The Range of a Contractive Projection in $L_p(H)$

Yves Raynaud

Institut de Mathématiques de Jussieu (CNRS),
Projet Analyse Fonctionnelle (Case 186),
Université Paris-VI,
4, place Jussieu,
75252 Paris Cedex 05, France.
yr@ccr.jussieu.fr

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ABSTRACT

We show that the range of a contractive projection on a Lebesgue-Bochner space of Hilbert valued functions $L_p(H)$ is isometric to a ℓ_p-direct sum of Hilbert-valued L_p-spaces. We explicit the structure of contractive projections. As a consequence for every $1 < p < \infty$ the class $\mathcal{C}_p$ of ℓ_p-direct sums of Hilbert-valued L_p-spaces is axiomatizable (in the class of all Banach spaces).

Key words: Contractive projections, Vector-valued L_p-spaces.

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Introduction

It was a remarkable achievement in the isometric theory of Banach spaces of the years 1960's to characterize the contractive linear projections of Lebesgue L_p spaces ($p \neq 2$). In the case of L_p spaces of a probability space it was done by Douglas [4] in the case $p = 1$ and Andô [1] in the case $1 < p < \infty$, $p \neq 2$. They showed that the range of such a contractive projection is itself isometric to a L_p space (for the same p, but a different measure space); if moreover the projection is positive then its range is a sublattice of the initial L_p space and is lattice isomorphically isometric to a L_p space. This was extended to the non-sigma-finite measure space setting by Tzafriri ([17]). In the case of a probability space, the structure of contractive projections is elucidated by Douglas-Andô works: a general contractive projection P on $L_p(\Omega, \Sigma, \mu)$ has the form

$$P = M_\varepsilon \widehat{P} M_\varepsilon^{-1} + V \tag{1}$$

where M_ε is the multiplication operator by a function ε with $|\varepsilon| = \mathbf{1}$, $\widehat{P}$ is a positive contractive projection, and $V = 0$ if $p > 1$, while if $p = 1$, then V is a contraction from L_1 into the range $R(P)$ of P which vanishes on the band generated by $R(P)$. Moreover $\widehat{P}$ is a weighted conditional expectation, i.e. there exist a sub-sigma algebra $\mathcal{B}$, an element $B \in \mathcal{B}$ and a nonnegative function $w \in L^p$ such that $\mathbb{E}(w^p \mid \mathcal{B}) = \mathbf{1}$ and

$$\widehat{P} f = w \mathbb{E}(\mathbf{1}_B f \cdot w^{p-1} \mid \mathcal{B})$$

for every $f \in L_p$ (in particular if $P\mathbf{1} = \mathbf{1}$ then P is a conditional expectation). This last formula can also be written

$$\widehat{P} f = w \mathbb{E}_\nu(\mathbf{1}_B f w^{-1} \mid \mathcal{B})$$

where $\mathbb{E}_\nu$ is the conditional expectation relative to the measure $\nu = w^p \cdot \mu$. If we denote by S the isometric isomorphism $f \mapsto w \cdot f$ of $L_p(\Omega, \Sigma, \nu)$ onto $L_p(\Omega, \Sigma, \nu)$ and by M_B the multiplication operator by the indicator function $\mathbf{1}_B$, we have:

$$\widehat{P} = S M_B \mathbb{E}_\nu(\ \mid \mathcal{B}) S^{-1}. \tag{2}$$

The structure of contractive projections in the non-sigma finite case was treated by Bernau and Lacey ([3]); their main result can be rephrased in saying that if we assume (as we may) that the measure space (Ω, Σ, μ) is localizable ([7]) then formulas (1) and (2) are still valid; now w is some Σ-measurable positive function, $\nu = w^p \cdot \mu$ and $\mathcal{B}$ is some semi-finite sigma-subalgebra of Σ.

The task of extending these results to various classical spaces was considered by numerous authors; see the recent survey paper [15] and the references inside. Here we are more specifically interested in the case of vector-valued Lebesgue L_p spaces, in particular mixed norm spaces $L_p(L_q)$. Since the survey paper [5] on this specific subject, several partial results appeared. In particular B. Randrianantoanina ([14]) succeeded in solving thoroughly the complex sequential case $\ell_p(\ell_q)$ using hermitian operator techniques introduced in the subject by Kalton and Wood. More recently the case of finite dimensional real Banach spaces with C^2 norm was considered by the authors of [12]; under some additional conditions on the dual norm (in particular it is assumed to be C^2 on the complementary set of the coordinate hyperplanes associated to a distinguished basis) the contractively complemented subspaces are shown to be necessarily generated by a block-basis of the given basis. This can be applied in particular to the real spaces $\ell_p^n(\ell_q^m)$, when $2 < p, q < \infty$ (or by duality when $1 < p, q < 2$), obtaining the same description of their contractively complemented subspaces as in the complex case [16].

In the present paper we examine the case of Lebesgue spaces of Hilbert valued functions $L_p(H)$; this is done in the most general case (without any assumption of sigma-finiteness of L_p-space or separability of the Hilbert space; in fact we have in mind some applications to the ultrapowers of such spaces, which are neither separable nor sigma-finite). It turns out that the range of a contractive projection is a ℓ_p-direct sum of spaces of the type $L_p(H)$. More precisely:

Theorem 0.1. *Let $1 \le p < \infty$, $p \neq 2$; H be a Hilbert space and $L_p = L_p(\Omega, \Sigma, \mu)$. The range of every contractive projection $P : L_p(H) \to L_p(H)$ is isometric to a ℓ_p-direct sum of Hilbert-valued L_p-spaces, i.e.*

$$R(P) \approx_1 \Big(\bigoplus_{i\in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i; H_i)\Big)_{\ell_p}$$

where $(\Omega_i)_i$ is a family of pairwise almost disjoint members of Σ, each $\mathcal{B}_i$ is a sub-sigma-algebra of the trace Σ_i of Σ on Ω_i; μ_i is the trace on Ω_i of the measure μ; and the Hilbert spaces H_i have Hilbertian dimension not greater than the Hilbertian dimension of H.

Conversely a ℓ_p-sum $\big(\bigoplus_{i\in I} L_p(\Omega_i, \Sigma_i, \mu_i; H_i)\big)_{\ell_p}$ embeds isometrically into $L_p(H)$, where $L_p = \big(\bigoplus_{i\in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i)\big)_{\ell_p}$ and $H = \big(\bigoplus_{i\in I} H_i\big)_{\ell_2}$. Hence a contractively complemented subspace of a ℓ_p-direct sum of Hilbert-valued L_p-spaces is still a ℓ_p-direct sum of Hilbert-valued L_p-spaces. In other words:

Corollary 0.2. *The class $\mathcal{C}_p$ of ℓ_p-direct sums of Hilbert-valued L_p-spaces is stable under contractive projections.*

The structure of the contractive projection P can be easily explained in the case where the space H is separable (the non-separable case is analogous and will be described in Section 5). Recall that given two Banach spaces X, Y, a family of operators $T_\omega : X \to Y$ is said to be strong-operator Σ-measurable if for every $x \in X$, the map $\omega \mapsto T_\omega x$ is Σ-measurable as a map $\Omega \to Y$. If moreover $\operatorname{Ess\,sup}_\omega \|T_\omega\| < \infty$, such a measurable family induces a bounded linear map T from $L_p(\Omega, \Sigma, \mu; X)$ into $L_p(\Omega, \Sigma, \mu; Y)$ by the equation:

$$(Tf)(\omega) = T_\omega(f_\omega)$$

Theorem 0.3. *Under the conditions of Thm. 0.1, if moreover H is separable, then*

$$P = \sum_{i\in I} S_i(\widetilde{P}_i \otimes \mathrm{Id}_{H_i}) S_i^\sharp M_{\Omega_i} + V$$

where $\widetilde{P}_i$ is a positive contractive projection in $L_p(\Omega_i, \Sigma_i, \mu_i)$; S_i is an isometric embedding of $L_p(\Omega_i, \Sigma_i, \mu_i; H_i)$ into $L_p(\Omega_i, \Sigma_i, \mu_i; H)$ associated with a (strong-operator)-measurable family $(S_{i,\omega})_{\omega\in\Omega_i}$ of isometric embeddings $H_i \to H$, while $S_i^\sharp$ is associated with the adjoint family $(S_{i,\omega}^*)_{\omega\in\Omega_i}$ of projections $H \to H_i$; $M_{\Omega_i} : L_p(\Omega; H) \to L_p(\Omega_i; H_i)$ is the multiplication operator by the indicator function $\mathbf{1}_{\Omega_i}$; and $V = 0$ if $p > 1$, while if $p = 1$ then V is a contraction of $L_1(\Omega, \Sigma, \mu; H)$ vanishing on every $L_1(\Omega_i, \Sigma_i, \mu_i; H)$ and taking values in the range of P.

Let us present shortly an application of the Thm. 0.1 which was in fact our main motivation for starting this study. If X, Y are Banach spaces, we say that X is an *ultraroot* of Y if Y is isometric to some ultrapower of X. Recall that a Banach space X embeds canonically isometrically in every of its ultrapowers $X_{\mathcal{U}}$, and that if X is reflexive, then this canonical image is contractively complemented in $X_{\mathcal{U}}$. As a consequence of Thm. 0.1 we see that every ultraroot of a $L_p(H)$ space, $p > 1$ is a member of $\mathcal{C}_p$. By Cor. 0.2 the same is true for ultraroots of members of $\mathcal{C}_p$. On the other hand it was proved in [13] that every ultraproduct of $L_p(H)$ spaces is isometric to a ℓ_p-direct sum of Hilbert-valued L_p-spaces. More generally every ultraproduct of members of $\mathcal{C}_p$ is itself isometric to a member of $\mathcal{C}_p$. Hence we obtain:

Corollary 0.4. *For every $1 < p < \infty$ the class $\mathcal{C}_p$ of ℓ_p-direct sums of Hilbert-valued L_p-spaces is stable under ultraproducts and ultraroots.*

In other words the class $\mathcal{C}_p$ is *axiomatizable* in the sense of Henson-Iovino [9] in their language of normed spaces structures (see [9], Thm. 13.8).

The paper is organized as follows: after a section devoted to definitions, notations and a general result on orthogonally complemented subspaces of $L_p(H)$, we have two sections of preliminary results distinguishing the case $p = 1$ (Section 2) from the case $p > 1$ (Section 3). In these sections it is proved that if f belongs to the range of a contractive projection P, then the whole subspace $Z_f := \overline{L_\infty(\Omega) \cdot f}$ is preserved by P (i.e. $PZ_f \subset Z_f$) which suggests clearly a possible reduction to the scalar case. It is also proved that the "orthogonal projection" onto Z_f preserves the range of P. This allows to find an "orthogonal system" in $R(P)$ which generates $Z_P := \overline{L_\infty(\Sigma) \cdot R(P)}$ over $L_\infty(\Sigma)$ which will furnish the orthogonal bases of the Hilbert spaces H_i of Thm. 0.1. Section 4 is devoted to the proof of Thm. 0.1; a key point consists in proving that the different subalgebras of Σ given by the scalar theorem (applied to each Z_f) are induced by the same sigma-subalgebra $\mathcal{F}$ of Σ. Finally Thm 0.3 is proved in Section 5 (in a more general version not requiring separability).

1. General preliminaries

1.1. Definitions and notations

Let $1 \leq p < \infty$, H be an Hilbert space and (Ω, Σ, μ) be a measure space. In the following we denote (when there is no ambiguity) by $L_p(H)$ the Lebesgue-Bochner space $L_p(\Omega, \Sigma, \mu; H)$ of classes of H-valued p-integrable functions (for μ-a.e. equality). Similarly $L_\infty(H)$ will be the space of classes of Bochner measurable, essentially bounded H-valued functions. These spaces can be defined directly from the Banach lattices L_p (resp. L_∞) and the Hilbert space H, but we adopt the functional point of view for the simplicity of the exposition. In the case where (Ω, Σ, μ) is not sigma-finite, it is preferable to suppose that this measure space is localizable: the measure μ is semifinite (every set in Σ of positive measure contains a further one of positive and finite measure) and $L_\infty(\Omega,\Sigma,\mu)$ is order complete. In particular every family $(A_i)_{i\in I}$ in Σ has a supremum A, denoted by $\bigvee_{i\in I} A_i$. The set A is defined (up to a μ-null set) by the conditions:

$$A\dot{\supset}A_i \text{ for every } i\in I,$$
$$\text{If } B\in\Sigma \text{ and } B\dot{\supset}A_i \text{ for every } i\in I \text{ then } B\dot{\supset}A,$$

where $B\dot{\supset}A$ means $\mu(A\setminus B)=0$ (define similarly $A\dot{\subset}B$ and $A\doteq B$). We say that B,C are almost disjoint if $A\cap B\doteq\emptyset$.

