Sharp Edge-Homotopy on Spatial Graphs

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ABSTRACT

A sharp-move is known as an unknotting operation for knots. A self sharp-move is a sharp-move on a spatial graph where all strings in the move belong to the same spatial edge. We say that two spatial embeddings of a graph are *sharp edge-homotopic* if they are transformed into each other by self sharp-moves and ambient isotopies. We investigate how is the sharp edge-homotopy strong and classify all spatial theta curves completely up to sharp edge-homotopy. Moreover we mention a relationship between sharp edge-homotopy and delta edge (resp. vertex)-homotopy on spatial graphs.

Key words: spatial graph, sharp move, delta move.

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1. Introduction

Let G be a finite graph which is considered as a topological space in the usual way. An embedding $f : G \to S^3$ is called a *spatial embedding of* G or simply a *spatial graph*. We call a subgraph of G a *cycle* if it is homeomorphic to S^1 and denote the set of all cycles of G by $\Gamma(G)$. A graph G is said to be *planar* if there exists an embedding of G into S^2, and a spatial embedding of a planar graph G is said to be *trivial* if it is ambient isotopic to an embedding of G into $S^2 \subset S^3$.

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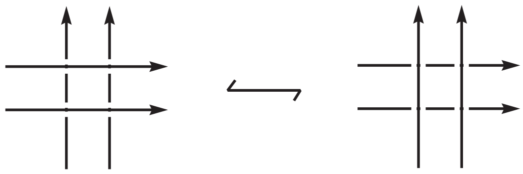


Figure 1

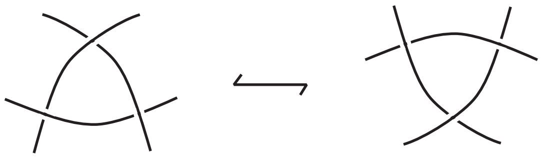
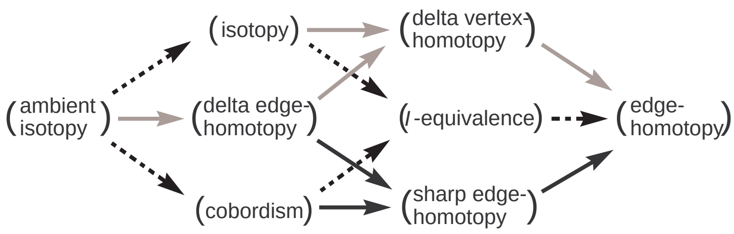


Figure 2

A graph is said to be *oriented* if an orientation is given for each edge. A *sharp-move* is a local move on a spatial oriented graph as illustrated in Fig. 1. It is known that the sharp-move is an unknotting operation [10]. We say that a sharp-move is a *self sharp-move* if all four strings in the move belong to the same spatial edge. We say that two spatial embeddings of a graph G are *sharp edge-homotopic* if they are transformed into each other by self sharp-moves and ambient isotopies. It is easy to see that this definition does not depend on the edge orientations. If G is homeomorphic to the disjoint union of 1-spheres, then it coincides with *self sharp-equivalence* on oriented links [14, 22, 23, 25].

First we investigate how strong is sharp edge-homotopy. In [27], equivalence relations *cobordism*, *isotopy*, *I-equivalence* and *edge-homotopy* on spatial graphs were introduced. A *delta move* is a local move on a spatial graph as illustrated in Fig. 2. It is known that the delta move is also an unknotting operation [7, 12]. We say that a delta move is a *self delta move* if all three strings in the move belong to the same spatial edge and a *quasi adjacent-delta move* if it is on exactly two adjacent spatial edges. Two spatial embeddings of a graph are said to be *delta edge (resp. vertex)-homotopic* if they are transformed into each other by self delta moves (resp. quasi adjacent-delta moves) and ambient isotopies [15]. This is a generalization of *delta link-homotopy* (or *self delta-equivalence*) on oriented links [13, 24].

Theorem 1.1. *The following implications hold:*



Moreover the converse of each implication does not hold and if there does not exist any arrow between two equivalence relations then there does not exist any implication between them.

We note that the dotted parts have already proved in [27] and the gray parts have already proved by the author [15]. We prove Theorem 1.1 in the next section.

In section 3, we show that there exists a theta curve which is not trivial up to sharp edge-homotopy, where a theta curve is a spatial embedding of the graph θ as illustrated in Fig. 3.

Moreover, we classify all theta curves completely up to sharp edge-homotopy. For a theta curve f, let $\tilde{\alpha}(f)$ be the modulo two reduction of $\sum_{\gamma\in\Gamma(\theta)} a_2(f(\gamma))$, where $a_i(L)$ denotes the i-th coefficient of the *Conway polynomial* $\nabla_L(z)$ of a link L. We note that the modulo two reduction of $a_2(K)$ coincides with the Arf *invariant* [21] of a knot K. From the beginning this was introduced as a delta edge-homotopy invariant of theta curves [15]. We give the following classification in section 3.

Theorem 1.2. *Two theta curves f and g are sharp edge-homotopic if and only if $\tilde{\alpha}(f) = \tilde{\alpha}(g)$.*

A theta curve is said to be *almost unknotted* if $f(\gamma)$ is a trivial knot for any cycle $\gamma \in \Gamma(\theta)$. As a corollary of Theorem 1.2, we have the following:

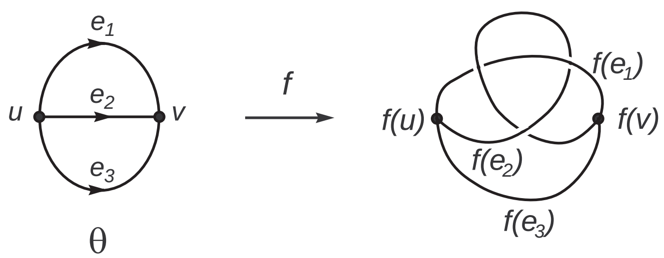


Figure 3

Corollary 1.3. *Any almost unknotted theta curve is trivial up to sharp edge-homotopy.*

It is known that there exist infinitely many oriented links up to delta edge-homotopy which are mutually sharp edge-homotopic [14]. On the other hand, the author showed that there exist infinitely many almost unknotted theta curves up to delta edge-homotopy [17]. Thus we have the following by Corollary 1.3.

Corollary 1.4. *There exist infinitely many theta curves up to delta edge-homotopy which are mutually sharp edge-homotopic.*

This implies that in general there exist infinitely many spatial embeddings of a graph G up to delta edge-homotopy which are mutually sharp edge-homotopic even in the case that G does not contain any disjoint cycles. In section 4, we give such an example in case G is the *complete graph* on four vertices K_4 as follows.

Theorem 1.5. *There exist spatial embeddings f_m of K_4 for $m \in \mathbf{N} \cup \{0\}$ such that*

(i) *f_i and f_j are not delta edge-homotopic for $i \neq j$.*

(ii) *For any subgraph H of K_4 which is homeomorphic to θ, $f_i|_H$ and $f_j|_H$ are delta edge-homotopic for $i \neq j$.*

(iii) *f_i and f_j are sharp edge-homotopic for $i \neq j$.*

Besides we show that there exist infinitely many spatial embeddings of the complete graph on five vertices K_5 up to delta vertex-homotopy which are mutually sharp edge-homotopic.

Theorem 1.6. *There exist spatial embeddings f_m of K_5 for $m \in \mathbf{N} \cup \{0\}$ such that*

(i) *f_i and f_j are not delta vertex-homotopic for $i \neq j$.*

(ii) *For any subgraph H of K_5 which is homeomorphic to K_4, $f_i|_H$ and $f_j|_H$ are delta vertex-homotopic for $i \neq j$.*

(iii) *f_i and f_j are sharp edge-homotopic for $i \neq j$.*

In particular, by Theorem 1.1 we have the following:

Corollary 1.7. *There exist infinitely many spatial embeddings of K_5 up to isotopy which are mutually sharp edge-homotopic.*

Remark 1.8. Classification of spatial graphs up to sharp-moves (resp. pass-moves) and ambient isotopies was studied by Y. Ohyama [19], but all strings in the move does not have to belong to the same spatial edge. The author and R. Shinjo have already classified all *boundary spatial embeddings* of a graph completely up to self sharp-moves (resp. self pass-moves) and ambient isotopies [18].

