called a *Hirzebruch-Jung string* with self-intersection numbers p_i .

Define also the coprime numbers $(n, q) \in \mathbf{N}^2$, $0 < q < n$ by the formula

$$\frac{n}{q} = |p_1| - \frac{1}{|p_2| - \frac{1}{\dots - \frac{1}{|p_r|}}} \tag{13}$$

Then one has the following theorem ((5.1) in [3]):

Theorem 6.1 (Hirzebruch). *If $C \subset X$ is a Hirzebruch-Jung string with self-intersection numbers p_i satisfying the relation (13), then the germ obtained by contracting C to a point is analytically isomorphic to the normalization of the germ at the origin of the surface with equation $Y^n = X_1 X_2^{n-q}$.*

This motivates the following definition given in [3]:

Definition 6.2. A normal germ of surface is said to be a *Hirzebruch-Jung singularity of type $\mathcal{A}_{n,q}$* if it is analytically isomorphic to the normalization at the origin of the surface with equation $Y^n = X_1 X_2^{n-q}$.

Remark. In [2], Aroca and Snoussi showed more generally that any normal quasi-ordinary singularity (i.e. Hirzebruch-Jung singularity in our terms) is the normalization of a complete intersection germ defined by binomial equations.

Let us see the relation between the classical normal form of Definition 6.2 and the one we introduced in the Definition 3.8.

Proposition 6.3. *The Hirzebruch-Jung singularity of type $\mathcal{A}_{n,q}$ following Definition 6.2 is of type $(\frac{1}{0} \ -q \ / \ n)$ following Definition 3.8.*

Proof. The polynomial $Y^n - X_1 X_2^{n-q}$ is quasi-ordinary with only one characteristic exponent $A_1 = (\frac{1}{n}, 1 - \frac{q}{n})$. Applying Lemma 4.3, with $\mathfrak{m}(W_0, \sigma_0, \prec_0; W_1) = \begin{pmatrix} r_{1,1} & r_{1,2} \\ 0 & r_{2,2} \end{pmatrix}$ and $\mathfrak{B}(W_0, \sigma_0, \prec_0; W_1) = (w_1^1, w_2^1)$, we get $l_1 = \text{den}(\frac{1}{n}) = n$, $l_2 = \text{lcm}(\text{den}(\frac{1}{n}), \text{den}(1 - \frac{q}{n})) = n$, $r_{1,1} = l_1 = n$, $r_{2,2} = \frac{l_2}{l_1} = 1$, $r_{1,2} = -(l_1 A_1^2 r_{2,2})(l_1 A_1^1)^{-1}$ in $\mathbf{Z}/r_{1,1}\mathbf{Z}$, which implies $r_{1,2} = q$. So,

$$\begin{cases} w_1^1 = n w_1, \\ w_2^1 = q w_1 + w_2, \end{cases}$$

which implies

$$\begin{cases} w_1 = \frac{1}{n} w_1^1, \\ w_2 = w_2^1 - \frac{q}{n} w_1^1. \end{cases}$$

Then, following the notations of Proposition 3.7, if v_1, v_2 are the minimal generators of W_1 situated on the edges $\mathbf{R}_+ w_1, \mathbf{R}_+ w_2$ of σ_0 , we deduce

$$\begin{cases} v_1 = w_1^1, \\ v_2 = n w_2^1 - q w_1^1. \end{cases} \tag{14}$$

This shows that

$$m(W_1, \sigma_0, \prec_0) = \begin{pmatrix} 1 & -q \\ 0 & n \end{pmatrix}.$$

The Proposition is proved. □

In dimension 2, there are only two possible choices of the ordering of the edges of σ , and so only two matrices for the type of a Hirzebruch-Jung singularity (see the comments following Definition 3.8). The following Proposition which relates them was probably first proved by Hirzebruch [15]:

Proposition 6.4 (Hirzebruch). *If two Hirzebruch-Jung singularities of types $\mathcal{A}_{n,q}$ and $\mathcal{A}_{n',q'}$ are isomorphic, then $n = n'$ and $(q = q'$ or $qq' \equiv 1 \pmod{n})$.*

Proof. Hirzebruch proved this result by looking at the minimal desingularizations of the singularities. Both have as exceptional divisors Hirzebruch-Jung strings with the same sequences of self-intersection numbers, but possibly reversed. Then one makes computations using formula (13).