To every $f\in L_p(H)$ we associate its "*random norm*" $N(f)\in L_p^+$ defined by $N(f)(\omega)=\|f(\omega)\|_H$, its *vectorial function support* $\mathbf{VS}(f)=\operatorname{Supp}(N(f))$ and its "random direction", i.e. the element u_f of $L_\infty(H)$ defined by $u_f(\omega)=\frac{f(\omega)}{N(f)(\omega)}$ if $\omega\in\mathbf{VS}(f)$, $=0$ if $\omega\notin\mathbf{VS}(f)$. If $M\subset L_p(H)$ we set $\mathbf{VS}(M)=\bigvee\{\mathbf{VS}(f)\mid f\in M\}$. If $f\in L_p(H)$, $g\in L_q(H)$ we define their random scalar product $\langle\langle f,g\rangle\rangle\in L_r$ (where $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$) by $\langle\langle f|g\rangle\rangle(\omega)=\langle f(\omega)|g(\omega)\rangle_H$, where $\langle\cdot\,|\,\cdot\rangle_H$ denotes the scalar product in H (which we suppose left linear, right antilinear in the complex case). When p,q are conjugate ($\frac{1}{p}+\frac{1}{q}=1$), we obtain a sesquilinear pairing

$$\langle f,g\rangle=\int_\Omega\langle\langle f\mid g\rangle\rangle d\mu \tag{3}$$

which gives rise to a canonical antilinear identification of $L_q(H)$ with $L_p(H)^*$ (if $1<p,q<\infty$; the case $p=1$, $q=\infty$ is more delicate); it is the usual duality pairing in the real spaces case. We have also

$$\forall f\in L_p(H), \langle\langle f|u_f\rangle\rangle=N(f).$$

We say that two elements $f,g\in L_p(H)$ are *orthogonal*, and we write $f\perp g$ if $\langle\langle f\mid g\rangle\rangle=0$. A related notation is the following. We set

$$\{\,f\perp g\,\}=\{\,\omega\in\Omega\mid\langle\langle f\mid g\rangle\rangle(\omega)=0\,\}$$

We have then $f\perp g \iff \{\,f\perp g\,\}\doteq\Omega$.

Let H, K two Hilbert spaces. We say that a linear operator $T:L_p(H)\to L_p(K)$ is Σ-*modular* iff $T(\varphi.f)=\varphi.Tf$ for every $f\in L_p(H)$ and $\varphi\in L_\infty(\Omega,\Sigma,\mu)$. It is *modularly contractive*, resp. *modularly isometric* iff $N(Tf)\le N(f)$, resp. $N(Tf)=N(f)$ for every $f\in L_p(H)$: it is then automatically Σ-modular (and, of course, contractive, resp. isometric). If H is separable, then a modularly contractive, resp. modularly isometric operator T is associated with a measurable family of contractions, resp. isometries $T_\omega:H\to K$.

Let $\mathcal{F}$ be a sub-sigma-algebra of Σ; a linear subspace Z of $L_p(H)$ is a $L_\infty(\mathcal{F})$-*submodule* iff $\varphi\cdot f\in Z$ for every $f\in Z$ and $\varphi\in L_\infty(\Omega,\mathcal{F},\mu)$. To every $f\in L_p(H)$ we associate the bounded Σ-modular operator:

$$E_f:L_p(H)\to L_p(H),\qquad g\mapsto\langle\langle g|u_f\rangle\rangle\,u_f.$$

We have $N(E_f g) = |\langle\langle g|u_f\rangle\rangle|\mathbf{1}_{\mathbf{VS}(f)} \leq N(g)$, hence E_f is modularly contractive.

We have clearly $E_f(f) = N(f)u_f = f$. Consequently for every $\varphi \in L_\infty$, we have

$$E_f((\varphi N(f)) \cdot u_f) = E_f(\varphi f) = \varphi f = (\varphi N(f)) \cdot u_f$$

and by density we deduce that $E_f(\psi \cdot u_f) = \psi \cdot u_f$ for every $\psi \in L_p$. In particular $E_f(E_f g) = E_f g$, so E_f is a projection (with range $R(E_f) = L_p(\Omega).u_f$). It is not hard to see that $R(E_f)$ is exactly the closed $L_\infty(\Sigma)$-submodule generated by f. Note also that if $f, g \in L_p(H)$,

$$f \perp g \quad \Longleftrightarrow \quad E_f g = 0 \quad \Longleftrightarrow \quad E_g f = 0.$$

1.2. Orthogonal projections

We end this section by considering a special class of contractive projections, namely the orthogonal ones. A projection Q in $L_p(H)$ is said to be *orthogonal* if $(f-Qf) \perp Qf$ for every $f \in L_p(H)$. Such a projection is trivially modularly contractive since

$$N(f)^2 = N(Qf)^2 + N((I-Q)f)^2 \geq N(Qf)^2.$$

Note that by polarization we have for every $f, g \in L_p(H)$:

$$\langle\langle f \mid g\rangle\rangle = \langle\langle Qf \mid Qg\rangle\rangle + \langle\langle (I-Q)f \mid (I-Q)g\rangle\rangle$$

Replacing g by Qg, we have

$$\langle\langle f \mid Qg\rangle\rangle = \langle\langle Qf \mid Qg\rangle\rangle$$

that is $(I-Q)f \perp Qg$; hence $\ker Q = R(I-Q) \perp R(Q)$.

Conversely if $f \perp R(I-Q)$ then $f - Qf \perp R(I-Q)$ and in particular $f - Qf \perp f - Qf$, i.e. $f = Qf \in R(Q)$. Hence $R(Q) = \ker Q^{\perp} := \{f \in L_p(H) \mid f \perp \ker Q\}$ and similarly (exchanging the roles of Q and $I-Q$) we have: $\ker Q = R(Q)^{\perp}$.

If A is a subset of $L_p(H)$ then $A^{\perp}$ is a closed $L_\infty(\Sigma)$-submodule of $L_p(H)$. In particular the range of any orthogonal projection in $L_p(H)$ is a closed $L_\infty(\Sigma)$-submodule. The converse is true:

Lemma 1.1. *If Z is a closed $L_\infty(\Sigma)$-submodule of $L_p(\Omega, \Sigma, \mu; H)$ there exists a unique orthogonal projection Q_Z in $L_p(H)$ with range Z.*

Proof. Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of Z. For every family $(\varphi_\alpha)_\alpha$ in $L_p(\Omega)$ and every finite subset B of A we have

$$\Big\|\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha}\Big\|_{L_p(H)} = \Big\|N\Big(\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha}\Big)\Big\|_p = \Big\|\Big(\sum_{\alpha \in B} \mathbf{1}_{\mathbf{VS}(f_\alpha)} |\varphi_\alpha|^2\Big)^{1/2}\Big\|_p.$$

Hence, by Cauchy's criterion, $\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}$ converges in $L_p(H)$ iff $(\sum_{\alpha\in A}\mathbf{1}_{\mathbf{VS}(f_\alpha)}$ $|\varphi_\alpha|^2)^{1/2}$ exists in L_p and

$$\Big\|\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}\Big\|_{L_p(H)}=\Big\|\Big(\sum_{\alpha\in A}\mathbf{1}_{\mathbf{VS}(f_\alpha)}|\varphi_\alpha|^2\Big)^{1/2}\Big\|_p.$$

If now $f\in L_p(H)$ and B is a finite subset of A we have

$$\begin{aligned}N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big)^2&=\sum_{\alpha\in B}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\\&=\Big\langle\!\!\Big\langle f,\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big\rangle\!\!\Big\rangle\le N(f)N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big),\end{aligned}$$

whence

$$N\Big(\sum_{\alpha\in B}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}\Big)=\Big(\sum_{\alpha\in B}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\Big)^{1/2}\le N(f),$$

so

$$\Big(\sum_{\alpha\in A}|\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle|^2\Big)^{1/2}\le N(f).$$

Consequently $Qf:=\sum_{\alpha\in A}\langle\!\langle f|u_{f_\alpha}\rangle\!\rangle u_{f_\alpha}=\sum_{\alpha\in A}E_{f_\alpha}f$ converges in $L_p(H)$ (with $\|Qf\|\le\|f\|$). Since $R(E_{f_\alpha})$ is the closed $L_\infty(\Sigma)$-submodule generated by f_α, we have $R(E_{f_\alpha})\subset Z$ for each α and consequently $Qf\in Z$ for every $f\in L_p(H)$. The map Q is modular for the action of $L_\infty(\Omega)$, and clearly $Qf_\beta=f_\beta$ for every $\beta\in A$. It results easily that $Qf=f$ for every $f=\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}$ (when this series converges), i.e. Q is a contractive projection in $L_p(H)$ with range

$$\begin{aligned}R(Q)&=\Big\{\sum_{\alpha\in A}\varphi_\alpha u_{f_\alpha}\ \Big|\ \Big(\sum_\alpha|\varphi_\alpha|^2\Big)^{1/2}\in L_p(\Omega)\Big\}\\&=\Big\{\sum_{\alpha\in A}\psi_\alpha f_\alpha\ \Big|\ \Big(\sum_\alpha|\psi_\alpha|^2N(f_\alpha)^2\Big)^{1/2}\in L_p(\Omega)\Big\}.\end{aligned}$$

Since clearly $\langle\!\langle Qf|f_\alpha\rangle\!\rangle=\langle\!\langle f|f_\alpha\rangle\!\rangle$ for every $\alpha\in A$ we have $(f-Qf)\perp f_\alpha$ for every $\alpha\in A$. By maximality of the system (f_α) we deduce that

$$f=Qf\text{ for every }f\in Z$$

so $R(Q)$ contains Z, hence coincides with Z. Note also that $f-Qf\perp Z$ for all $f\in L_p(H)$, and so Q is orthogonal.

The unicity of the orthogonal projection onto Z is a consequence of the fact that its image and kernel are uniquely determined ($R(Q)=Z$ and $\ker Q=Z^\perp$). □

2. Preliminary results: the case $p = 1$

Lemma 2.1. *Let P be a contractive projection in $L_1(H)$. Then for every $f \in R(P)$ we have*

$$PE_f = E_f P E_f$$

Proof. For every $\varphi \in L_1(\Omega)$ with $0 \le \varphi \le N(f)$ we have

$$\begin{aligned}\|f\| - \|\varphi \cdot u_f\| &= \int N(f)\, d\mu - \int N(\varphi u_f)\, d\mu = \int (N(f) - \varphi)\, d\mu \\ &= \|(N(f) - \varphi) \cdot u_f\| = \|f - \varphi . u_f\| \\ &\ge \|P(f - \varphi \cdot u_f)\| = \|f - P(\varphi \cdot u_f)\| \\ &\ge \|f\| - \|P(\varphi . u_f)\| \\ &\ge \|f\| - \|\varphi \cdot u_f\|.\end{aligned}$$

Hence all the inequalities are equalities, and in particular

$$\|f - P(\varphi \cdot u_f)\| = \|f\| - \|P(\varphi \cdot u_f)\|,$$

that is,

$$\int N(f - P(\varphi \cdot u_f))\, d\mu = \int [N(f) - N(P(\varphi \cdot u_f))]\, d\mu.$$

Note that the function in the left-hand integral is greater than the one in the right-hand integral. Thus,

$$N(f - P(\varphi \cdot u_f)) = N(f) - N(P(\varphi \cdot u_f))$$

(equality as elements of $L_1(\Omega)$). Since H is strictly convex this implies that

$$P(\varphi \cdot u_f) = \alpha \cdot f$$

for some $\alpha \in L_\infty^+(\Omega)$. Hence

$$E_f P(\varphi \cdot u_f) = E_f(\alpha \cdot f) = \alpha \cdot f = P(\varphi \cdot u_f).$$

This property has been proved for $\varphi \in L_1(\Omega)$ with $0 \le \varphi \le N(f)$; it is extended by linearity and density to every $\varphi \in L_1(\Omega)$. In particular if we take $\varphi = \langle\langle h | u_f \rangle\rangle$, we obtain

$$\forall h \in L_1(H), \qquad E_f P E_f h = P E_f h,$$

that is, $E_f P E_f = P E_f$. □

Lemma 2.2. *Let P be a contractive projection in $L_1(H)$. Then for every $f, g \in R(P)$ we have: $E_g f \in R(P)$. In other words $E_g P = P E_g P$.*

Proof. We have $(f - E_g f) \perp g$, while (by Lemma 2.1) $E_g f - PE_g f = E_g(f - PE_g f) \in L_1(\Omega) \cdot u_g$. Hence $(f - E_g f) \perp (E_g f - PE_g f)$. It results that

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2]^{1/2} \geq N(f - E_g f). \tag{4}$$

Hence:

$$\begin{aligned} \|f - PE_g f\| &\geq \|f - E_g f\| \\ &\geq \|P(f - E_g f)\| \\ &= \|f - PE_g f\| \end{aligned}$$

Hence the inequalities are equalities. In view of (4), the equality $\|f - PE_g f\| = \|f - E_g f\|$ implies

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2]^{1/2} = N(f - E_g f),$$

which implies in turn that $N(E_g f - PE_g f) = 0$, that is $E_g f = PE_g f$. So $E_g f \in R(P)$. □

3. Preliminary results: the case $p > 1$

Notations. Let p_* be the conjugate exponent of p. If $T : L_p(H) \to L_p(H)$ is a bounded operator, we define its adjoint $T^* : L_{p_*}(H) \to L_{p_*}(H)$ by

$$\forall f \in L_{p_*}(H), \forall g \in L_p(H) \qquad \langle T^* f, g \rangle = \langle f, Tg \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the sesquilinear pairing given by eq. (3).