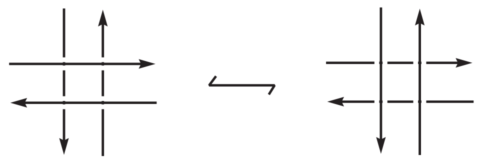


Figure 4

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We give the definitions of cobordism, isotopy and I-equivalence on spatial graphs roughly. We refer the reader to [27] for precise definitions. Two spatial embeddings f and g of a graph G are said to be *I-equivalent* if there exists an embedding $\Phi : G \times [0,1] \to S^3 \times [0,1]$ between f and g. On that occasion, if Φ is *locally flat* then f and g are said to be *cobordant*. On the other hand if Φ is *level preserving* then f and g are said to be *isotopic*. We remark here that f and g are ambient isotopic if Φ is locally flat and level preserving.

We give the definition of edge-homotopy on spatial graphs precisely. We say that a crossing change is a *self crossing change* if all two strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *edge-homotopic* [27] if they are transformed into each other by self crossing changes and ambient isotopies. (In [27], edge-homotopy is called simply a *homotopy*.) This is a generalization of *link-homotopy* on oriented links [8].

Lemma 2.1. *Let f and g be spatial embeddings of a graph.*

(i) *If f and g are ambient isotopic then they are sharp edge-homotopic.*

(ii) *If f and g are delta edge-homotopic then they are sharp edge-homotopic.*

(iii) *If f and g are sharp edge-homotopic then they are edge-homotopic.*

Proof. (i) and (iii) are clear. It is known that a delta move is realized by sharp-moves on the strings in the delta move and ambient isotopies [14]. Thus we have (ii) immediately. □

Lemma 2.2. *If two spatial embeddings of a graph are cobordant then they are sharp edge-homotopic.*

To prove Lemma 2.2, we prepare two kinds of specific local moves on a spatial graph. A *pass-move* on a spatial oriented graph is a local move as illustrated in Fig. 4 [5]. We say that a pass-move is a *self pass-move* if all four strings in the move

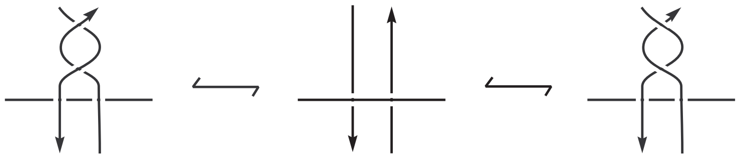


Figure 5

belong to the same spatial edge. It is known that a pass-move is realized by sharp-moves on the strings in the pass-move and ambient isotopies [12]. Thus we have the following:

Lemma 2.3. *A self pass-move is realized by sharp edge-homotopy.*

A Γ-*move* [5] is a local move on a spatial oriented graph as illustrated in Fig. 5. We call a Γ-move a *self* Γ-*move* if all three strings in the move belong to the same spatial edge. It is known that a Γ-move is realized by a pass-move on the strings in the Γ-move and ambient isotopies [5]. Thus by Lemma 2.3 we have the following:

Lemma 2.4. *A self* Γ*-move is realized by sharp edge-homotopy.*

The following is a normalization of cobordism between two cobordant spatial embeddings of a graph.

Lemma 2.5 ([27]). *Let f and g be cobordant spatial embeddings of a graph G. Then there exists an embedding $\Phi : G \times [0,1] \to S^3 \times [0,1]$ between f and g satisfying the following conditions:*

(i) *$\pi \circ \Phi|_{v\times[0,1]} : v \times [0,1] \to [0,1]$ is a homeomorphism for any vertex v of G, where $\pi : S^3 \times [0,1] \to [0,1]$ is the natural projection.*

(ii) *The image of Φ has only finitely many critical points in $\mathrm{int}(e \times [0,1])$ for any edge e of G, consisting of minimal points, saddle points and maximal points.*

(iii) *All of the minimal points lie in $S^3 \times \{\frac{1}{6}\}$ and all of the maximal points lie in $S^3 \times \{\frac{5}{6}\}$.*

(iv) *All of the saddle points lie in $S^3 \times \{\frac{1}{3}\}$ and $S^3 \times \{\frac{2}{3}\}$ such that the cross-section $\Phi(G \times [0,1]) \cap S^3 \times \{\frac{1}{2}\}$ is homeomorphic to G.*

Proof of Lemma 2.2. Let f and g be cobordant spatial embeddings of a graph G and Φ is a normalization of cobordism between f and g as in Lemma 2.5. We can regard the cross-section $\Phi(G \times [0,1]) \cap S^3 \times \{\frac{1}{2}\}$ as the image of a spatial embedding of G which is denoted by $h_{1/2}$.

Claim 1. f and $h_{1/2}$ are sharp edge-homotopic.

We remark here that the number of minimal points in $S^3 \times \{\frac{1}{6}\}$ equals the number of saddle points in $S^3 \times \{\frac{1}{3}\}$ as it was pointed out in [27]. We can deform $\Phi(G \times [0,1])$ by an ambient isotopy of $S^3 \times [0,1]$ so that the minimal points in $S^3 \times \{\frac{1}{6}\}$ and the saddle points in $S^3 \times \{\frac{1}{3}\}$ change into *minimal bands* $D_1, D_2, \ldots, D_l$ and *saddle bands* $b_1, b_2, \ldots, b_l$, respectively and each saddle band intersects the spatial graph by an arc. Then we have that $h_{1/2}$ is a band fusion of an l-component trivial link $L = J_1 \cup J_2 \cup \cdots \cup J_l$ and f. Then by using self Γ-moves and ambient isotopies if necessary, we can shrink each band with the component J_i one by one, see Fig. 6. By shrinking all bands in such a way, we obtain the spatial embedding f. Therefore by Lemma 2.4 we have that f and $h_{1/2}$ are sharp edge-homotopic.

Claim 2. g and $h_{1/2}$ are sharp edge-homotopic.

We can see Claim 2 in the same way as Claim 1. This completes the proof. □

Remark 2.6. The proof of Lemma 2.2 shows that if two spatial embeddings of a graph are cobordant then they are transformed into each other by self pass-moves and ambient isotopies.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have the desired implications. Let f and g be theta curves as illustrated in Fig. 7. It is known that any theta curve is trivial up to isotopy [27]. But f is not trivial up to sharp edge-homotopy, see Example 3.6. We note that g is an almost unknotted theta curve which is called *Kinoshita's theta curve.* It is known that g is not trivial up to delta edge-homotopy and cobordism [17]. But we have that g is trivial up to sharp-edge homotopy by Corollary 1.3. Let L be an oriented link as illustrated in Fig. 7. By a calculation we have that $\nabla_L(z) = -4z^3 - 5z^5 - z^7$. Since the first non-vanishing coefficient of the Conway polynomial of a link is an I-equivalence invariant [1], we have that L is not trivial up to I-equivalence and isotopy. Since the third coefficient of the Conway polynomial of a 2-component algebraically split link is a delta link-homotopy invariant [13], we have that L is not trivial up to delta vertex-homotopy. But L is trivial up to sharp edge-homotopy, see Fig. 8. Therefore we have that the converse of each implication does not hold and if there does not exist any arrow between two equivalence relations then there does not exist any implication between them. This completes the proof. □

3. Sharp edge-homotopy classification of theta curves

In this section we prove Theorem 1.2. We first show the sharp edge-homotopy invariance of the $\tilde{\alpha}$-invariant of theta curves.