Here we give another proof, which uses the orbifold map instead of the minimal desingularization one. As showed by Theorem 3.4, the couple (\mathcal{W}, σ) is well-defined up to isomorphism by the analytical structure of the singularity. If one chooses the reverse order of \prec_0 in the previous computations, one gets

$$\begin{cases} v_2 = e_1^1, \\ v_1 = n'e_2^1 - q'e_1^1, \end{cases}$$

where (e_1^1, e_2^1) is a basis of W_1 . Combining these relations with (14), we get first $n = n'$, as both measure the index $(W_1 : \mathbf{Z}v_1 + \mathbf{Z}v_2)$. Then $w_1^1 = v_1 = ne_2^1 - q'e_1^1 = ne_2^1 - q'v_2 = ne_2^1 - q'(nw_2^1 - qw_1^1) \Rightarrow (1 - qq')w_1^1 = n(e_2^1 - q'w_2^1)$. As w_1^1 is a primitive element of W_1 , this implies that $1 - qq' \equiv 0 \pmod{n}$, which proves the Proposition.

Another method would have been to apply the algorithm of normalization as in the proof of Proposition 6.3, but starting with $A_1 = (1 - \frac{q}{n}, \frac{1}{n})$. □

The computations we have done in order to prove Proposition 6.3 are a particular case of the normalization algorithm 4.5 presented in the previous section. By using Lemma 4.3, we can give in a more explicit form this algorithm, as we published it (but with slightly different notations) in [21, 22]:

Proposition 6.5. *Let $f \in \mathbf{C}\{X_1, X_2\}[Y]$ be an irreducible quasi-ordinary polynomial with characteristic exponents A_1, \dots, A_G . If*

$$m(W_0, \sigma_0, \prec_0; W_k) = \begin{pmatrix} r_{1,1}^k & r_{1,2}^k \\ 0 & r_{2,2}^k \end{pmatrix},$$

$$m(W_{k-1}, \sigma_{k-1}, \prec_{k-1}; W_k) = \begin{pmatrix} s_{1,1}^k & s_{1,2}^k \\ 0 & s_{2,2}^k \end{pmatrix}, \quad \forall k \in \{1, \dots, G\},$$

and

$$\begin{pmatrix} r_{1,1}^0 & r_{1,2}^0 \\ 0 & r_{2,2}^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\begin{aligned} s_{1,1}^k &= \text{den}(A_k^1 r_{1,1}^k), \\ s_{2,2}^k &= \frac{\text{lcm}(\text{den}(A_k^1 r_{1,1}^k), \text{den}(A_k^1 r_{1,2}^{k-1} + A_k^2 r_{2,2}^{k-1}))}{\text{den}(A_k^1 r_{1,1}^k)}, \\ s_{1,2}^k &= \begin{cases} \begin{aligned} & - \text{den}(A_k^1 r_{1,2}^{k-1} + A_k^2 r_{2,2}^{k-1}) \\ & \cdot \text{lcm}(\text{den}(A_k^1 r_{1,1}^k), \text{den}(A_k^1 r_{1,2}^{k-1} + A_k^2 r_{2,2}^{k-1})) \\ & \cdot \text{num}(A_k^1 r_{1,1}^{k-1})^{-1} \text{ in } \mathbf{Z} / \text{den}(A_k^1 r_{1,1}^{k-1}) \mathbf{Z}, \end{aligned} & \text{if } \text{den}(A_k^1 r_{1,1}^{k-1}) \neq 1, \\ 0, & \text{if } \text{den}(A_k^1 r_{1,1}^{k-1}) = 1, \end{cases} \\ r_{1,1}^k &= s_{1,1}^k r_{1,1}^{k-1}, \\ r_{2,2}^k &= s_{2,2}^k r_{2,2}^{k-1}, \\ r_{1,2}^k &= s_{1,1}^k r_{1,2}^{k-1} + s_{2,2}^k r_{2,2}^{k-1} \text{ in } \mathbf{Z} / r_{1,1}^k \mathbf{Z}. \end{aligned}$$