If $f \in L_p(H)$, $f \neq 0$, let $Jf \in L_{p_*}(H)$ be the unique norm-one element such that $\langle f, Jf \rangle = \|f\|$. In fact it will be easier to consider the $(p-1)$-homogeneous functional $J_p(h) = \|h\|^{p-1} J(h)$. We have $J_p(h) = N(h)^{p-1} \cdot u_h = N(h)^{p-2} h$, hence J_p is random direction preserving. Note that pJ_p is the derivative of the p^{th} power of the norm.

Lemma 3.1. *Let $1 < p < \infty$, $p \neq 2$, and P be a contractive projection in $L_p(H)$. Then for every $f, g \in R(P)$ the function $F(f, g) := \operatorname{sgn}\langle\langle g \mid f \rangle\rangle f + \gamma_p \mathbf{1}_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$, where γ_p is a positive constant depending only on p.*

Proof. a) Case $2 < p < \infty$.

Recall that since $L_p(H)$ is smooth the duality map J maps $R(P)$ into $R(P^*)$ (see e.g. [6, Lemma 4.8]); hence $J_p(f + tg) \in R(P^*)$ for every $t \geq 0$. The derivative $\frac{\partial}{\partial t} J_p(f + tg)$ exists at $t = 0$ (since the norm to the power p is twice differentiable) and it belongs to $R(P^*)$ too. We have

$$\frac{\partial}{\partial t} J_p(f + tg) = N(f + tg)^{p-2} g + \left(\frac{p-2}{2} \frac{\partial}{\partial t} N(f + tg)^2 \right) N(f + tg)^{p-4} (f + tg).$$

Hence

$$A(f,g) := \frac{\partial}{\partial t} J_p(f+tg)\,|_{t=0} = N(f)^{p-2}g + (p-2)\operatorname{Re}(\langle\!\langle f \mid g\rangle\!\rangle)N(f)^{p-4}f \\ = N(f)^{p-2}[g + (p-2)\operatorname{Re}(\,\langle\!\langle u_f, g\rangle\!\rangle\,)\,u_f] \in R(P^*) \tag{5}$$

In the complex case, replacing f by if, we obtain

$$B(f,g) := N(f)^{p-2}[g - i(p-2)\operatorname{Im}(\,\langle\!\langle u_f, g\rangle\!\rangle\,)\,u_f] \in R(P^*) \tag{5bis}$$

adding

$$N(f)^{p-2}[2g + (p-2)\langle\!\langle g, u_f\rangle\!\rangle\,u_f] \in R(P^*)$$

With $E_f g = \langle\!\langle g, u_f\rangle\!\rangle\,u_f$ we obtain

$$N(f)^{p-2}[2(g - E_f g) + pE_f g] \in R(P^*).$$

In the case of a real space (5) is valid without the symbol Re and we obtain

$$N(f)^{p-2}[(g - E_f g) + (p-1)E_f g] \in R(P^*).$$

If $h \in R(P^*)$ then $J_{p_*}h = N(h)^{p_*-1}u_h \in R(P)$, hence if we set $Tg = \alpha_p(g - E_f g) + E_f g$, with $\alpha_p = \frac{2}{p}$ in the complex case, $\alpha_p = \frac{1}{p-1}$ in the real case, we obtain:

$$\Phi(g) := N(f)^{(p-2)(p_*-1)}N(Tg)^{(p_*-1)}u_{Tg} \in R(P).$$

Since T is Σ-modular we have $u_{T(\varphi\cdot u_h)} = \mathbf{1}_{\operatorname{Supp}\varphi}\cdot u_{Th}$ for every $h \in L_p(H)$ and $\varphi \in L_p$, and more generally $u_{T^k(\varphi\cdot u_h)} = \mathbf{1}_{\operatorname{Supp}\varphi}\cdot u_{T^k h}$ for every $k \geq 1$. It is easily deduced that: $u_{T^k\Phi(g)} = \mathbf{1}_{\mathbf{VS}(f)}\cdot u_{T^{k+1}g}$ for every $k \geq 0$. Then

$$u_{\Phi^n(g)} = u_{\Phi(\Phi^{n-1}(g))} = \mathbf{1}_{\mathbf{VS}(f)}\cdot u_{T\Phi^{n-1}(g)} \\ = \mathbf{1}_{\mathbf{VS}(f)}\cdot u_{T\Phi(\Phi^{n-2}(g))} = \mathbf{1}_{\mathbf{VS}(f)}\cdot u_{T^2\Phi^{n-2}(g)}\cdots \\ = \mathbf{1}_{\mathbf{VS}(f)}\cdot u_{T^n g} \tag{6}$$

for every $n \geq 1$. If $E_f g(\omega) \neq 0$ we have

$$u_{T^n g}(\omega) = \frac{\alpha_p^n(g - E_f g)(\omega) + E_f g(\omega)}{N(\alpha_p^n(g - E_f g) + E_f g)(\omega)} \longrightarrow \frac{E_f g(\omega)}{N(E_f g)(\omega)} = u_{E_f g}(\omega) \tag{7}$$

(norm convergence in H) while if $E_f g(\omega) = 0$

$$u_{T^n g}(\omega) = \frac{(g - E_f g)(\omega)}{N((g - E_f g))(\omega)} = u_{(g - E_f g)}(\omega) = u_g(\omega). \tag{7'}$$

Since $g - E_f g \perp E_f g$ we have $N(Tg) \leq N(g)$. Hence

$$N(\Phi(g)) = N(f)^{2-p_*}N(Tg)^{p_*-1} \leq N(f)^{2-p_*}N(g)^{p_*-1}. \tag{8}$$

In particular

$$N(\Phi(g)) \leq \max(N(f), N(g)). \tag{9}$$

Reiterating (8) we obtain for every $n \geq 1$

$$N(\Phi^n(g)) \leq N(f)^{(2-p_*)\Sigma_{k=0}^{n-1}(p_*-1)^k} N(g)^{(p_*-1)^n} = N(f)^{1-(p_*-1)^n} N(g)^{(p_*-1)^n}.$$

Since $0 < p_* - 1 < 1$ we obtain

$$\overline{\lim_{n\to\infty}} N(\Phi^n(g)) \leq \mathbf{1}_{\mathbf{VS}(g)} N(f). \tag{10}$$

We try now to be more precise. If $E_f g(\omega) = 0$ we have $N(Tg)(\omega) = \alpha_p N(g)(\omega)$. Hence

$$N(\Phi(g))(\omega) = N(f)(\omega)^{2-p_*}(\alpha_p N(g)(\omega))^{p_*-1}.$$

Moreover, since in this case $u_{\Phi^n(g)}(\omega) = u_g(\omega)$, we have $E_f\Phi^n(g)(\omega) = 0$ for every n, and we can reiterate. We obtain

$$N(\Phi^n(g))(\omega) = (\alpha_p^{p_*-1} N(f)(\omega)^{(2-p_*)})^{\Sigma_{k=0}^{n-1}(p_*-1)^k} N(g)(\omega)^{(p_*-1)^n}.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} N(\Phi^n(g))(\omega) &= \alpha_p^{(p_*-1)/(2-p_*)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega) \\ &= \alpha_p^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega).\end{aligned} \tag{11}$$

If now $E_f(g)(\omega) \neq 0$, we have also $E_f(\Phi^n(g))(\omega) \neq 0$ for every $n \geq 0$. Set

$$\beta_n(\omega) = \frac{N(E_f\Phi^n(g))(\omega)}{N(\Phi^n(g))(\omega)}$$

We have then

$$N(T\Phi^n(g))(\omega) \geq \beta_n(\omega) N(\Phi^n(g))(\omega)$$

and consequently:

$$N(\Phi^{n+1}(g))(\omega) \geq N(f)^{2-p_*}(\beta_n(\omega) N(\Phi^n(g)(\omega))^{p_*-1}. \tag{12}$$

On the other hand

$$\beta_n(\omega) = |\langle\langle u_{\Phi^n(g)}, u_f\rangle\rangle(\omega)| = |\langle\langle u_{T^n(g)}, u_f\rangle\rangle(\omega)| = \frac{N(E_f T^n(g))(\omega)}{N(T^n(g))(\omega)} = \frac{N(E_f g)(\omega)}{N(T^n(g))(\omega)}$$

and since $N(T^n g) = (\alpha_p^{2n} N(g - E_f g)^2 + N(E_f g)^2)^{1/2} \searrow N(E_f g)$ pointwise (as $\alpha_p < 1$) we have $\beta_n(\omega) \nearrow 1$ on the set $\{\omega \mid E_f g(\omega) \neq 0\}$. Reiterating (12) from the step $n = n_0$ we obtain then

$$\underline{\lim_{n\to\infty}} N(\Phi^n(g))(\omega) \geq (\beta_{n_0}(\omega))^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(\Phi_{n_0}(g))}(\omega) N(f)(\omega)$$

and letting $n_0 \to \infty$, we have, since $\mathbf{VS}(\Phi_n(g)) = \mathbf{VS}(g) \cap \mathbf{VS}(f)$ for every n,

$$\varliminf_{n\to\infty} N(\Phi^n(g))(\omega) \geq \mathbf{1}_{\mathbf{VS}(g)}(\omega)N(f)(\omega). \tag{13}$$

From (6), (7), (7'), and (11), (10), (13) we deduce that

$$\Phi^n(g) \to N(f)[u_{E_f(g)} + \alpha_p^{1/(p-2)}\mathbf{1}_{\{f\perp g\}}u_g\,] \tag{14}$$

almost everywhere in H-norm, hence in $L_p(H)$-norm by (9) and Lebesgue's Theorem. Hence the right-hand member of (14) belongs to $R(P)$. Since $u_{E_fg} = \operatorname{sgn}\langle\langle g \mid f\rangle\rangle u_f$ the right member of (14) can be written

$$\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f + \gamma_p\mathbf{1}_{\{f\perp g\}}N(f)u_g = F_p(f,g) \tag{15}$$

where we have set $\gamma_p = \alpha_p^{1/(p-2)}$.

b) Case $1 < p < 2$.

This case is treated by duality. Set $\gamma_p = \gamma_{p_*}^{p_*-1}$ and define $F_p(f,g)$ by the formula (15). If $g = J_{p_*}g'$, $f = J_{p_*}h'$ with $f', g' \in L_{p_*}(H)$ we have

$$\operatorname{sgn}\langle\langle g \mid f\rangle\rangle = \operatorname{sgn}\langle\langle g' \mid f'\rangle\rangle.$$

Hence

$$\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f = J_{p_*}(\operatorname{sgn}\langle\langle g' \mid f'\rangle\rangle f')$$

and similarly

$$N(f) = N(J_{p^*}f') = N(f')^{p_*-1}.$$

Hence

$$N(f)u_g = N(f')^{p_*-1}u_g = J_{p_*}(N(f')u_{g'}).$$

Finally, since $\{f \perp g\} = \{f' \perp g'\}$ and J_{p_*} is additive on elements with disjoint functional supports, and positively homogeneous of degree $p_* - 1$,

$$F_p(f,g) = J_{p_*}(F_{p_*}(f',g')).$$

Then since $f' = J_pf, g' = J_pg$ belong to $R(P^*)$, the function $F_{p_*}(f',g'))$ belongs to $R(P^*)$ too by the case (a), and $F_p(f,g)$ belongs to $R(P)$. □

Corollary 3.2. *Let p and P be as in Lemma 3.1. Then for every $f, g \in R(P)$ the three elements* $\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f$, $\mathbf{1}_{\{f\perp g\}}f$ *and* $\mathbf{1}_{\{f\perp g\}}N(f)u_g$ *belong to $R(P)$.*

Proof. The set Λ of scalars λ such that the set $\{\,\omega \in \mathbf{VS}(f) \mid \frac{\langle\langle g|f\rangle\rangle(\omega)}{\langle\langle f|f\rangle\rangle(\omega)} = -\lambda\,\}$ has positive measure is at most countable. This set is also the set of λ's such that $\{(g+\lambda f) \perp f\} \cap \mathbf{VS}(f)$ has positive measure. Choose a sequence (ε_n) of positive numbers not in $\Lambda \cup (-\Lambda)$ which converges to 0. Then by Lemma 3.1

$$\operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f\rangle\rangle f \in R(P)$$

for every $n \geq 1$. Since

$$\operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f\rangle\rangle(\omega) \to \begin{cases} \operatorname{sgn}\langle\langle g \mid f\rangle\rangle(\omega) & \text{if } \langle\langle g \mid f\rangle\rangle(\omega) \neq 0, \\ \pm 1 & \text{if } \langle\langle g \mid f\rangle\rangle(\omega) = 0 \quad \text{and } f(\omega) \neq 0, \end{cases}$$

we have

$$\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f \pm \mathbf{1}_{\{f\perp g\}} f = \lim_n \operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f\rangle\rangle f \in R(P)$$

and consequently $\operatorname{sgn}\langle\langle g \mid f\rangle\rangle f$ and $\mathbf{1}_{\{f\perp g\}} f$ belong to $R(P)$. Then $F_p(f,g) - \operatorname{sgn}\langle\langle g \mid f\rangle\rangle f = \gamma_p \mathbf{1}_{\{f\perp g\}} N(f) u_g$ belongs to $R(P)$ too. □

Corollary 3.3. *Let p and P be as in Lemma 3.1. Then for every $f, g \in R(P)$ we have $\mathbf{1}_{\mathbf{VS}(g)} f \in R(P)$.*

Proof. By Cor. 3.2, $h := G(f,g) := \mathbf{1}_{\{f\perp g\}} N(f) u_g$ belongs to $R(P)$. Then $G(h,f) = \mathbf{1}_{\{f\perp g\}} \mathbf{1}_{\{u_g \neq 0\}} N(f) u_f = \mathbf{1}_{\mathbf{VS}(g)\cap\{f\perp g\}} f$ belongs to $R(P)$ too. By Cor. 3.2, $f - \mathbf{1}_{\{f\perp g\}} f = \mathbf{1}_{\{f\not\perp g\}} f \in R(P)$, thus $\mathbf{1}_{\mathbf{VS}(g)} f = \mathbf{1}_{\{f\not\perp g\}} f + \mathbf{1}_{\mathbf{VS}(g)\cap\{f\perp g\}} f \in R(P)$. □

Remark 3.4. In the complex case, for every $f, g \in R(P)$ the elements $\operatorname{sgn}(\operatorname{Re}\langle\langle g \mid f\rangle\rangle) f$ and $\mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g|f\rangle\rangle)=0\}} f$ belong to $R(P)$ too. Indeed, $L_p(H)$ is a real Hilbert-valued $L_p(K)$ space, where K is the real vector space H equipped with the scalar product $(x,y)_K = \operatorname{Re}(x \mid y)_H$. As a consequence, the element $\mathbf{1}_{\{\operatorname{Re}\langle\langle g|f\rangle\rangle>0\}} f = \frac{1}{2}(\operatorname{sgn}\operatorname{Re}\langle\langle g \mid f\rangle\rangle + \mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g|f\rangle\rangle)\neq 0\}}) f$ belongs to $R(P)$.