Proposition 3.1. *If two theta curves f and g are sharp edge-homotopic then $\tilde{\alpha}(f) = \tilde{\alpha}(g)$.*

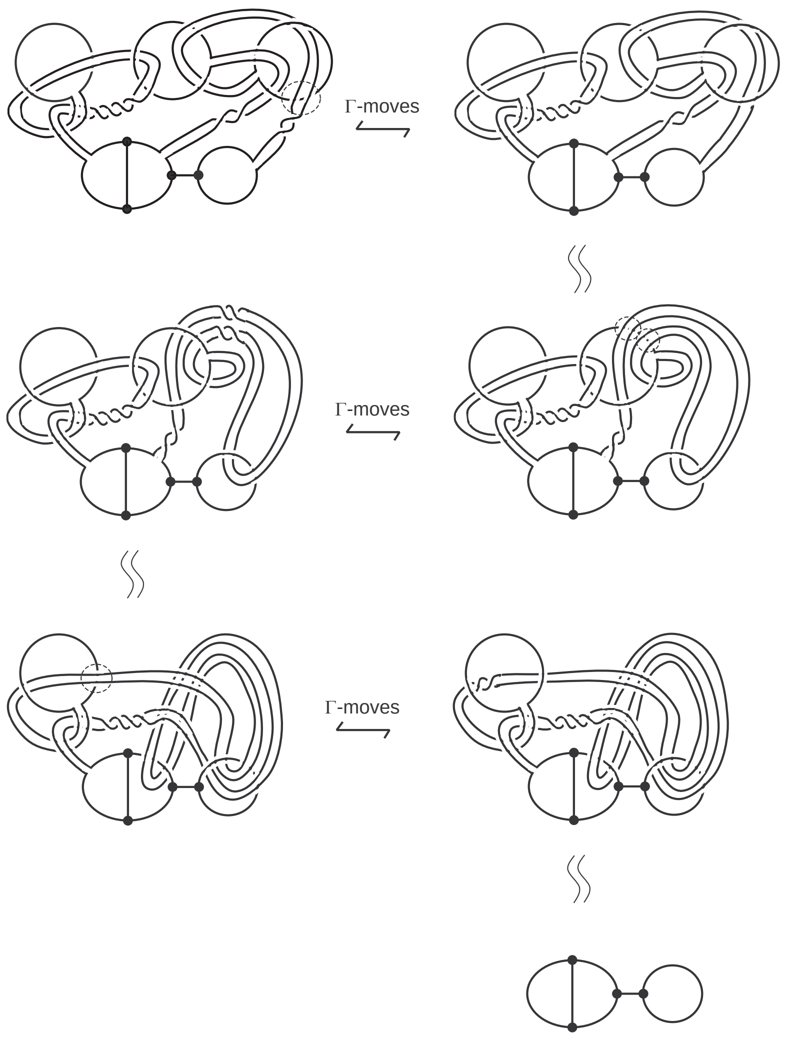


Figure 6

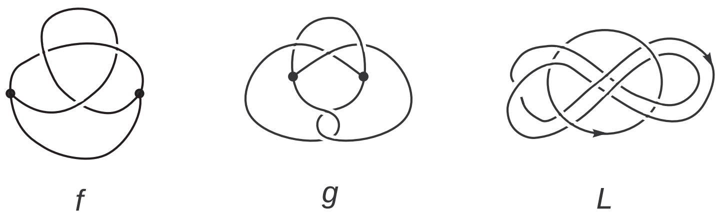


Figure 7

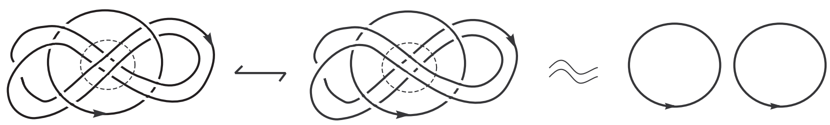


Figure 8

To prove Proposition 3.1, we recall the a_3-*invariant* of theta curves [17]. For a theta curve f, the associated 3-component link L_f is defined, which is the boundary of a compact, connected and orientable surface S_f with zero *Seifert linking form* having f as a spine [6], see Fig. 9. We order and orient $L_f = K_f^1 \cup K_f^2 \cup K_f^3$ so that K_f^i is freely homotopic to $f(e_{i+1}) - f(e_{i+2})$, where suffixes are taken modulo 3. We denote the sublink $K_f^{i+1} \cup K_f^{i+2}$ by $l_i(f)$ $(i = 1, 2, 3)$. Since it is known that $a_3(l_1(f)) = a_3(l_2(f)) = a_3(l_3(f))$ for any theta curve f [4, 26], we can define that $a_3(f) = a_3(S)$, where S is arbitrary 2-component sublink of L_f. We remark here that the a_3-invariant is a complete delta edge-homotopy invariant of theta curves.

Theorem 3.2 ([17])**.** *Two theta curves f and g are delta edge-homotopic if and only if $a_3(f) = a_3(g)$.*

We also recall the calculation of the Arf invariant of totally proper links. The following is a direct consequence of the results in [2, 9, 29].

Proposition 3.3. *Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ be an n-component totally proper link, namely* $\mathrm{lk}(J_i \cup J_j) \equiv 0 \pmod 2$ *for any $i \neq j$, where* lk *denotes the linking number. Then we have the following:*

(i) $a_{n+1}(L) \equiv \sum_{m=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \mathrm{Arf}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \pmod 2$,

(ii) $\mathrm{Arf}(L) \equiv \sum_{m=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} a_{m+1}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \pmod 2$.

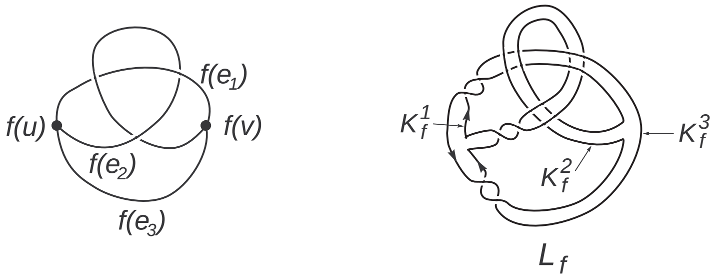


Figure 9

Lemma 3.4. $\tilde{\alpha}(f) \equiv a_3(f) \pmod 2$ *for any theta curve* f.

Proof. By Proposition 3.3 we have that

$$\begin{aligned}
0 = a_4(L_f) &\equiv \mathrm{Arf}(L_f) + \sum_{i=1}^{3} \mathrm{Arf}(l_i(f)) + \sum_{i=1}^{3} \mathrm{Arf}(K_f^i) \\
&\equiv \mathrm{Arf}(L_f) + \sum_{i=1}^{3} a_3(l_i(f)) + \sum_{i=1}^{3} a_2(K_f^i) \\
&\equiv \mathrm{Arf}(L_f) + a_3(f) + \tilde{\alpha}(f) \pmod 2.
\end{aligned}$$

Since we can see that $\mathrm{Arf}(L_f) = 0$, we have the result. □

Let $\overline{\mathrm{Arf}}(L)$ be the *reduced Arf invariant* [22] of a 2-component proper link $L = J_1 \cup J_2$, namely $\overline{\mathrm{Arf}}(L)$ is the modulo two reduction of $\mathrm{Arf}(L) - \mathrm{Arf}(J_1) - \mathrm{Arf}(J_2)$.

Lemma 3.5. (i) ([23]) $\overline{\mathrm{Arf}}(L)$ *is a sharp edge-homotopy invariant of a 2-component proper link* L.

(ii) $a_3(L) \equiv \overline{\mathrm{Arf}}(L) \pmod 2$.