The normalization of the germ defined by $f = 0$ is a Hirzebruch-Jung singularity of type

$$\begin{pmatrix} 1 & -\frac{r_{1,2}^G}{\text{gcd}(r_{1,2}^G, r_{1,2}^G)} \\ 0 & \frac{r_{1,1}^G}{\text{gcd}(r_{1,2}^G, r_{1,2}^G)} \end{pmatrix}.$$

7. Questions

If $(\mathcal{S}, 0)$ is a reduced germ of complex analytical space, we denote by $K(\mathcal{S})$ its *abstract boundary* (also called the *link* of $(\mathcal{S}, 0)$). It is defined as the intersection of a representative of $(\mathcal{S}, 0)$ with a sufficiently small Euclidean sphere centered at 0 in an arbitrary system of local coordinates at 0. It is independent of these choices (Durfee’s proof in [9] for algebraic varieties extends to analytical ones).

Hirzebruch [15] noticed that the abstract boundary of a bidimensional Hirzebruch-Jung singularity $(\mathcal{Z}, 0)$ of type $\mathcal{A}_{n,q}$ is a lens space $L(n, q)$. As it was known since Reidemeister [26] that $L(n, q)$ is homeomorphic (preserving the orientations) to $L(n', q')$ if and only if $n = n'$ and $(q = q'$ or $qq' \equiv 1 \pmod{n})$, this showed by Proposition 6.4 that in this case the homeomorphism type of $K(\mathcal{Z})$ determines the analytical type of $(\mathcal{Z}, 0)$. More generally, we ask:

Question 7.1. *Let $(\mathcal{Z}, 0)$ be a Hirzebruch-Jung singularity of dimension ≥ 3 . Denote by $K(\mathcal{Z})$ its abstract boundary. Is it true that the homeomorphism type of $K(\mathcal{Z})$ determines the analytical type of $(\mathcal{Z}, 0)$?*

If the answer to the previous question is negative, it would be interesting to know what supplementary structure one should add to the boundary $K(\mathcal{Z})$ in order to make it affirmative (e.g. should one consider it rather as an orbifold, or add some stratified smooth structure?)

In the case when the canonical representation $\rho(\mathcal{Z})$ associated to the singularity (see (1)) is a cyclic fixed-point free action outside the origin, the answer to the question is affirmative. Indeed, in this case the boundary is a *generalized lens space* and the corresponding result was obtained by Franz (see Dieudonné [7, page 246]). If $d \geq 3$, the action $\rho(\mathcal{Z})$ may be non-cyclic, and even if it is cyclic, it may have fixed points. One can decide if $\Gamma(\mathcal{Z})$ is cyclic by computing the invariant factors of a matrix of presentation of $\widehat{\mathcal{W}}$ with respect to \mathcal{W} , for example $\mathfrak{m}(\mathcal{W}, \sigma, \prec)$ for an arbitrary ordering \prec .

Consider now more general pairs (\mathcal{W}, σ) than the simplicial ones. By analogy with Question 7.1, we ask also:

Question 7.2. *Let \mathcal{Z} be an affine (not necessarily simplicial) toric variety. Is it true that the homeomorphism type (possibly enriched with supplementary structure) of $K(\mathcal{Z})$ determines the analytical type of $(\mathcal{Z}, 0)$?*

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