Lemma 3.5. *Let p and P be as in Lemma 3.1. For every $f, g \in R(P)$ denote by $\Sigma_{f,g}$ the σ-field generated by the element $\frac{\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle}$. Then for every $\Sigma_{f,g}$-measurable function φ such that $\varphi \cdot N(f) \in L_p(\Omega, \Sigma, \mu)$, the element $\varphi \cdot f$ belongs to $R(P)$.*

Proof. Since $R(P)$ is a closed linear subspace, it is sufficient to prove this for indicator functions of $\Sigma_{f,g}$-measurable sets. The sigma-algebra $\Sigma_{f,g}$ is generated by the sets $\{\frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\}$, $\{-\frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\}$, $\{\frac{\operatorname{Im}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\}$, and $\{-\frac{\operatorname{Im}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda\}$, $\lambda \in \mathbb{R}_+$. If $A_{f,g,\lambda} = \{ \frac{\operatorname{Re}\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle} > \lambda \}$ we have $A_{f,g,\lambda} = \{\operatorname{Re}\langle\langle g - \lambda f \mid f\rangle\rangle > 0\}$, hence $\mathbf{1}_{A_{f,g,\lambda}} f \in R(P)$ by Rem. 3.4. The conclusion is the same for the three others kinds of sets (replacing g by $-g$ or $\pm ig$). Now if $\mathbf{1}_B f \in R(P)$ then $A_{f,g,\lambda} \cap B = A_{f',g,\lambda}$ with $f' = \mathbf{1}_B f$, hence $\mathbf{1}_{A_{f,g,\lambda}\cap B} f = \mathbf{1}_{A_{f',g,\lambda}} f' \in R(P)$. It results that the class $\mathcal{C}$ of the sets $A \in \Sigma$ such that $\mathbf{1}_A f \in R(P)$ contains finite intersections of sets of the four preceding types. Since $\mathcal{C}$ is closed by complementation and monotone limits, it contains the sigma-algebra $\Sigma_{f,g}$. □

Corollary 3.6. *Let p and P be as in Lemma 3.1. For every $f \in R(P)$ we have $E_f P = P E_f$.*

Proof. Let $g \in R(P)$. Applying Lemma 3.5 to the function $\varphi = \frac{\langle\langle g|f\rangle\rangle}{\langle\langle f|f\rangle\rangle}$ we obtain that $E_f g \in R(P)$. Hence for every $h \in L_p(H)$, we have $E_f Ph \in R(P)$, i.e. $E_f Ph = PE_f Ph$; thus $E_f P = PE_f P$. Similarly, reasoning with the contractive projection P^* in $L_{p_*}(H)$, and the element $J_p f$ of $R(P^*)$, we have $E_{J_p f}P^* = P^* E_{J_p f} P^*$. Dualizing we obtain $PE^*_{J_p f} = PE^*_{J_p f}P$. We claim that $E^*_f = E_{J_p f}$. This will show that $PE_f = PE_f P = E_f P$. Let us show this claim. Since $u_{J_p f} = u_f$, we have for every $g \in L_p(H)$ and $h' \in L_{p_*}(H)$

$$\begin{aligned}\langle E_f g, h'\rangle &= \int \langle\langle E_f g, h'\rangle\rangle \, d\mu = \int \langle\langle \langle\langle g, u_f\rangle\rangle u_f, h'\rangle\rangle \, d\mu \\ &= \int \langle\langle g, u_f\rangle\rangle \langle\langle u_f, h'\rangle\rangle \, d\mu \\ &= \int \langle\langle g, \langle\langle h', u_f\rangle\rangle u_f\rangle\rangle \, d\mu \\ &= \int \langle\langle g, E_{J_p f} h'\rangle\rangle \, d\mu = \langle g, E_{J_p f} h'\rangle \qquad \square\end{aligned}$$

Remark. The preceding proof of Cor. 3.6 is essentially a real one. In the complex case it can be replaced by a shorter one, of more algebraic nature, due to Arazy and Friedman in the context of spaces C_p (see [2]). It seemed interesting to us to reproduce this proof in the Annex (see §6), after simplifying it considerably by eliminating the unnecessary non-commutative apparatus.

4. The range of a contractive projection

This section is devoted to the proof of Thm. 0.1, which consists in four lemmas.

Lemma 4.1. *The closed $L_\infty(\Sigma)$-module Z generated by $R(P)$ in $L_p(H)$ is generated (as L_∞-module) by a family $(f_\alpha)_{\alpha\in A}$ of pairwise orthogonal elements of $R(P)$. We have in fact a Schauder (orthogonal) decomposition*

$$Z = \bigoplus_{\alpha\in A} L_p(\Omega) \cdot u_{f_\alpha}$$

Proof. Let $(f_\alpha)_{\alpha\in A}$ be a maximal family of pairwise orthogonal non zero elements of $R(P)$ and Z_0 be the closed $L_\infty(\Sigma)$-submodule generated by the family $(f_\alpha)_{\alpha\in A}$. Let Q_{Z_0} be the orthogonal projection onto Z_0. By the proof of Lemma 1.1 we know that $Q_{Z_0} = \sum_{\alpha\in A} E_{f_\alpha}$ (convergence in strong operator topology). Hence, by Lemma 2.2 if $p = 1$, resp. Cor. 3.6 if $p > 1$, $Q_{Z_0} f \in R(P)$ for every $f \in R(P)$. Since Q_{Z_0} is orthogonal and $f_\alpha \in R(Q_{Z_0})$ we have $(f - Q_{Z_0} f) \perp f_\alpha$ for every $\alpha \in A$. By maximality of the system (f_α) we deduce that

$$f = Q_{Z_0} f \text{ for every } f \in R(P)$$

i.e. $Q_{Z_0}P = P$. Then $Z_0 = R(Q_{Z_0})$ is a closed L_∞-module containing $R(P)$ and generated by a subset of $R(P)$; hence it coincides with the closed L_∞-module generated by $R(P)$. □

Lemma 4.2. *There exists a sub-σ-algebra $\mathcal{F}$ of Σ containing the vectorial function supports of all elements of $R(P)$ such that for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$, the product $\varphi \cdot u_f$ belongs to $R(P)$ iff $\mathbf{1}_{\mathbf{VS}(f)} N(f)^{-1}\varphi$ is $\mathcal{F}$-measurable. In particular $R(P)$ is a $L_\infty(\Omega, \mathcal{F}, \mu)$-submodule.*

Proof. Since $PE_f = E_f PE_f$ by Lemma 2.1 (if $p = 1$) or by Cor. 3.6 (if $p > 1$), we have $P(\varphi \cdot u_f) \in L_p(\Omega) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. We may write $P(\varphi \cdot u_f) = (\widetilde{P}_f \varphi) \cdot u_f$, with $\mathrm{Supp}(\widetilde{P}_f \varphi) \subset \mathbf{VS}(f)$. Clearly $\widetilde{P}_f$ is linear, $\widetilde{P}_f^2 = \widetilde{P}_f$ and

$$\|\widetilde{P}_f \varphi\|_p = \|P(\varphi \cdot u_f)\| \leq \|\varphi \cdot u_f\| \leq \|\varphi\|_p,$$

hence $\widetilde{P}_f$ is a contractive projection in $L_p(\Omega, \Sigma, \mu)$. Moreover $\widetilde{P}_f(N(f)) = N(f)$ and $\widetilde{P}_f \psi = 0$ for every $\psi \in L_p(\Omega, \Sigma, \mu)$ disjoint from $N(f)$.

It results from Douglas' theorem (in case $p = 1$) or Andô's theorem (in case $p > 1$) that $\widetilde{P}_f$ is positive and

$$\widetilde{P}_f(\varphi) = N(f)\mathbb{E}^{\mathcal{F}_f}_{\nu_f}\left(\frac{\mathbf{1}_{\mathrm{Supp}(N(f))}\varphi}{N(f)}\right)$$

where $\mathbb{E}^{\mathcal{F}_f}_{\nu_f}$ is the conditional expectation with respect to some subalgebra $\mathcal{F}_f$ of Σ containing $\mathbf{VS}(f)$ and to the measure $\nu_f = N(f)^p \, d\mu$. (We may assume that $\Omega \setminus \mathbf{VS}(f)$ is an atom of $\mathcal{F}_f$). In particular $L_p(\Omega, \Sigma, \mu) \cdot u_f \cap R(P) = L_p(\Omega, \mathcal{F}_f, \nu_f) \cdot f$ is a $L_\infty(\Omega, \mathcal{F}_f, \mu)$-module.

Let us denote $\mathbb{E}^f \psi = \mathbb{E}^{\mathcal{F}_f}_{\nu_f}(\mathbf{1}_{\mathbf{VS}(f)}\psi)$, we have then $P(\psi \cdot f) = \mathbb{E}^f(\psi) \cdot f$ for every $\psi \in L_\infty(\Omega, \Sigma, \mu)$. Let now $f, g \in R(P)$. If $g = h \cdot u_f$ with $h \in L^p(\Omega)$ then $\frac{h}{N(f)}$ is $\mathcal{F}_f$-measurable and for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ we have

$$\mathbb{E}^g(\varphi) \cdot g = P(\varphi h \cdot u_f) = N(f)\mathbb{E}^f\Big(\frac{\varphi \cdot h}{N(f)}\Big) \cdot u_f = h\mathbb{E}^f(\varphi) \cdot u_f = \mathbb{E}^f(\varphi) \cdot g,$$

Hence

$$\mathbb{E}^g(\varphi) = \mathbf{1}_{\mathbf{VS}(g)}.\mathbb{E}^f(\varphi) = \mathbf{1}_{\mathrm{Supp}\, h}\mathbb{E}^f(\varphi). \tag{16}$$

Let now g be a general element of $R(P)$. For every $\varphi \in L_\infty(\Omega)$ the equation

$$P(\varphi \cdot (f + g)) = P(\varphi \cdot f) + P(\varphi \cdot g)$$

is equivalent to

$$\mathbb{E}^{f+g}(\varphi) \cdot (f + g) = \mathbb{E}^f(\varphi) \cdot f + \mathbb{E}^g(\varphi) \cdot g. \tag{17}$$

Let $g = h \cdot u_f + g'$ be the orthogonal decomposition, i.e. $h = \langle\!\langle g \mid u_f \rangle\!\rangle$ and $g' \perp f$. Note that $h \cdot u_f = E_f g \in R(P)$. Set $A = \mathbf{VS}(f)$, $B = \mathbf{VS}(g)$ and $B' = \mathbf{VS}(g')$. Taking the images of both sides of (17) by the orthogonal projection $I - E_f$ we obtain

$$\mathbb{E}^{f+g}(\varphi) \cdot g' = \mathbb{E}^g(\varphi) \cdot g',$$

hence $\mathbf{1}_{B'}\mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{B'}\mathbb{E}^g(\varphi)$. Then by (17) again, $\mathbf{1}_{B'}\mathbb{E}^{f+g}(\varphi)f = \mathbf{1}_{B'}\mathbb{E}^f(\varphi)f$ and finally

$$\mathbf{1}_{A\cap B'}\mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{A\cap B'}\mathbb{E}^f(\varphi) = \mathbf{1}_{A\cap B'}\mathbb{E}^g(\varphi). \tag{18}$$

On the other hand similarly to (17) we have

$$\mathbb{E}^{h\cdot u_f - g}(\varphi) \cdot (h \cdot u_f - g) = \mathbb{E}^{hu_f}(\varphi) \cdot hu_f - \mathbb{E}^g(\varphi) \cdot g.$$