Proof. We show (ii). By Proposition 3.3 (ii) we have that

$$\begin{aligned}
\overline{\mathrm{Arf}}(L) &\equiv \mathrm{Arf}(L) - \mathrm{Arf}(J_1) - \mathrm{Arf}(J_2) \\
&\equiv a_3(L) + a_2(J_1) + a_2(J_2) - a_2(J_1) - a_2(J_2) \\
&= a_3(L) \pmod 2.
\end{aligned}$$

□

Proof of Proposition 3.1. Let f and g be theta curves such that g is obtained from f by a single self sharp move on $f(e_1)$. Then we have that $l_3(g)$ is obtained from $l_3(f)$

by a single self sharp-move on K_f^2, namely $l_3(g)$ and $l_3(f)$ are self sharp-equivalent. Then by Lemmas 3.4 and 3.5 (i) and (ii), we have that

$$\begin{aligned}\tilde{\alpha}(f) &\equiv a_3(f) = a_3(l_3(f)) \equiv \overline{\mathrm{Arf}}(l_3(f)) \\ &= \overline{\mathrm{Arf}}(l_3(g)) \equiv a_3(l_3(g)) = a_3(g) \equiv \tilde{\alpha}(g) \pmod 2.\end{aligned}$$

This completes the proof. □

Example 3.6. Let f be the theta curve as illustrated in Fig. 7. Then by a calculation we have that $\tilde{\alpha}(f) = 1$. Thus f is not trivial up to sharp edge-homotopy.

In the rest of this section we show that the converse of Proposition 3.1 is also true. For quasi adjacent-delta moves, we have the following:

Lemma 3.7. *Let G be a graph and v a vertex of G such that exactly three edges e_1, e_2 and e_3 of G are incident to v. Let f be a spatial embedding of G. Then, any quasi adjacent-delta move on two of the three spatial edges $f(e_1), f(e_2)$ and $f(e_3)$ are realized by quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies.*

Proof. It is sufficient to show that a quasi adjacent-delta move on $f(e_2)$ and $f(e_3)$ can be realized by quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies.

Step 1. If two of the three strings in a quasi adjacent-delta move belong to $f(e_2)$, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by a self delta move on $f(e_2)$, namely self sharp-moves on $f(e_2)$, and a quasi adjacent-delta move on $f(e_1)$ and $f(e_2)$ and ambient isotopies. So we have the result.

Step 2. If two of the three strings in a quasi adjacent-delta move belong to $f(e_3)$, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by a quasi adjacent-delta move on $f(e_2)$ and $f(e_3)$, a delta move on exactly three spatial edges $f(e_1), f(e_2)$ and $f(e_3)$ and ambient isotopies. The former move can be realized by a quasi adjacent-delta move on $f(e_1)$ and $f(e_2)$ and sharp edge-homotopies in the same way as Step 1. For the latter move, by an application of the move in Fig. 10 with respect to $f(e_3)$, we have that it is realized by two quasi adjacent-delta moves on $f(e_1)$ and $f(e_2)$ and ambient isotopies. This completes the proof. □

Proof of Theorem 1.2. We have the 'only if' part by Proposition 3.1. We show the 'if' part. Let f and g be two theta curves such that $\tilde{\alpha}(f) = \tilde{\alpha}(g)$. It is known that any two theta curves are delta vertex-homotopic [15]. Thus we have that f and g are delta vertex-homotopic. Then by Lemma 3.7 we have that f can be obtained from g by quasi adjacent-delta moves on $g(e_1)$ and $g(e_2)$ and sharp edge-homotopies. We note that a delta move can be regarded as a band fusion of Borromean rings, see Fig. 11. So we have that f is sharp edge-homotopic to f_0 which is a band fusion of Borromean rings and g, where the roots of three fusion bands with each Borromean rings belong to $g(e_1)$ and $g(e_2)$. We note that $\tilde{\alpha}(f) = \tilde{\alpha}(f_0)$ by Lemma 3.1.

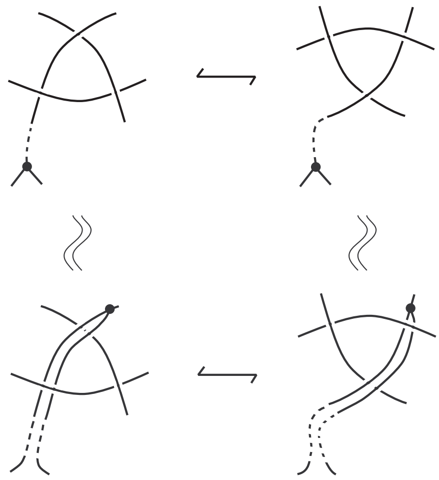


Figure 10

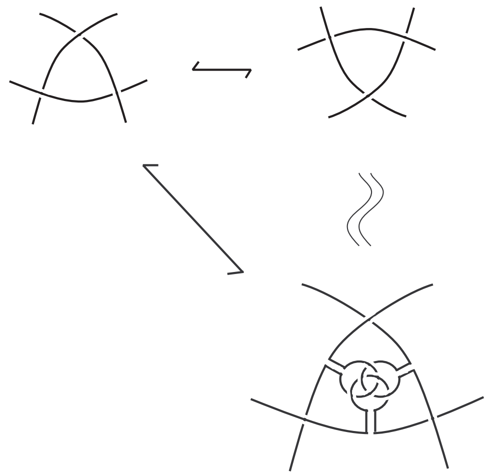


Figure 11

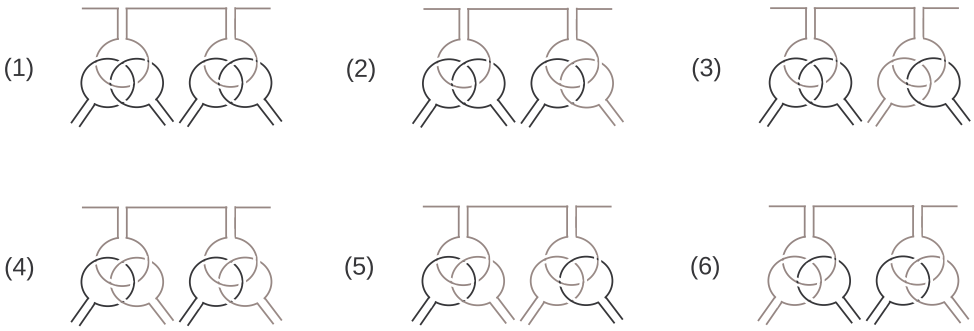


Figure 12

All pairs of adjacent two Borromean rings attached to $g(e_1)$ and $g(e_2)$ with respect to $f_0(e_1)$ are divided into the six patterns as illustrated in Fig. 12, where the gray spatial edge is $f_0(e_1)$ and the black spatial edge is $f_0(e_2)$. By sliding the roots of a fusion band with the Borromean rings if necessary, we can reduce (4), (5), and (6) to (1). Then we can see that each pair of adjacent two Borromean rings attached to $g(e_1)$ and $g(e_2)$ as illustrated in Fig. 12 (1), (2), and (3) can be omitted by self Γ-moves and ambient isotopies, namely up to sharp edge-homotopy by Lemma 2.4 as illustrated in Figs. 13, 14, and 15, respectively. Therefore we have that f_0 is sharp edge-homotopic to f_1 which is a band fusion of Borromean rings and g, where the number of the Borromean rings is at most one. Assume that there exists exactly one Borromean rings. We remark here that if a knot K can be obtained from J by a single delta move then $|a_2(K)-a_2(J)| = 1$ [20]. Then we have that $|a_2(f_1(e_1\cup e_2))-a_2(g(e_1\cup e_2))| = 1$, $a_2(f_1(e_2 \cup e_3)) = a_2(g(e_2 \cup e_3))$ and $a_2(f_1(e_2 \cup e_3)) = a_2(g(e_2 \cup e_3))$. This implies that $\tilde{\alpha}(f) = \tilde{\alpha}(f_0) = \tilde{\alpha}(f_1) \neq \tilde{\alpha}(g)$. This is a contradiction. Thus we have that f_1 is sharp edge-homotopic to g. Therefore f and g are sharp edge-homotopic. □