Since $h \cdot u_f - g = -g'$ we deduce that

$$\mathbf{1}_{\Omega\setminus B'}\mathbb{E}^{hu_f}(\varphi) \cdot hu_f = \mathbf{1}_{\Omega\setminus B'}\mathbb{E}^g(\varphi) \cdot g,$$

hence

$$\mathbf{1}_{B\setminus B'}\mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{B\setminus B'}\mathbb{E}^g(\varphi). \tag{19}$$

We have $\mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{\operatorname{Supp} h}\mathbb{E}^f(\varphi)$ by eq. (16). Hence since $B \setminus B' \subset \operatorname{Supp} h$, eq. (19) gives

$$\mathbf{1}_{B\setminus B'}\mathbb{E}^f(\varphi) = \mathbf{1}_{B\setminus B'}\mathbb{E}^g(\varphi)$$

which together with eq. (18) gives

$$\mathbf{1}_{A\cap B}\mathbb{E}^f(\varphi) = \mathbf{1}_{A\cap B}\mathbb{E}^g(\varphi)$$

for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$. In particular

$$\begin{aligned} \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)} &= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^g(\mathbf{1}_{\mathbf{VS}(g)}) \\ &= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(g)}) \\ &= \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}), \end{aligned}$$

hence

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}) \geq \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}$$

and since $\mathbb{E}^f$ is a contraction in $L_p(\Omega, \Sigma, N(f)^p \cdot \mu)$ we have in fact

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)}) = \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)},$$

that is, $\mathbf{VS}(f) \cap \mathbf{VS}(g) \in \mathcal{F}_f$. In particular $\mathbf{1}_{\mathbf{VS}(g)} \cdot f = \mathbf{1}_{\mathbf{VS}(f)\cap\mathbf{VS}(g)} \cdot f \in R(P)$. More generally for every $A \in \mathcal{F}_g$ its trace $\mathbf{VS}(f) \cap A$ belongs to $\mathcal{F}_f$ (as is easily seen by treating separately the cases $A \subset \mathbf{VS}(g)$ and $A = \Omega \setminus \mathbf{VS}(g)$). Let $\mathcal{F}$ be the σ-algebra consisting of sets $A \in \Sigma$ such that $A \cap \mathbf{VS}(f)$ belongs to $\mathcal{F}_f$ for every $f \in R(P)$. Then for every $f \in R(P)$ and $\varphi \in L_0(\Omega, \Sigma, \mu)$ the function $\mathbf{1}_{\mathbf{VS}(f)}\varphi$ is $\mathcal{F}$ measurable iff it is $\mathcal{F}_f$-measurable, and the Lemma follows. □

Lemma 4.3. *There is a weight $w \in L_0(\Omega,\Sigma,\mu)$ with support $\mathbf{VS}(R(P))$ such that for every $f \in R(P)$, $w^{-1}N(f)$ is $\mathcal{F}$-measurable.*

Proof. a) First we claim that for every $f, g \in R(P)$ then $\mathbf{1}_{\mathbf{VS}(f)}\frac{N(g)}{N(f)}$ is $\mathcal{F}$-measurable. Since $E_f g = \langle\!\langle g \mid u_f\rangle\!\rangle u_f \in R(P)$ by Lemma 2.2, it results from Lemma 4.2 that $N(f)^{-1}\langle\!\langle g \mid u_f\rangle\!\rangle = N(f)^{-2}\langle\!\langle g \mid f\rangle\!\rangle$ is $\mathcal{F}$-measurable; hence its absolute value $N(f)^{-2}$ $|\langle\!\langle g \mid f\rangle\!\rangle|$ is $\mathcal{F}$-measurable, and similarly $N(g)^{-2}|\langle\!\langle f \mid g\rangle\!\rangle|$ is $\mathcal{F}$-measurable too. Then the ratio of these functions, that is $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g|f\rangle\!\rangle}N(g)^2N(f)^{-2}$ is $\mathcal{F}$-measurable, and so is its square root $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g|f\rangle\!\rangle}N(g)N(f)^{-1}$. Replacing g by $g_\varepsilon = g + \varepsilon f$, $\varepsilon > 0$ we obtain that $\mathbf{1}_{\mathrm{Supp}\langle\!\langle g_\varepsilon|f\rangle\!\rangle}N(g_\varepsilon)N(f)^{-1}$ is $\mathcal{F}$ measurable. When $\varepsilon \to 0$ we have $g_\varepsilon \to g$, $N(g_\varepsilon) \to N(g)$ (in L_p-norm) and $\mathrm{Supp}\langle\!\langle g_\varepsilon \mid f\rangle\!\rangle \to \mathrm{Supp}\, N(f) = \mathbf{VS}(f)$ (in probability). At the limit $\mathbf{1}_{\mathbf{VS}(f)}\frac{N(g)}{N(f)}$ is $\mathcal{F}$-measurable.

b) Let $(f_i)_{i\in I}$ be a maximal family of non zero elements in $R(P)$ with pairwise almost disjoint functional supports $\mathbf{VS}(f_i)$. Then $\mathbf{VS}(R(P)) = \bigvee_{i\in I}\mathbf{VS}(f_i)$: if $f \in R(P)$ then, since $S = \bigvee_{i\in I}\mathbf{VS}(f_i)$ belongs to $\mathcal{F}$, so does its complementary set S^c, and thus $\mathbf{1}_{S^c}f \in R(P)$; then, by maximality of the family (f_i), we have $\mathbf{1}_{S^c}.f = 0$, that is, $f = \mathbf{1}_S \cdot f$. We set $w = \sum_{i\in I} N(f_i)$ (which converges in $L_0(\Omega,\Sigma,\mu)$): this is a Σ-measurable weight with support $\mathbf{VS}(R(P))$. For every $f \in R(P)$ and every $i \in I$, $\mathbf{1}_{\mathbf{VS}(f_i)}w^{-1}N(f) = \mathbf{1}_{\mathbf{VS}(f_i)}N(f_i)^{-1}N(f)$ is $\mathcal{F}$-measurable; hence $w^{-1}N(f) = \sum_{i\in I}\mathbf{1}_{\mathbf{VS}(f_i)}w^{-1}N(f)$ is $\mathcal{F}$-measurable. □

We can now give the

Proof of the Thm. 0.1. Consider the new measure $\nu = w^p \cdot \mu$, which has support $\Omega_P = \mathbf{VS}(R(P))$ and set $T : L_p(\Omega_P,\Sigma_P,\mu) \to L_p(\Omega_P,\Sigma_P,\nu)$, defined by $Tf = w^{-1}f$ (we denote by Σ_P the trace of Σ on Ω_P). Then T is an isometry; $Y := (T\otimes\mathrm{Id_H})(R(P))$ is a $L_\infty(\mathcal{F}_P)$-module isometric to $R(P)$ and for every $f \in Y$ its new random norm $\widetilde{N}(f) = w^{-1}N(f)$ belongs to $L_p(\Omega_P,\mathcal{F}_P,\nu)$. It results from an argument in [13] that Y is isometric to $\left(\bigoplus_{i\in I} L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};H_i)\right)_{\ell_p}$, for some families (Ω_i) of pairwise almost disjoint sets in $\mathcal{F}$ and (H_i) of Hilbert spaces. Set then $\widehat{w}_i = (\mathbb{E}(\mathbf{1}_{\Omega_i}\cdot w^p \mid \mathcal{F}))^{1/p}$, and define an isometry $S_i : L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i}) \to L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\mu_{|\Omega_i})$ by $S_i f = \widehat{w}_i \cdot f$. Then each $S_i\otimes\mathrm{Id_{H_i}}$ is an onto isometry $L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};H_i) \to L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\mu_{|\Omega_i};H_i)$; the collection of these isometries induces an isometry of the corresponding ℓ_p-direct sums. The proof of Thm. 0.1 is complete. □

Let us finally adapt to the present situation the argument of [13] for the commodity of the reader (and for further reference in Section 5).

Lemma 4.4. *Let (Ω,Σ,ν) be a localizable measure space, $\mathcal{F}$ be a sub-sigma algebra such that $(\Omega,\mathcal{F},\nu)$ is still localizable and H be a Hilbert space. Let Y be a closed $L_\infty(\mathcal{F})$-submodule of $L_p(\Omega,\Sigma,\nu;H)$ such that for every $f \in Y$ its random norm $N(f)$ is $\mathcal{F}$-measurable. Then there exist a family $(\Omega_i)_{i\in I}$ of pairwise almost disjoint members of $\mathcal{F}$, a family $(\mathcal{H}_i)$ of Hilbert spaces (of lower Hilbertian dimension than H) and a random norm preserving isometry from Y onto $\left(\bigoplus_{i\in I} L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)\right)_{\ell_p}$.*

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Proof. Note that by an elementary polarization argument all the scalar products $\langle\!\langle f \mid g\rangle\!\rangle$, $f, g \in Y$ are $\mathcal{F}$-measurable. Hence for every $f \in Y$, the projection E_f restricts to a projection from Y onto $L_p(\Omega,\mathcal{F}) \cdot u_f$. It results that for every closed $L_\infty(\mathcal{F})$-submodule Z of Y there is an orthogonal projection from Y onto Z (which is the restriction of the orthogonal projection from $L_p(\Omega,\Sigma;H)$ onto the closed $L_\infty(\Sigma)$-submodule generated by Z). In particular $Y = Z \oplus (Z^\perp \cap Y)$.

Remark that if $A \in \mathcal{F}$ is ν-sigma-finite and $M \subset Y$ is a closed $L_\infty(\mathcal{F})$-submodule such that $\mathbf{VS}(M) \supset A$ then there exists $g \in M$ such that $\mathbf{VS}(g) = A$: take a maximal family (g_n) in M of norm-one elements with almost disjoint functional supports included in A; this family is necessarily at most countable and $\bigvee_n \mathbf{VS}(g_n) = A$; then set $g = \sum_n 2^{-n} g_n$.

Now we claim that for every $A \in \mathcal{F}$, $A \subset \mathbf{VS}(Y)$ with positive measure, there exists a $\mathcal{F}$-measurable subset B of A of positive measure and a family of pairwise orthogonal element $(f_\gamma)_{\alpha\in\Gamma_B}$, such that $\mathbf{VS}(f_\gamma) = B$ for every $\gamma \in \Gamma_B$, which generates $\mathbf{1}_B \cdot Y$ as closed $L_\infty(\mathcal{F})$-submodule. For, let $A' \subset A$ be a sigma-finite $\mathcal{F}$-measurable subset with positive measure, and $(g_\gamma)_{\gamma\in\Gamma}$ be a maximal family of pairwise orthogonal elements of Y with $\mathbf{VS}(g_\gamma) = A'$. If this family generates $\mathbf{1}_{A'}.Y$ as closed $L_\infty(\mathcal{F})$-submodule we can take $B = A'$. If not, consider the set $M = \{\, f \in Y \mid f \perp g_\gamma, \forall \gamma \in \Gamma \,\}$. Then M is a closed $L_\infty(\mathcal{F})$-submodule of Y, and $\mathbf{VS}(M) \not\supset A'$ by the maximality of $(g_\gamma)_{\gamma\in\Gamma}$ (and the preceding remark). Let $B = A' \setminus \mathbf{VS}(M)$, then $(\mathbf{1}_B g_\gamma)_{\gamma\in\Gamma}$ is a maximal family in $\mathbf{1}_B \cdot Y$ of nonzero, pairwise orthogonal elements of $\mathbf{1}_B \cdot Y$. Consequently it generates $\mathbf{1}_B \cdot Y$ as $L_\infty(\mathcal{F})$-submodule, and moreover $\mathbf{VS}(\mathbf{1}_B g_\gamma) = B$ for every $\gamma \in \Gamma$.

Let now $(\Omega_i)_{i\in I}$ be a maximal family of $\mathcal{F}$-measurable almost disjoint subsets of $\mathbf{VS}(Y)$ of positive measure, such that there exists for each $i \in I$ a family $(f^i_\gamma)_{\gamma\in\Gamma_i}$ of pairwise orthogonal elements with $\mathbf{VS}(f^i_\gamma) = \Omega_i$ for every $\gamma \in \Gamma_i$, which generates $\mathbf{1}_{\Omega_i} \cdot Y$ as closed $L_\infty(\mathcal{F})$-submodule. By the claim, we have $\bigvee_{i\in I} \Omega_i = \mathbf{VS}(Y)$. Every $f \in \mathbf{1}_{\Omega_i} \cdot Y$ can be written $f = \sum_{\gamma\in\Gamma_i} \varphi_\gamma f^i_\gamma$ with $\varphi_\gamma \in L_0((\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$; then $N(f) = (\sum_{\gamma\in\Gamma_i} |\varphi_\gamma|^2 N(f^i_\gamma)^2)^{1/2} \in L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$.