4. Sharp edge-homotopy vs. delta edge (resp. vertex)-homotopy

In this section we give some examples of infinitely many spatial embeddings of K_4 (resp. K_5) up to delta edge (resp. vertex)-homotopy which are mutually sharp edge-homotopic. To detect the non-trivial delta edge (resp. vertex)-homotopy classes, we recall the *n-invariant* [15] of spatial embeddings of K_4 and K_5. A cycle of a graph G is called a *k-cycle* if it contains exactly k edges. Let $\omega_4 : \Gamma(K_4) \to \mathbf{Z}$ be a map defined by $\omega_4(\gamma) = 1$ if γ is a 4-cycle and -1 if γ is a 3-cycle. Let $\omega_5 : \Gamma(K_5) \to \mathbf{Z}$ be a map defined by $\omega_5(\gamma) = 1$ if γ is a 5-cycle, -1 if γ is a 4-cycle and 0 if γ is a 3-cycle. For

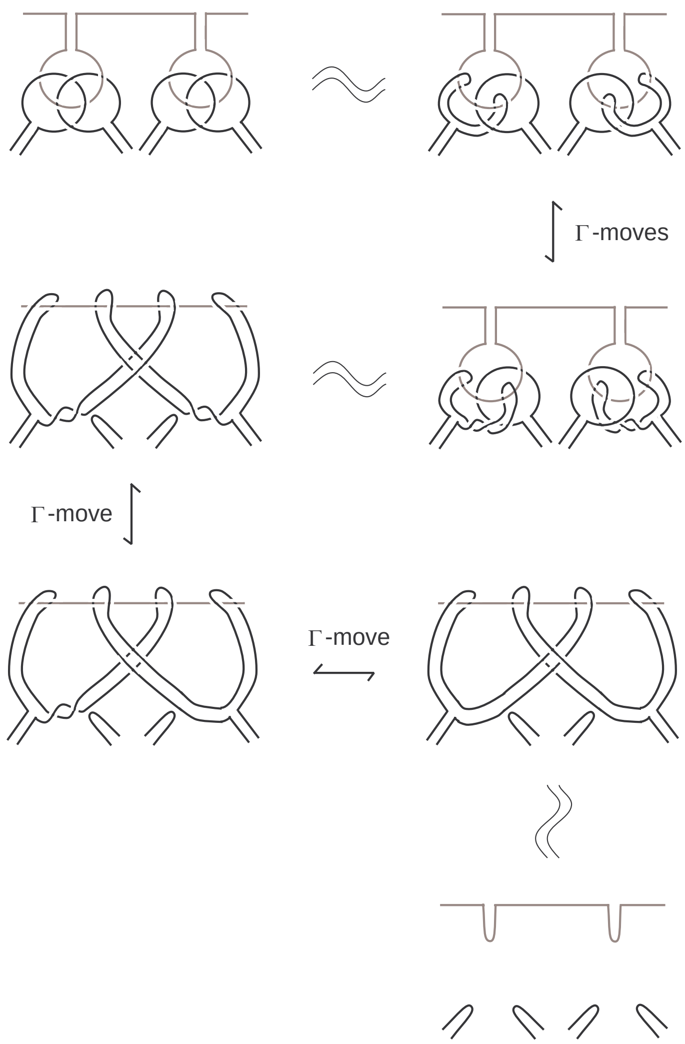


Figure 13

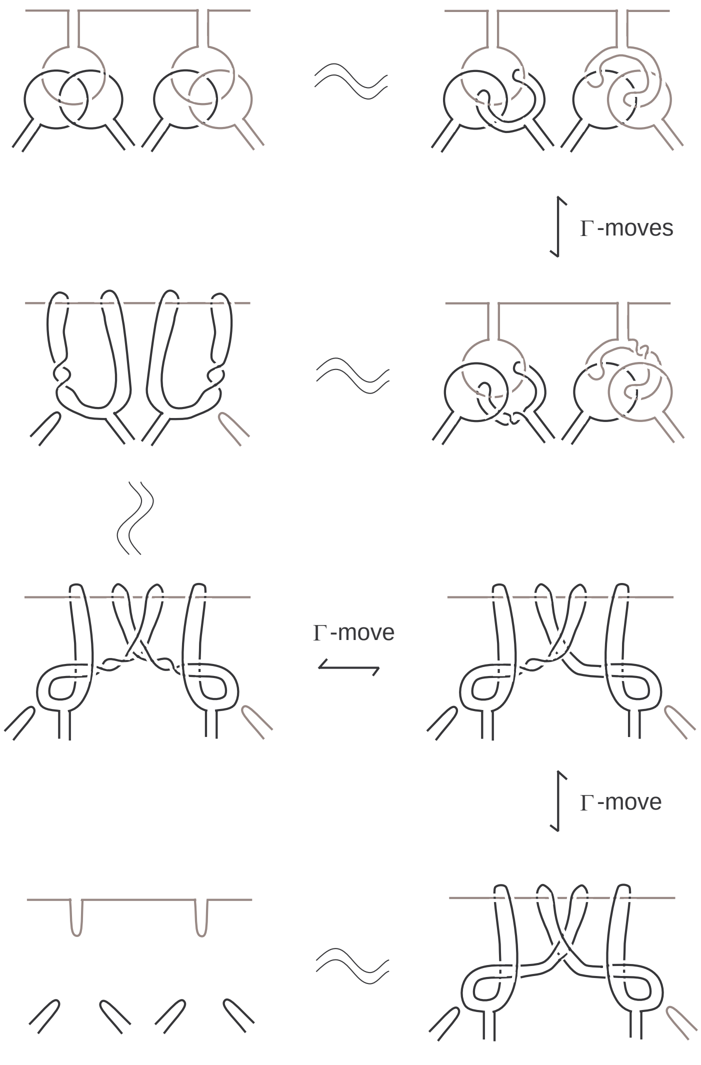


Figure 14

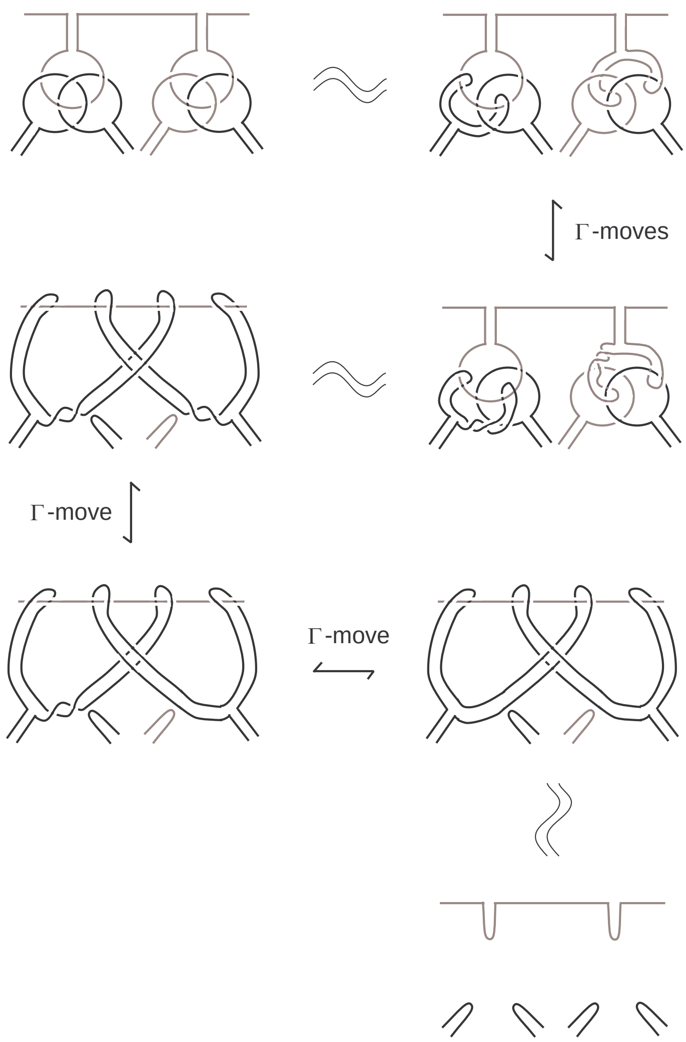


Figure 15

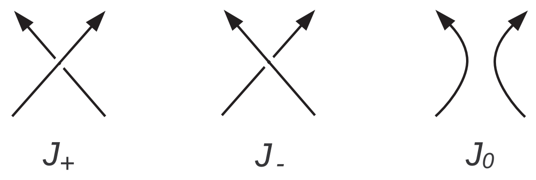


Figure 16

a spatial embedding f of K_i for $i = 4, 5$, we put

$$n_{\omega_i}(f) = \frac{1}{18} \sum_{\gamma \in \Gamma(K_i)} \omega_i(\gamma) V^{(3)}_{f(\gamma)}(1),$$

where $V_L^{(i)}(1)$ denotes the i-th derivative at 1 of the *Jones polynomial* $V_L(t)$ of a link L. (In this paper we calculate the Jones polynomial of a link by the skein relation $tV_{J_+}(t) - t^{-1}V_{J_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{J_0}(t)$.) Then the following was shown in [15] by the author:

Theorem 4.1. (i) *$n_{\omega_4}(f)$ is an integer-valued delta edge-homotopy invariant of a spatial embedding f of K_4.*

(ii) *$n_{\omega_5}(f)$ is an integer-valued delta vertex-homotopy invariant of a spatial embedding f of K_5.*

We prepare the formula of the variation of $V^{(3)}(1)$ of knots which differed by a single crossing change. The following formula is convenient to calculate the n-invariant.

Proposition 4.2.

$$V^{(3)}_{J_+}(1) - V^{(3)}_{J_-}(1) = 36a_2(J_+) + 18\{\mathrm{lk}(J_0)\}^2 - 36\{a_2(K_1) + a_2(K_2)\},$$

where J_+, J_- and $J_0 = K_1 \cup K_2$ are two knots and a 2-component link as illustrated in Fig. 16.

To prove Proposition 4.2, we use the following results.

Lemma 4.3. *Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ be an n-component link.*

(i) ([3]) $V_L(1) = (-2)^{n-1}$.

(ii) ([3, 11])

$$V_L^{(1)}(1) = \begin{cases} 0 & \text{if } n = 1, \\ 3(-2)^{n-2} \sum_{1 \le i < j \le n} \mathrm{lk}(J_i \cup J_j) & \text{if } n \ge 2. \end{cases}$$

(iii) ([11])

$$V_L^{(2)}(1) = \begin{cases} -6a_2(L) & \textit{if } n = 1, \\ 12\{a_2(J_1) + a_2(J_2)\} - 6\{\mathrm{lk}(L)\}^2 - 3\,\mathrm{lk}(L) - \frac{1}{2} & \textit{if } n = 2. \end{cases}$$

Proof of Proposition 4.2. By differentiating both sides of the skein relation at 1 about three times and Lemma 4.3, it is easy to check that

$$V_{J_+}^{(3)}(1) - V_{J_-}^{(3)}(1) = -3V_{J_+}^{(2)}(1) - 3V_{J_-}^{(2)}(1) + 3V_{J_0}^{(1)}(1) - 3V_{J_0}^{(2)}(1) - \frac{3}{2}. \tag{1}$$

By Lemma 4.3 we have that

$$\begin{aligned} V_{J_+}^{(2)}(1) &= -6a_2(J_+), \\ V_{J_-}^{(2)}(1) &= -6a_2(J_-), \\ V_{J_0}^{(1)}(1) &= 3\,\mathrm{lk}(J_0), \\ V_{J_0}^{(2)}(1) &= 12\{a_2(K_1) + a_2(K_2)\} - 6\{\mathrm{lk}(J_0)\}^2 - 3\,\mathrm{lk}(J_0) - \frac{1}{2}. \end{aligned}$$

Besides it is well known that ([5])

$$a_2(J_+) - a_2(J_-) = \mathrm{lk}(J_0). \tag{2}$$

Thus by substituting them to (1) we have the result. □

Proof of Theorem 1.5. Let $f_m (m \in \mathbf{N} \cup \{0\})$ be a spatial embedding of the complete graph on four vertices K_4 as illustrated in Fig. 17.

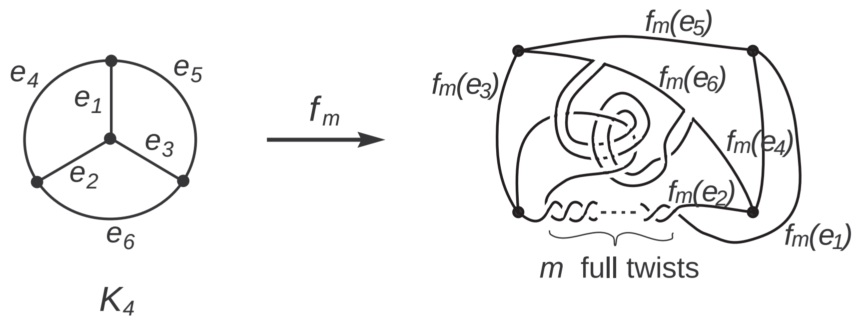


Figure 17

We first show (iii). We can see that f_m is sharp edge-homotopic to a trivial spatial embedding for any $m \in \mathbf{N} \cup \{0\}$ as illustrated in Fig. 18.

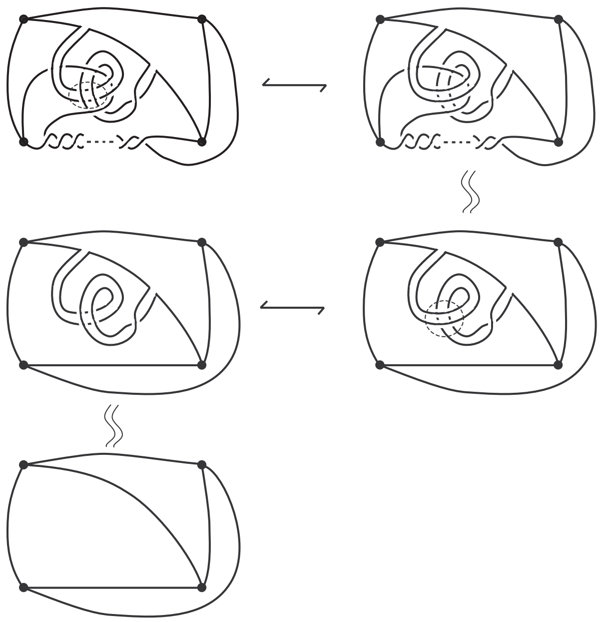


Figure 18

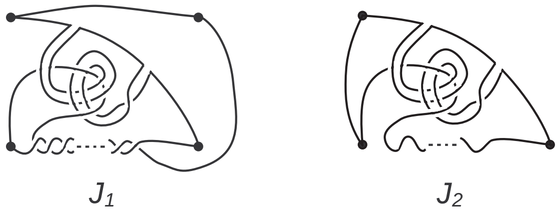


Figure 19

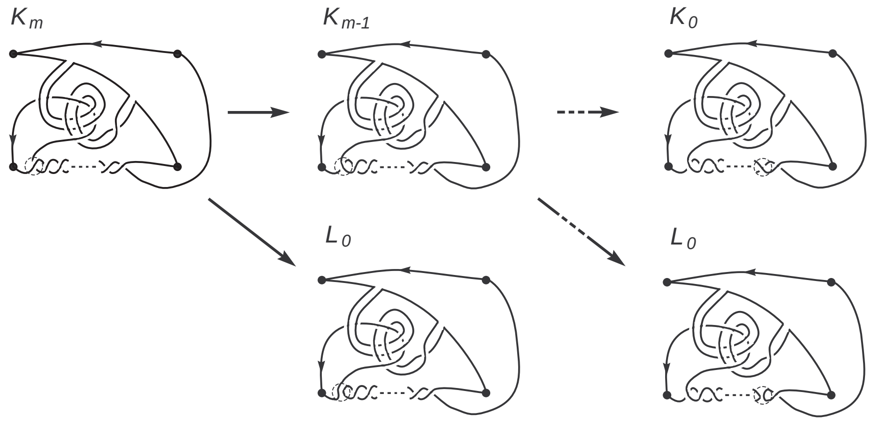


Figure 20

Next we show (i). It is not hard to see that f_m contains exactly two non-trivial knots $J_1 = K_m$ and J_2 as illustrated in Fig. 19. We note that J_2 is ambient isotopic to K_0 and

$$\begin{aligned}\nabla_{J_2}(z) &= 1 + 2z^2, \\ V_{J_2}(t) &= -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}.\end{aligned} \tag{3}$$