Note that, by refining if necessary the "partition" (Ω_i) we may suppose that each Ω_i has finite ν-measure. Then, replacing each f^i_γ by $u_{f^i_\gamma} = N(f^i_\gamma)^{-1} f^i_\gamma$, we may assume that $N(f^i_\gamma) = \mathbf{1}_{\Omega_i}$. We have then $N(f) = (\sum_{\gamma\in\Gamma_i} |\varphi_\gamma|^2)^{1/2}$ for each $f = \sum_{\gamma\in\Gamma_i} \varphi_\gamma f^i_\gamma$ in $\mathbf{1}_{\Omega_i} \cdot Y$. Let $\mathcal{H}_i = \ell^2(\Gamma_i)$. Then $T_i : L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu; \mathcal{H}_i) \to \mathbf{1}_{\Omega_i} \cdot Y$, $\sum_{\gamma\in\Gamma_i} \varphi_\gamma e_\gamma \mapsto \sum_{\gamma\in\Gamma_i} \varphi_\gamma u_{f^i_\gamma}$ is an (onto) isometry (preserving the random norm), and finally Y is isometric to $\big(\bigoplus_{i\in I} L_p(\Omega_i; \mathcal{H}_i)\big)_{\ell_p}$.

For proving the assertion about the Hilbertian dimension of $\mathcal{H}_i$, suppose that for some $i \in I$, the Hilbertian dimension d_H of H is strictly smaller than that of $\mathcal{H}_i$, $d_{\mathcal{H}_i}$. We distinguish two cases:

(i) if H is finite dimensional: select a finite subset Γ'_i of Γ_i with cardinality $d_H + 1$; since $\langle\!\langle f^i_\gamma \mid f^i_\delta\rangle\!\rangle = 0$ for every $\gamma \neq \delta \in \Gamma'_i$, there exists $\omega \in \Omega$ such that $\langle\!\langle f^i_\gamma \mid f^i_\delta\rangle\!\rangle(\omega) = 0$, i.e. the vectors $f^i_\gamma(\omega)$, $\gamma \in \Gamma'_i$ of H are pairwise orthogonal: a contradiction.

(ii) if H is infinite dimensional: for every $x \in H$ the set $\{\gamma \in \Gamma_i \mid \langle\langle f^i_\gamma \mid \mathbf{1}_{\Omega_i} x\rangle\rangle \neq 0\}$ is at most countable (since $\sum_\gamma |\langle\langle f^i_\gamma \mid \mathbf{1}_{\Omega_i} x\rangle\rangle|^2 \leq N(\mathbf{1}_{\Omega_i} \cdot x)^2 = \mathbf{1}_{\Omega_i}\|x\|^2$), hence if D is a dense set in H of cardinality d_H, the set $\{\, \gamma \in \Gamma_i \mid \exists x \in D, \langle\langle f^i_\gamma \mid \mathbf{1}_{\Omega_i} x\rangle\rangle \neq 0 \,\}$ has cardinality $d_H < d_{\mathcal{H}_i} = \#\Gamma_i$. Hence there exists some $\gamma \in \Gamma_i$ such that $f^i_\gamma \perp \mathbf{1}_{\Omega_i} x$ for every $x \in D$, and thus for every $x \in H$, which means $f^i_\gamma = 0$, a contradiction. □

Remark 4.5. The final argument in the proof of Lemma 4.4 shows indeed that if $L_p(\Omega, \Sigma, \nu; \mathcal{H})$ embeds in $L_p(\Omega, \Sigma, \nu; H)$ by a modularly isometric map then $\dim \mathcal{H} \leq \dim H$.

Remark 4.6. In a forthcoming paper ([10]) it will be proved that contractively complemented sublattices of $L_p(L_q)$ are isometric to "abstract $L_p(L_q)$ spaces", i.e. bands in $L_p(L_q)$ spaces. Let us show how this permits to deduce shortly the essence of Thm. 0.1 from Lemma 4.1.

As in the proof of Lemma 4.1 let $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a maximal family of non zero, pairwise orthogonal elements of $R(P)$ and $Z = \bigoplus_\alpha L_p(\Omega, \Sigma, \mu) \cdot u_{f_\alpha}$ be the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. There is clearly a Σ-modular isometry U from the closed submodule Z onto a band Y of the Banach lattice $L_p(\Omega, \Sigma, \mu; \mathcal{H})$ where $\mathcal{H}$ is the discrete Banach lattice $\ell_2(A)$, such that $Ue_\alpha = N(f_\alpha)e_\alpha$, where $(e_\alpha)_{\alpha \in A}$ is a Hilbertian basis of $\mathcal{H}$. Then $P\,|_Z$ is similar by U to a contractive projection $\widehat{P}$ of Y which preserves the spaces $Y_\alpha = L_p(A_\alpha) \cdot e_\alpha$ (where $A_\alpha = \operatorname{Supp} N(f_\alpha)$) by Lemma 2.1 if $p = 1$ and Cor. 3.6 if $p > 1$, as well as the elements $N(f_\alpha) \cdot e_\alpha$. By the classical (scalar) theorem of Douglas if $p = 1$, Andô if $p > 1$, $\widehat{P}\,|_{Y_\alpha}$ is positive and its image is a sublattice of Y_α. Since $Y = \bigoplus_\alpha Y_\alpha$ is a decomposition in disjoint subbands, $\widehat{P}$ is itself positive and its range is a sublattice of Y, hence of $L_p(\mathcal{H})$. By the analysis of contractive projections on sublattices in $L_p(L_q)$-spaces developed in [10], the range $R(\widehat{P})$ is an abstract $L_p(L_2)$-space, hence by [13] it is Banach-isometric to a ℓ_p-direct sum $\bigoplus_{i \in I} L_p(\Omega_i, H_i)$, where the H_i are Hilbert spaces. □

5. Structure of the contractive projections

Theorem 5.1. *Let $1 \leq p < \infty$, $p \neq 2$. For every contractive projection P of $L_p(H)$ there exist a family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, a positive contractive projection $\widetilde{P}$ of $L_p(\Omega)$ and, if $p = 1$, a contractive linear operator $V : L_1(H) \to L_1(H)$ verifying $\ker V \supset \mathbf{1}_A L_1(H)$ where $A = \bigvee_{\gamma \in \Gamma} \mathbf{VS}(u_\gamma)$, and $R(V) \subset \sum_\gamma R(\widetilde{P}) \cdot u_\gamma$, such that:*

$$Pf = \begin{cases} \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle) u_\gamma & \text{if } p \neq 1, \\ \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle) u_\gamma + V(f) & \text{if } p = 1, \end{cases} \tag{20}$$

for every $f \in L_p(H)$.

Conversely for every family $(u_\gamma)_{\gamma\in\Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, every positive contractive projection $\widetilde{P}$ of $L_p(\Omega)$ [and every linear contraction V of $L_1(H)$ satisfying the previous conditions of kernel and range in the case $p = 1$], the formula (20) *defines a contractive projection P of $L_p(H)$.*

Moreover if $p \neq 1$ the inequality $N(Pf) \leq \widetilde{P}(N(f))$ holds for every $f \in L_p(H)$ [this happens also for a contractive projection of $L_1(H)$ for which the operator V of formula (20) *is zero]. ($\widetilde{P}$ is a "majorizing L_p-contraction" for P in the terminology of [8]).*

The proof of Thm. 5.1 will require the two following Lemmas, the first of which is specific to the $p = 1$ case:

Lemma 5.2. *Let P be a contractive projection in $L_1(H)$. Then $Pf = 0$ for every $f \in L_1(H)$ with $\mathbf{VS}(f) \subset \mathbf{VS}(R(P))$ and $f \perp R(P)$.*

Proof. Assume that $f \perp R(P)$ and $\mathbf{VS}(f) \subset \mathbf{VS}(h)$ for some $h \in R(P)$. Then $g := Pf + \mathbf{1}_{(\mathbf{VS}(Pf))^c} h$ belongs to $R(P)$ and $\mathbf{VS}(g) \supset \mathbf{VS}(f) \cup \mathbf{VS}(Pf)$. We have for every $t > 0$:

$$\begin{aligned}\int (N(g)^2 + t^2 N(f)^2)^{1/2}\, d\mu &= \|g + tf\| \\ &\geq \|P(g + tf)\| = \|g + tPf\| \\ &= (1 + t)\|Pf\| + \|\mathbf{1}_{(\mathbf{VS}(Pf))^c} \cdot h\| \\ &= \|g\| + t\|Pf\|.\end{aligned}$$

Hence:

$$\|Pf\| \leq \lim_{t\to 0} \left(\frac{\|g + tf\| - \|g\|}{t} \right) = \lim_{t\to 0} \int \frac{(N(g)^2 + t^2 N(f)^2)^{1/2} - N(g)}{t}\, d\mu = 0. \quad \square$$

Lemma 5.3. *Let P be a contractive projection in $L_p(H)$. There exists a positive contractive projection $\widetilde{P}$ on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi \cdot u_f) = (\widetilde{P}\varphi) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$.*

Proof. Let $\mathcal{F}$ be the σ-algebra of Lemma 4.2 and w be the weight of Lemma 4.3. Define $\widetilde{P}_f(\varphi)$ as in the proof of Lemma 4.2. Recall that for every $f \in R(P)$ the function $w^{-1}N(f)$ is $\mathcal{F}$-measurable. We have then for every $h \in L_\infty(\Omega, \mathcal{F}, \mu)$:

$$\begin{aligned}\int \widetilde{P}_f(\varphi) h N(f)^{p-1}\, d\mu &= \int \mathbb{E}^{\mathcal{F}}_{N(f)^p\cdot\mu}(N(f)^{-1}\mathbf{1}_{\mathbf{VS}(f)}\varphi) h\, N(f)^p \cdot d\mu \\ &= \int \mathbf{1}_{\mathbf{VS}(f)}\varphi \cdot h N(f)^{p-1} \cdot d\mu\end{aligned}$$

$$= \int (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) \cdot (\mathbf{1}_{\mathbf{VS}(f)} h(w^{-1}N(f))^{p-1}) w^p \cdot d\mu$$
$$= \int \mathbb{E}^{\mathcal{F}}_{w^p\mu}(\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) \mathbf{1}_{\mathbf{VS}(f)} h(w^{-1}N(f))^{p-1} w^p \cdot d\mu$$
$$= \int \mathbf{1}_{\mathbf{VS}(f)} w \mathbb{E}^{\mathcal{F}}_{w^p\mu}(\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1}\varphi) h N(f)^{p-1} d\mu.$$

Hence $\widetilde{P}_f\varphi = \mathbf{1}_{\mathbf{VS}(f)}\widetilde{P}\varphi$ if we set $\widetilde{P}\varphi = w\mathbb{E}^{\mathcal{F}}_{w^p\cdot\mu}(\mathbf{1}_{\mathbf{VS}(R(P))}w^{-1}\varphi)$ for every $\varphi \in L_p(\Omega,\Sigma,\mu)$. Then $\widetilde{P}$ is a positive contractive projection in $L_p(\Omega,\Sigma,\mu)$ and $P(\varphi.u_f) = \widetilde{P}(\varphi)\cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega,\Sigma,\mu)$. □

Proof of Thm. 5.1. Let Q be the orthogonal projection from $L_p(H)$ onto the closed submodule generated by $R(P)$. It results from Lemma 4.1 that if $(f_\gamma)_{\gamma\in\Gamma}$ is a maximal family of pairwise orthogonal elements of $R(P)$ then $Q = \sum_{\gamma\in\Gamma} E_{f_\gamma}$ (convergence for s.o.t.), hence $PQ = \sum_{\gamma\in\Gamma} PE_{f_\gamma}$. If $p > 1$ we know by Cor. 3.6 that $E_{f_\gamma}P = PE_{f_\gamma}$ for every γ, hence $P = QP = PQ$. If $p = 1$ let $\Pi : L_1(H) \to L_1(H)$ the projection defined by $\Pi f = \mathbf{1}_{\mathbf{VS}(R(P))}\cdot f$, then Π and $I-\Pi$ are contractive. We have $Q\Pi = \Pi Q = Q$ and it results from the preceding Lemma 5.2 that $P(I-Q)\Pi = 0$. Hence $P = PQ + V$, where $V = P(I-\Pi)$.

Let us express now PE_f when $f \in R(P)$. If $\widetilde{P}$ is the positive projection in $L_p(\Omega,\Sigma,\mu)$ defined in Lemma 5.3 we have for every $g \in L_p(\Omega,\Sigma,\mu;H)$

$$PE_f g = P(\langle\langle g \mid u_f\rangle\rangle \cdot u_f) = \widetilde{P}(\langle\langle g \mid u_f\rangle\rangle)\cdot u_f$$

The formula (20) in Thm. 5.1 is now clear if we set $u_\gamma = u_{f_\gamma}$.