For a skein triple (K_i, K_{i-1}, L_0) $(i = 1, 2, \ldots, m)$ as illustrated in Fig. 20, we have that $a_2(K_m) = a_2(K_{m-1}) = \cdots = a_2(K_0) = 2$ because $\mathrm{lk}(L_0) = 0$. We note that the components M_0^1 and M_0^2 of L_0 are trivial knots. Then by Proposition 4.2 we have that

$$\frac{1}{18}V_{K_i}^{(3)}(1) - \frac{1}{18}V_{K_{i-1}}^{(3)}(1) = 2a_2(K_i) = 4$$

for $i = 1, 2, \ldots, m$. Thus we have that

$$\frac{1}{18}V_{K_m}^{(3)}(1) = \frac{1}{18}V_{K_{m-1}}^{(3)}(1) + 4 = \cdots = \frac{1}{18}V_{K_0}^{(3)}(1) + 4m.$$

Therefore we have that

$$n_{\omega_4}(f_m) = \frac{1}{18}V_{J_1}^{(3)}(1) - \frac{1}{18}V_{J_2}^{(3)}(1) = \frac{1}{18}V_{K_m}^{(3)}(1) - \frac{1}{18}V_{K_0}^{(3)}(1) = 4m.$$

This implies that f_i and f_j are not delta edge-homotopic for $i \neq j$.

Finally we show (ii). Fig. 21 illustrates all constituent theta curves of f_m. We can see that $f_m|_{K_4-e_1}$ and $f_m|_{K_4-e_6}$ are trivial theta curves for any $m \in \mathbf{N} \cup \{0\}$ and $f_i|_{K_4-e_2}$ and $f_j|_{K_4-e_2}$ are ambient isotopic for $i \neq j$.

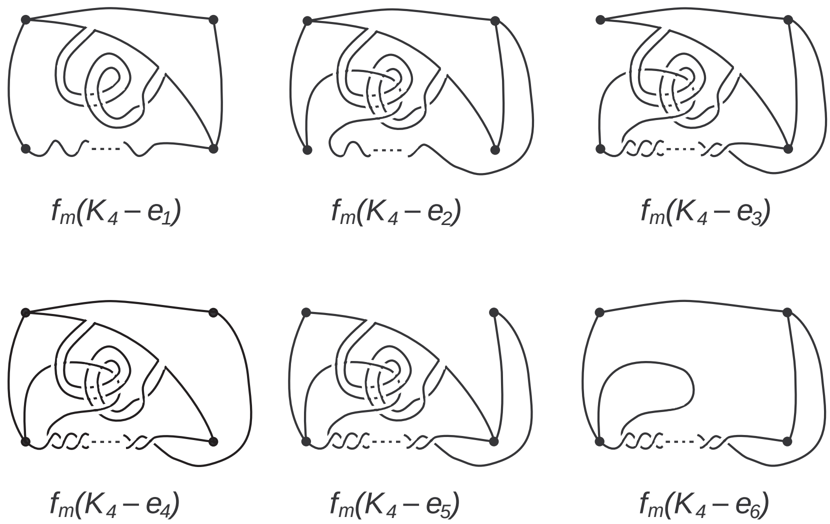


Figure 21

We take a 2-component sublink L_m^3, L_m^4, and L_m^5 of the associated 3-component link of $f_m|_{K_4-e_3}$, $f_m|_{K_4-e_4}$ and $f_m|_{K_4-e_5}$, respectively as illustrated in Fig. 22. Since L_i^3 and L_j^3 are ambient isotopic for $i \neq j$, we have that $a_3(f_i|_{K_4-e_3}) = a_3(f_j|_{K_4-e_3})$ for $i \neq j$. It is easy to see that L_m^4 is delta edge-homotopic to a trivial theta curve. Thus we have that $a_3(f_m|_{K_4-e_4}) = 0$ for any $m \in \mathbf{N} \cup \{0\}$. For L_m^5, we can see that the knot which can be obtained from L_m^5 by a smoothing on any crossing point of full twists is trivial. Then it is not hard to see that $a_3(L_m^5) = -4$. Thus we have that $a_3(f_m|_{K_4-e_5}) = -4$ for any $m \in \mathbf{N} \cup \{0\}$. Hence by Theorem 3.2 we have that $f_i|_{K_4-e_k}$ and $f_j|_{K_4-e_k}$ are delta edge-homotopic for $i \neq j$ and $k = 3, 4, 5$. This completes the proof. □

Proof of Theorem 1.6. Let f_m ($m \in \mathbf{N} \cup \{0\}$) be a spatial embedding of the complete graph on five vertices K_5 as illustrated in Fig. 23.

We first show (iii). We can see that f_m is sharp edge-homotopic to f_0 for any $m \in \mathbf{N}$ in a similar way as in Fig. 18.

Next we show (i). It is not hard to see that f_m ($m \neq 0$) contains exactly six non-trivial knots $J_1, J_2, \ldots, J_6$ as illustrated in Fig. 24. We note that each of J_1, J_3, and J_5 is ambient isotopic to the m times connected sum of the knot J_2 in Fig. 19. Since $V^{(3)}(1)$ is additive under the connected sum, by (3) we have that

$$\frac{1}{18}V_{J_1}^{(3)}(1) = \frac{1}{18}V_{J_3}^{(3)}(1) = \frac{1}{18}V_{J_5}^{(3)}(1) = 8m. \qquad (4)$$

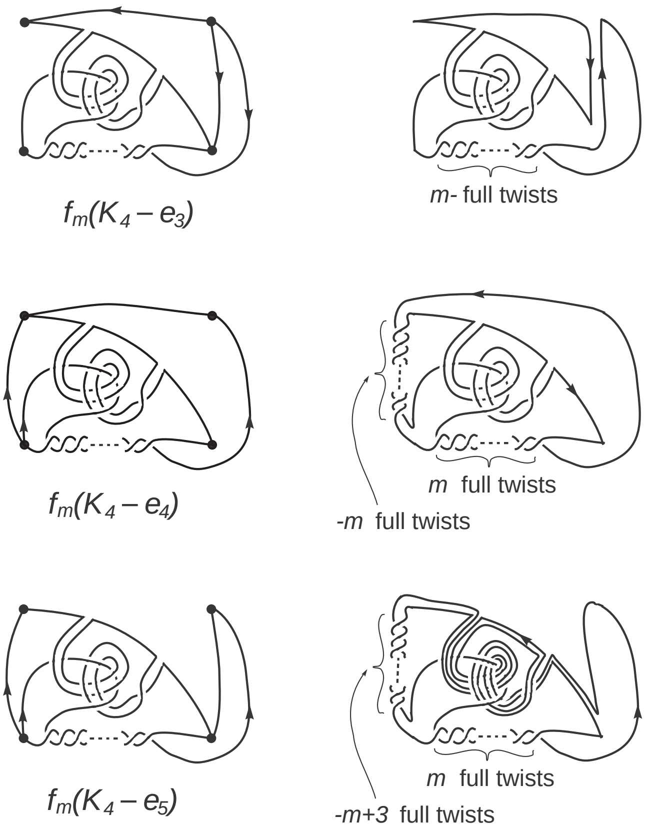

$f_m(K_4 - e_3)$
m- full twists
$f_m(K_4 - e_4)$
m full twists
$-m$ full twists
$f_m(K_4 - e_5)$
m full twists
$-m+3$ full twists

Figure 22

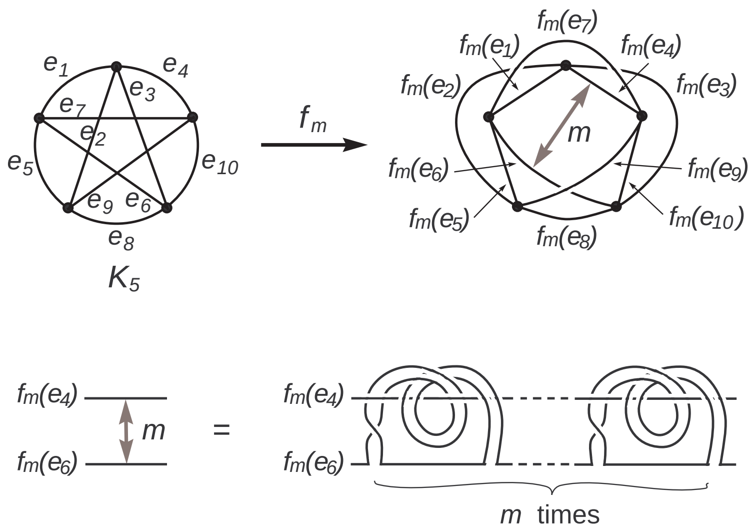


Figure 23

Besides we can see that J_2, J_4, and J_6 are mutually ambient isotopic. Let us consider a skein tree of $J_2 = K_m$ as illustrated in Fig. 25. Then by Proposition 4.2 and (2), we have that

$$\begin{aligned}
\frac{1}{18}V^{(3)}_{K_m}(1) - \frac{1}{18}V^{(3)}_{M_2}(1) &= 2a_2(K_m),\\
\frac{1}{18}V^{(3)}_{M_2}(1) - \frac{1}{18}V^{(3)}_{M_3}(1) &= 2a_2(M_2) + 2,\\
\frac{1}{18}V^{(3)}_{M_3}(1) - \frac{1}{18}V^{(3)}_{M_4}(1) &= -2a_2(M_4),\\
\frac{1}{18}V^{(3)}_{M_4}(1) - \frac{1}{18}V^{(3)}_{M_5}(1) &= -2a_2(M_5) - 4,\\
a_2(K_m) = a_2(M_2) = a_2(M_3) = a_2(M_4) &= a_2(M_5) - 2.
\end{aligned}$$

Thus we have that

$$\frac{1}{18}V^{(3)}_{K_m}(1) = \frac{1}{18}V^{(3)}_{K_{m-1}}(1) - 6 = \cdots = \frac{1}{18}V^{(3)}_{K_0}(1) - 6m = -6m. \tag{5}$$

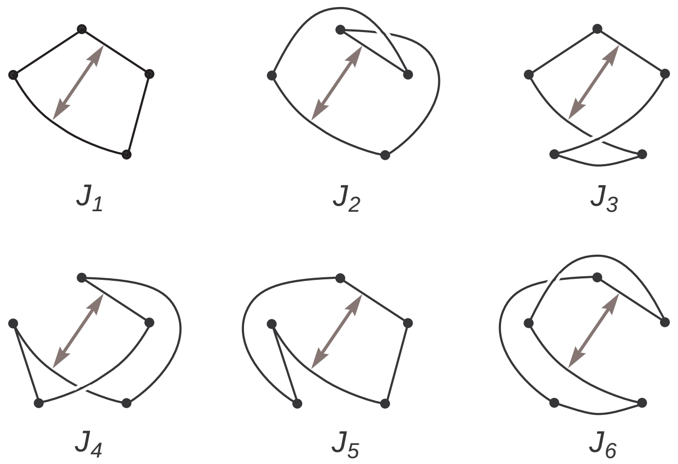


Figure 24

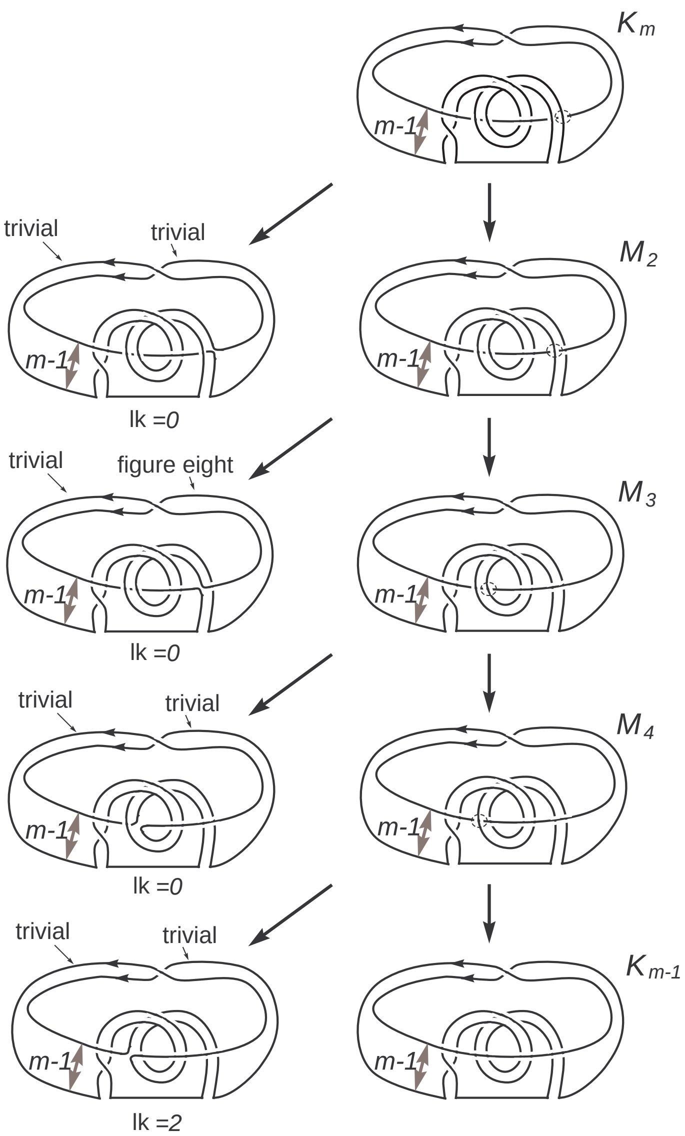
Therefore by (4) and (5) we have that

$$
\begin{aligned}
n_{\omega_5}(f_m) = & -\left\{\frac{1}{18}V^{(3)}_{J_1}(1) + \frac{1}{18}V^{(3)}_{J_2}(1)\right\} \\
& + \left\{\frac{1}{18}V^{(3)}_{J_3}(1) + \frac{1}{18}V^{(3)}_{J_4}(1) + \frac{1}{18}V^{(3)}_{J_5}(1) + \frac{1}{18}V^{(3)}_{J_6}(1)\right\} \\
= & -(8m - 6m) + (8m - 6m + 8m - 6m) \\
= & \, 2m.
\end{aligned}
$$

This implies that f_i and f_j are not vertex-homotopic for $i \neq j$.

Finally we show (ii). Since f_i and f_j are sharp edge-homotopic for $i \neq j$, by Theorem 1.1 we have that f_i and f_j are edge-homotopic for $i \neq j$. We note that $f_i|_H$ and $f_j|_H$ are also edge-homotopic for any subgraph H of K_5. It is known that if two spatial embeddings of K_4 are edge-homotopic then they are delta vertex-homotopic [16]. This completes the proof. □

Remark 4.4. This spatial embedding f_m of K_5 contains a theta curve T_m as illustrated in Fig. 26. We note that T_m is the m times vertex connected sum of the theta curve T_1. Then by a direct calculation we have that $a_3(T_1) = -4$. Since the a_3-invariant of theta curves is a cobordism invariant and additive under the vertex connected sum of theta curves [17], we have that $a_3(T_m) = -4m$ and f_i and f_j are not cobordant for $i \neq j$.



K_m
m-1
trivial
trivial
M_2
lk =0
trivial
figure eight
M_3
trivial
trivial
M_4
trivial
trivial
K_{m-1}
lk =2

Figure 25

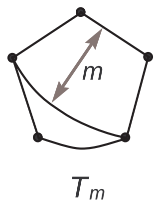


Figure 26

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