Conversely given (u_γ), $\widetilde{P}$ and V, let us prove first that P is a contraction. We have for every finite subset G of Γ (using the positivity of $\widetilde{P}$):

$$\begin{aligned} N\Big(\sum_{\gamma\in G}\widetilde{P}(\langle\langle f\mid u_\gamma\rangle\rangle)u_\gamma\Big) &= \Big(\sum_{\gamma\in G}|\widetilde{P}(\langle\langle f\mid u_\gamma\rangle\rangle)|^2\Big)^{1/2} \\ &= \bigvee\Big\{\Big|\sum_{\gamma\in G} a_\gamma\widetilde{P}\langle\langle f\mid u_\gamma\rangle\rangle\Big|\;\Big|\; a_\gamma\in\mathbb{C}, \sum_{\gamma\in G}|a_\gamma|^2\le 1\Big\} \\ &\le \widetilde{P}\Big(\bigvee\Big\{\Big|\sum_{\gamma\in G} a_\gamma\langle\langle f\mid u_\gamma\rangle\rangle\Big|\;\Big|\; a_\gamma\in\mathbb{C}, \sum_{\gamma\in G}|a_\gamma|^2\le 1\Big\}\Big) \\ &= \widetilde{P}\Big(\Big(\sum_{\gamma\in G}|\langle\langle f\mid u_\gamma\rangle\rangle|^2\Big)^{1/2}\Big). \end{aligned}$$

Hence $\|\sum_{\gamma\in G}\widetilde{P}(\langle\langle f \mid u_\gamma\rangle\rangle)u_\gamma\|^p \le \int(\sum_{\gamma\in G}|\langle\langle f\mid u_\gamma\rangle\rangle|^2)^{p/2}\,d\mu$ and the sum $P_0 f := \sum_{\gamma\in\Gamma}\widetilde{P}(\langle\langle f\mid u_\gamma\rangle\rangle)u_\gamma$ converges in $L_p(H)$. Moreover

$$N(P_0 f) \le \widetilde{P}\Big(\Big(\sum_{\gamma\in\Gamma}|\langle\langle f\mid u_\gamma\rangle\rangle|^2\Big)^{1/2}\Big) \le \widetilde{P}(N(\mathbf{1}_A\cdot f))$$

(see section 1.2 and the proof of Lemma 1.1) and

$$\|P_0 f\| \leq \|\mathbf{1}_A \cdot f\|$$

where $A = \bigvee_\gamma \mathbf{VS}(u_\gamma)$. That P_0 is a projection follows immediately from the fact that $\widetilde{P}$ is. If $p = 1$ we have to care with the contraction V. Since $\|Vf\| \leq \|\mathbf{1}_{A^c} \cdot f\|$ we obtain $\|Pf\| \leq \|\mathbf{1}_A \cdot f\| + \|\mathbf{1}_{A^c} \cdot f\| = \|f\|$. Then since $VP_0 = 0$, $P_0 V = V$, it follows clearly that $P = P_0 + V$ is a projection. □

We can now give the structure theorem for contractive projections:

Theorem 5.4. *For every contractive projection P of $L_p(\Omega, \Sigma, \mu; H)$ ($1 \leq p < \infty$, $p \neq 2$) there exist:*

- *a modularly isometric automorphism W of $L_p(H)$;*
- *a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint Σ-measurable subsets of Ω of positive measure;*
- *a family $(\mathcal{H}_i)_{i \in I}$ of Hilbert spaces;*
- *for every $i \in I$ a (strong operator) measurable family $(U_{i,\omega})_{\omega \in \Omega}$ of isometric embeddings of $\mathcal{H}_i$ into H;*
- *a positive contractive projection $\widetilde{P}$ of $L_p(\Omega, \Sigma, \mu)$ commuting with the band projections associated with the sets Ω_i;*
- *and if $p = 1$ a contraction V from $L_1(S, \Sigma_{|S}, \mu_{|S}; H)$ into $R(P)$, where $S = \Omega \setminus \bigvee_i \Omega_i$*

such that (setting $V = 0$ if $p > 1$):

$$P = WU\Big(\sum_i \widetilde{P} M_{\Omega_i} \otimes \mathrm{Id}_{\mathcal{H}_i}\Big) U^\sharp W^{-1} + V$$

where M_{Ω_i} denotes the multiplication operator by the characteristic function $\mathbf{1}_{\Omega_i}$; U is the modularly isometric embedding of $\bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i)$ into $L_p(H)$ naturally associated with the family $(U_{i,\omega})_{i \in I, \omega \in \Omega}$ by mean of the formula:

$$(Uf)(\omega) = U_{i,\omega}(f(\omega)) \quad \textit{when } \omega \in \Omega_i$$

and similarly $U^\sharp : L_p(H) \to \bigoplus L_p(\Omega_i, \Sigma_{|\Omega_i}, \mu_{|\Omega_i}; \mathcal{H}_i)$ is the modularly contractive map associated with the family $(U^*_{i,\omega})_{i \in I, \omega \in \Omega}$.

Remark 5.5. In fact the families $(U_{i,\omega})_{\omega \in \Omega}$ may be chosen locally constant, i.e. there is a partition of Ω_i in Σ-measurable subsets of positive measure on which $U_{i,\omega}$ is constant.

Remark 5.6. In the case where H is separable, it is a standard (and easy) fact that every modularly isometric automorphism W of $L_p(H)$ is associated with a measurable family $(W_\omega)_{\omega\in\Omega}$ of unitary operators on H; so we recover the Theorem 0.3 of the Introduction.

Proof. By the proof of Thm. 0.1 in Section 4, there are a sub-σalgebra $\mathcal{F}$ of Σ, a family $(\Omega_i)_{i\in I}$ of pairwise almost disjoint elements of $\mathcal{F}$, a positive weight w on Ω with support $\bigvee_{i\in I}\Omega_i$, a family $(\mathcal{H}_i)_{i\in I}$ of Hilbert spaces and for every $i\in I$ an isometry T_i from $L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu\,|_{\Omega_i};\mathcal{H}_i)$ into $L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};H)$ such that $R(P)=\bigoplus_{i\in I}w\cdot R(T_i)$ and moreover $N(T_if)=N(f)$ for all $f\in L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ (recall that $\nu=w^p\cdot\mu$). Moreover P commutes with the action of $L_\infty(\mathcal{F})$, in particular with the multiplication operators M_{Ω_i}.

Each T_i extends uniquely to a modularly isometric map $\widetilde{T}_i$ from $L_p(\Omega_i,\Sigma_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ onto the closed $L_\infty(\Sigma)$-submodule generated by $R(T_i)$ in $L_p(\Omega,\Sigma,\nu;H)$: set simply $\widetilde{T}_i(\sum_k\varphi_kf_k)=\sum_k\varphi_kT_i(f_k)$ when $\varphi_1,\dots,\varphi_n\in L_\infty(\Omega_i,\Sigma_{|\Omega_i})$ and $f_1,\dots,f_k\in L_p(\Omega_i,\mathcal{F}_{|\Omega_i},\nu_{|\Omega_i};\mathcal{H}_i)$ and verify that $N(\sum_k\varphi_kT_i(f_k))=N(\sum_k\varphi_kf_k)$ (since T_i preserves the random scalar products).

Now define $S_i:L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i};\mathcal{H}_i)\to L_p(\Omega,\Sigma,\mu;H)$ by $S_if=w\widetilde{T}_i(w^{-1}f)$: the range $R(S_i)=wR(\widetilde{T}_i)$ is exactly $\mathbf{1}_{\Omega_i}\cdot Z$, where Z is the closed $L_\infty(\Sigma)$-submodule generated by $R(P)$. We can glue up the maps S_i and obtain a modularly isometric embedding S from $\bigoplus_{i\in I}L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i};\mathcal{H}_i)$ into $L_p(\Omega,\Sigma,\mu;H)$, with range $R(S)=Z$.

By Lemma 5.3 there exists a positive projection $\widetilde{P}$ on $L_p(\Omega,\Sigma,\mu)$ such that $P(\varphi.u_f)=(\widetilde{P}\varphi).u_f$ for every $f\in R(P)$ and $\varphi\in L_p(\Omega,\Sigma,\mu)$. Note that $\widetilde{P}$ is $\mathcal{F}$-modular, in particular it commutes with every multiplication operator M_{Ω_i}, $i\in I$.

If $A\in\mathcal{F}$ is a ν-integrable subset of Ω_i and $e\in H_i$ we have $S_i(\mathbf{1}_Aw\otimes e)=wT_i(\mathbf{1}_A\otimes e)\in R(P)$, and $N(S_i(\mathbf{1}_A\cdot w\otimes e))=N(\mathbf{1}_A\cdot w\otimes e)=\mathbf{1}_A\cdot w$, and consequently for $f=S_i(\mathbf{1}_Aw\otimes e)$ we have $f=w\cdot u_f$. Thus for every $\psi\in L_\infty(\Omega,\Sigma,\mu)\cap L_p(\Omega,\Sigma,\mu)$ we have

$$\begin{aligned}PS_i(\psi\mathbf{1}_Aw\otimes e)&=P(\psi\cdot wu_f)=\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot u_f=\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot w^{-1}S_i(\mathbf{1}_Aw\otimes e)\\&=S_i(\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\cdot w^{-1}\cdot\mathbf{1}_Aw\otimes e)=S_i(\widetilde{P}(\psi\cdot w\mathbf{1}_{\Omega_i})\mathbf{1}_A\otimes e),\end{aligned}$$

hence by linearity and density we have for every $\varphi\in L_p(\Omega_i,\Sigma_{|\Omega_i},\mu_{|\Omega_i})$ and $e\in\mathcal{H}_i$:

$$PS_i(\varphi\otimes e)=S_i(\widetilde{P}(\varphi)\otimes e),$$

that is, the restriction of P to $\mathbf{1}_{\Omega_i}\cdot Z$ is similar by S_i to the projection $\widetilde{P}\otimes\mathrm{id}_{\mathcal{H}_i}$; consequently the restriction of P to Z is similar by S to the projection $\sum_{i\in I}\widetilde{P}M_{\Omega_i}\otimes\mathrm{id}_{\mathcal{H}_i}$.

In the case where $Z=L_p(\Omega,\Sigma,\mu;H)$ we have necessarily $\dim H=\dim\mathcal{H}_i$ for every $i\in I$ since S_i is a modularly isometric map from $L_p(\Omega_i;\mathcal{H}_i)$ onto $\mathbf{1}_{\Omega_i}.Z=$

$L_p(\Omega_i;H)$ (see Remark 4.5). Thus we may assume that $\mathcal{H}_i = H$ and the conclusion of Thm. 5.4 is obtained with $W = S$ and $U = \mathrm{Id}$.

In the general case we apply Lemma 4.4 to the $L_\infty(\Sigma)$ submodule $Z^\perp$. We find a family $(\Omega'_j)_{j\in J}$ of pairwise almost disjoint members of Σ, a family $(\mathcal{K}_j)_{j\in J}$ of Hilbert spaces and a modularly isometric map S' from $(\bigoplus_{j\in J} L_p(\Omega'_j, \Sigma_{|\Omega'_j}, \mu_{|\Omega'_j}; \mathcal{K}_j))_{\ell_p}$ onto $Z^\perp$. Note that now the sets Ω'_j have no reason to belong to the smaller σ-algebra $\mathcal{F}$. We have $\bigvee_j \Omega'_j = \mathbf{VS}(Z^\perp)$. For the commodity of the notation we may assume $\bigvee_j \Omega'_j = \Omega$, adding if necessary one extra set $\Omega'_0 = \Omega \setminus \bigvee_j \Omega'_j$ for which we set $\mathcal{K}_0 = \{0\}$, the 0-dimensional Hilbert space. Similarly, up to the cost of adding one extra set $\Omega_0 = \Omega \setminus \Omega_P$ and setting $\mathcal{H}_0 = \{0\}$, we may assume that $\bigvee_i \Omega_i = \Omega$. We may also refine the partition (Ω'_j) by setting $\Omega'_{ij} = \Omega_i \cap \Omega'_j$ and removing the Ω'_{ij} which are almost void. This operation gives a doubly indexed family $(\Omega'_{ij})_{i\in I; j\in J_i}$.

For every $i \in I, j \in J_i$, set $L_{ij} = \mathcal{H}_i \oplus \mathcal{K}_j$ (direct Hilbertian sum). Then $L_p(\Omega'_{ij};\mathcal{H}_i)$ and $L_p(\Omega'_{ij};\mathcal{K}_j)$ identify naturally to a pair of mutually orthogonal $L_\infty(\Sigma)$-submodules of $L_p(\Omega'_{ij};L_{ij})$: if u^0_{ij} and u'^0_{ij} are the inclusion maps of $\mathcal{H}_i$, resp. $\mathcal{K}_j$ into L_{ij} then $U^0_{ij} = \mathrm{id} \otimes u^0_{ij}$ and $U'^0_{ij} = \mathrm{id} \otimes u'^0_{ij}$ are the corresponding embeddings of $L_p(\Omega'_{ij};\mathcal{H}_i)$ and $L_p(\Omega'_{ij};\mathcal{K}_j)$ into $L_p(\Omega'_{ij};L_{ij})$. Since u^{0*}_{ij} and u'^{0*}_{ij} are the orthogonal projections $L_{ij} \to \mathcal{H}_i$, resp. $L_{ij} \to \mathcal{K}_j$, we see that $U^{0\sharp}_{ij}$ and $U'^{0\sharp}_{ij}$ are the orthogonal projections (in the sense given in Section 1.2) onto $L_p(\Omega'_{ij};\mathcal{H}_i)$, resp. $L_p(\Omega'_{ij};\mathcal{K}_j)$.

Now define $W^0_{ij} : L_p(\Omega'_{ij};L_{ij}) \to L_p(\Omega'_{ij};H)$ by $W^0_{ij}f = S_i(U^{0\sharp}_{ij}f) + S'_j(U'^{0\sharp}_{ij}f)$: we have

$$N(W^0_{ij}f)^2 = N(S_i(U^{0\sharp}_{ij}f))^2 + N(S'_i(U^{0\sharp}_{ij}f))^2 = N(U^{0\sharp}_{ij}f)^2 + N(U'^{0\sharp}_{ij}f)^2 = N(f)^2$$

since S_i and S'_i are modularly isometric and have values in orthogonal subspaces Z, resp. $Z^\perp$. Hence W^0_{ij} is modularly isometric and $R(W^0_{ij}) = \mathbf{1}_{\Omega'_{ij}}Z + \mathbf{1}_{\Omega'_{ij}}Z^\perp = L_p(\Omega'_{ij};H)$.

We know by the proof of Thm. 5.1 that $P = PQ + V$, where Q is the orthogonal projection onto Z. Since V satisfies the requirements of the theorem, we look only for a representation of $P_0 = PQ$. From the first part of the proof we know that $P_0S_i = S_i(\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i})$; on the other hand $P_0S'_j = 0$ since $R(S'_j) \subset Z^\perp = \ker Q$. Hence, for every $f \in L_p(\Omega_{ij};L_{ij})$,

$$P_0W^0_{ij}f = P_0S_iU^{0\sharp}_{ij}f + P_0S'_jU'^{0\sharp}_{ij}f = S_i(\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i})U^{0\sharp}_{ij}f = W^0_{ij}U^0_{ij}(\widetilde{P} \otimes \mathrm{id}_{\mathcal{H}_i})U^{0\sharp}_{ij}f$$

i.e. P_0 is similar by W^0_{ij} to $U^0_{ij}(\widetilde{P}M_{\Omega'_{ij}} \otimes \mathrm{id}_{\mathcal{H}_i})U^{0\sharp}_{ij}$.

Since $L_p(\Omega_{ij};L_{ij})$ is modularly isometric to $L_p(\Omega_{ij};H)$ (by W^0_{ij}), we have $\dim L_{ij} = \dim H$ by Rem. 4.5, so we may identify L_{ij} with H by an isomorphism θ_{ij}. This isomorphism induces in turn a modular isometry $\Theta_{ij} = \mathrm{Id} \otimes \theta_{ij}$ from $L_p(\Omega'_{ij};L_{ij})$ onto $L_p(\Omega'_{ij};H)$. Set $W_{ij} = W^0_{ij}\Theta^{-1}_{ij}$: then W_{ij} is a modular automorphism of $L_p(\Omega'_{ij};H)$. Let also $u_{ij} = \theta_{ij} \circ u^0_{ij}$ be the embedding of $\mathcal{H}_i$ into H resulting from this identification and $U_{ij} = \mathrm{id}_{L_p(\Omega_{ij})} \otimes u_{ij} = \Theta_{ij} U^0_{ij}$ be the associated embedding of $L_p(\Omega_{ij}; \mathcal{H}_i)$ into $L_p(\Omega_{ij}; \mathcal{H})$. Since $\Theta_{ij}^{-1} = \Theta_{ij}^\sharp$ we see that P_0 is similar by W_{ij} to $U_{ij}(\widetilde{P} M_{\Omega'_{ij}} \otimes \mathrm{id}_{\mathcal{H}_i}) U_{ij}^\sharp$.

Finally we glue up the automorphisms W_{ij} to an automorphism W of $L_p(\Omega; \mu; H)$ by setting

$$Wf = \sum_{i \in I} \sum_{j \in J_i} W_{ij} M_{\Omega'_{ij}} f$$

and similarly we glue up the embeddings U_{ij} to an embedding U of $\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i)$ into $L_p(\Omega; H)$. The maps W and U are still modularly isometric and P_0 is similar by W to $U(\sum_{i \in I} \widetilde{P} M_{\Omega_i} \otimes \mathrm{id}_{\mathcal{H}_i}) U^\sharp$. □

6. Annex: a proof of Corollary 3.6 specific to the complex case

The following proof is an adaptation of that of Thm. 4.1 in [2]. We assume that $2 < p < \infty$ (the case $1 < p < 2$ follows by duality).

If $f \in R(P)$ we introduce besides the projection E_f (defined in §1) the operators F_f and G_f defined by

$$F_f g = \mathbf{1}_{\mathbf{VS}(f)^c} g \ ; \ G_f g = \mathbf{1}_{\mathbf{VS}(f)} g - E_f g.$$

Then E_f, G_f and F_f are commuting modularly contractive projections in $L_p(H)$ with $E_f + F_f + G_f = I$.

Let $f, g \in R(P)$, then the elements $A(f,g)$ and $B(f,g)$ defined in §3 (eqs. (5) and (5bis) belong to the range of P^*; so do the sum and difference: $M_f(g) := A(f,g) + B(f,g)$ and $\Gamma_f(g) := \frac{p}{p-2}[A(f,g) - B(f,g)]$ belong to $R(P^*)$. Set

$$Q_f g = \langle\!\langle u_f, g \rangle\!\rangle\, u_f.$$

We have then

$$M_f(g) = N(f)^{p-2}[2g + (p-2) E_f g]$$
$$\Gamma_f(g) = p N(f)^{p-2} Q_f g$$

Then M_f, resp. Γ_f are bounded linear, resp. antilinear operators from $L_p(H)$ into $L_{p_*}(H)$, and Q_f is a contractive antilinear endomorphism of $L_p(H)$ such that $Q_f^2 = E_f$; moreover:

$$M_f P = P^* M_f P, \qquad \Gamma_f P = P^* \Gamma_f P. \tag{21}$$

Consider the positive symmetric bounded bilinear form defined on $L_p(H)$ by

$$(g,h)_f := \langle M_f(g), h \rangle = \int N(f)^{p-2} \langle\!\langle (2I + (p-2)E_f) g \mid h \rangle\!\rangle \, d\mu$$

Note that $\Gamma_f = M_f Q_f = Q_{J_p f} M_f$ and $Q_f^* = Q_{J_p f}$; then Q_f is hermitian for $(\cdot,\cdot)_f$ since

$$\begin{aligned}(Q_f g, h)_f &= \langle M_f Q_f g, h\rangle = \langle Q_f g, M_f h\rangle \\ &= \langle Q_{J_p f} M_f h, g\rangle = \langle M_f Q_f h, g\rangle = (Q_f h, g)_f\end{aligned}$$

On the other hand P is hermitian for $(\cdot,\cdot)_f$ since (using (21))

$$\begin{aligned}(Pg, h)_f &= \langle M_f Pg, h\rangle = \langle P^* M_f Pg, h\rangle = \langle M_f Pg, Ph\rangle = (Pg, Ph)_f \\ &= \overline{(Ph, Pg)_f} = \overline{(Ph, g)_f} = (g, Ph)_f\end{aligned}$$

Let N_f be the kernel of the form $(\cdot,\cdot)_f$: we have $g \in N_f$ iff $(g,g)_f = 0$ iff $(g,h)_f = 0$ for all $h \in L_p(H)$ (by Cauchy-Schwartz inequality). Then $PN_f \subset N_f$ since

$$(Pg, Pg)_f = (g, Pg)_f = 0 \quad \text{if } g \in N_f$$

On the other hand the operator $2\cdot \mathbf{1}_{\mathbf{VS}(f)} + (p-2)E_f$ maps $L_p(H)$ onto $\mathbf{1}_{\mathbf{VS}(f)} L_p(H)$; hence $g \in N_f$ iff $\langle N(f)^{p-2} g, h\rangle = 0$ for every $h \in 1_{\mathbf{VS}(f)} L_p(H)$ iff $N(f)^{p-2} g = 0$ iff $\mathbf{1}_{\mathbf{VS}(f)} g = 0$.

We have thus $R(F_f) = N_f$ and consequently

$$PF_f = F_f P F_f$$

Since $L_p(H)$ is a strictly convex Banach space as well as its dual, we have by the auxiliary Lemma 6.1 below:

$$PF_f = F_f P$$

Let us show that $Q_f P$ is hermitian for $(\cdot,\cdot)_f$, using eq. (21) again:

$$\begin{aligned}(Q_f Pg, h)_f &= \langle M_f Q_f Pg, h\rangle = \langle \Gamma_f Pg, h\rangle \\ &= \langle P^* \Gamma_f Pg, h\rangle = \langle \Gamma_f Pg, Ph\rangle \\ &= (Q_f Pg, Ph)_f = (Q_f Ph, Pg)_f \\ &= (Q_f Ph, g)_f\end{aligned}$$

Since Q_f and P are separately hermitian for $(\cdot,\cdot)_f$ we have

$$(Q_f Pg, h)_f = (PQ_f h, g)_f,$$

hence $(PQ_f - Q_f P)h \in N_f$, i.e. $(I - F_f)PQ_f = (I - F_f)Q_f P$. Composing on the left by G_f and on the right by Q_f, or conversely, we obtain

$$G_f P E_f = 0 = E_f P G_f.$$

Since, on the other hand,

$$F_f P E_f = P F_f E_f = 0 = E_f F_f P = E_f P F_f,$$

we obtain

$$PE_f = E_f PE_f = E_f P. \qquad \square$$

We state now and give a proof of the announced auxiliary Lemma.

Lemma 6.1. *Let X be a strictly convex Banach space with strictly convex dual, and P, Q two contractive projections on X. The following conditions are equivalent:*

(i) *PQ is a projection.*

(ii) *$PQ = QPQ$.*

(iii) *$PQ = PQP$.*

If moreover the complementary projection $Q^{\perp}$ is contractive too then $PQ = QP$.

Proof. If (ii) is verified then $(PQ)^2 = PQPQ = P \cdot PQ = PQ$; while if (iii) is verified then $(PQ)^2 = PQPQ = PQ \cdot Q = PQ$. Hence both (ii) and (iii) imply (i) (without any contractiveness assumption). Conversely if (i) is verified then for every $x \in R(PQ)$ we have $x = Qx = PQx$ (by [2, Prop. 1.1 (iii)]; only the strict convexity of X is needed) so $x \in R(P) \cap R(Q)$. Since the converse is trivial, we see that $R(PQ) = R(P) \cap R(Q)$; in particular $QPQ = PQ$ and (ii) is verified. Dualizing we have that P^*, Q^* and Q^*P^* are contractive projections in X^*; hence $Q^*P^* = P^*Q^*P^*$, so $PQ = PQP$ and (iii) is verified. Now (iii) implies $PQ^{\perp} = PQ^{\perp}P$, and if $Q^{\perp}$ is contractive this implies $PQ^{\perp} = Q^{\perp}PQ^{\perp}$ by the preceding. Then

$$Q = PQ + PQ^{\perp} = QPQ + Q^{\perp}PQ^{\perp}$$

which in turn implies $QP = PQ = QPQ$. $\square$

Remark. The final assertion $PQ = QP$ of Lemma 6.1 is stated in [2] (for $X = C_p$) as Cor. 1.7 without the assumption that the complementary projection $Q^{\perp}$ is contractive. This statement is not correct: if $p \neq 2$ it is easy to construct rank 1 contractive projections P, Q in $X = \ell_p$ or C_p such that $PQ = 0 \neq QP$: choose non zero elements $a, b \in X$ such that their norming functionals Ja, Jb verify $\langle Ja, b\rangle = 0$ and $\langle Jb, a\rangle \neq 0$ and set $P = a \otimes Ja$, $Q = b \otimes Jb$.

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References

[1] T. Andô, *Contractive projections in L_p spaces*, Pacific J. Math. **17** (1966), 391–405.

[2] J. Arazy and Y. Friedman, *Contractive projections in C_p*, Mem. Amer. Math. Soc. **95** (1992).

[3] S. J. Bernau and H. E. Lacey, *The range of a contractive projection on an L_p-space*, Pacific J. Math. **53** (1974), 21–41.

[4] R. G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math. **15** (1965), 443–462.

[5] I. Doust, *Contractive projections on Lebesgue-Bochner spaces*, Function Spaces (Edwardsville, IL, 1994), Lecture Notes in Pure and Appl. Math., vol. 172, Dekker, New York, 1995, pp. 101–109.

[6] P. G. Dodds, C. B. Huijsmans, and B. de Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.

[7] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge University Press, London, 1974.

[8] S. Guerre and Y. Raynaud, *Sur les isométries de $L^p(X)$ et le théorème ergodique vectoriel*, Canad. J. Math. **40** (1988), 360–391.

[9] C. W. Henson and J. Iovino, *Ultraproducts in analysis*, Analysis and Logic (Mons, 1997) (C. Finet and C. Michaux, eds.), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110.

[10] C. W. Henson and Y. Raynaud, *On the theory of $L_p(L_q)$ Banach lattices*, in preparation.

[11] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, New York, 1974.

[12] B. Lemmens and O. van Gaans, *On one-complemented subspaces of Minkowski spaces with smooth Riesz norms*, Eurandom report (2002), preprint.

[13] M. Levy and Y. Raynaud, *Ultrapuissances de $L^p(L^q)$*, Seminar on Functional Analysis, 1983/1984, Publ. Math. Univ. Paris VII, vol. 20, Univ. Paris VII, Paris, 1984, pp. 69–79.

[14] B. Randrianantoanina, 1-*complemented subspaces of spaces with* 1-*unconditional bases*, Canad. J. Math. **49** (1997), 1242–1264.

[15] ______, *Norm-one projections in Banach spaces*, Taiwanese J. Math. **5** (2001), 35–95.

[16] ______ (2003), private communication.

[17] L. Tzafriri, *Remarks on contractive projections in L_p-spaces*, Israel J. Math. **7** (1969), 9–15.