Real Cubic Hypersurfaces and Group Laws

Johannes Huisman

Département de Mathématiques
UFR Sciences et Techniques
Université de Bretagne Occidentale
6, avenue Victor Le Gorgeu
B.P. 809
29285 Brest Cedex – France
johannes.huisman@univ-brest.fr

Recibido: 20 de Enero de 2003
Aceptado: 19 de Febrero de 2004

ABSTRACT

Let X be a real cubic hypersurface in $\mathbb{P}^n$. Let C be the pseudo-hyperplane of X, i.e., C is the irreducible global real analytic branch of the real analytic variety $X(\mathbb{R})$ such that the homology class $[C]$ is nonzero in $H_{n-1}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Let $\mathcal{L}$ be the set of real linear subspaces L of $\mathbb{P}^n$ of dimension $n-2$ contained in X such that $L(\mathbb{R}) \subseteq C$. We show that, under certain conditions on X, there is a group law on the set $\mathcal{L}$. It is determined by $L + L' + L'' = 0$ in $\mathcal{L}$ if and only if there is a real hyperplane H in $\mathbb{P}^n$ such that $H \cdot X = L + L' + L''$. We also study the case when these conditions on X are not satisfied.

Key words: real cubic hypersurface, real cubic curve, real cubic surface, pseudo-hyperplane, pseudo-line, pseudo-plane, linear subspace, group.

2000 Mathematics Subject Classification: 14J70, 14P25.

1. Introduction

The group law on the set of rational points of a cubic curve does not admit a generalization to cubic hypersurfaces [4]. That is, the set of rational points of a cubic hypersurface does not have a group law for which colinear points have zero sum. The idea of the present paper is that the higher dimensional analogue of a rational point of a cubic curve should not be a rational point of a cubic hypersurface, but should be a rational linear subspace of $\mathbb{P}^n$ of dimension $n-2$ that is contained in a cubic hypersurface.

2. Pseudo-hyperplanes of real hypersurfaces

Let n be a natural integer satisfying $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface, i.e., X is defined by a nonconstant homogeneous real polynomial. Note that we do not assume X to be reduced, irreducible or smooth. The set of real points $X(\mathbb{R})$ of X is a real analytic subvariety of $\mathbb{P}^n(\mathbb{R})$. Let C be an irreducible global real analytic branch of $X(\mathbb{R})$. Then C is a compact connected real analytic subvariety of $\mathbb{P}^n(\mathbb{R})$. Its dimension is at most $n-1$. By [1], C realizes a $\mathbb{Z}/2\mathbb{Z}$-homology class $[C]$ in $H_{n-1}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. This homology class vanishes if $\dim(C) < n-1$. We say that C is a *pseudo-hyperplane* of X if $[C] \neq 0$. In particular, the dimension of a pseudo-hyperplane of X is equal to $n-1$. If $n = 2$, a pseudo-hyperplane is called a *pseudo-line*. If $n = 3$, a pseudo-hyperplane is called a *pseudo-plane*.

Proposition 2.1. *Let n and d be natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, the number of pseudo-hyperplanes of X, when counted with multiplicities, is congruent to d* (mod 2).

Proof. We may assume that X is reduced. Denote by $[X(\mathbb{R})]$ the homology class of $X(\mathbb{R})$ in $H_{n-1}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. One has $[X(\mathbb{R})] = d[\mathbb{P}^{n-1}(\mathbb{R})]$. Let L be a general real projective line in $\mathbb{P}^n$. Then,

$$[X(\mathbb{R})] \cdot [L(\mathbb{R})] = d[\mathbb{P}^{n-1}(\mathbb{R})] \cdot [L(\mathbb{R})] = d$$

in $\mathbb{Z}/2\mathbb{Z}$. But the intersection number $[X(\mathbb{R})] \cdot [L(\mathbb{R})]$ is equal to the number of pseudo-hyperplanes of X. Therefore, the statement follows. □

Proposition 2.2. *Let n and d be natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, X has at most d pseudo-hyperplanes, when counted with multiplicities.*

Proof. Let $L \subseteq \mathbb{P}^n$ be a general real projective line. Let C be a pseudo-hyperplane of X. Since $[C] \neq 0$ and $[L(\mathbb{R})] \neq 0$, the homological intersection product $[C] \cdot [L(\mathbb{R})]$ is nonzero. In particular, the subsets C and $L(\mathbb{R})$ of $\mathbb{P}^n(\mathbb{R})$ intersect each other. Therefore, any pseudo-hyperplane of X intersects $L(\mathbb{R})$. Hence, the number of pseudo-hyperplanes of X, counted with multiplicities, is not greater than the degree of the intersection product $X \cdot L$. Since the latter degree is equal to d, the statement follows. □

Proposition 2.3. *Let n and d be natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, X has exactly d pseudo-hyperplanes if and only if X is the scheme-theoretic union of d real hyperplanes.*

Proof. Suppose that X is the scheme-theoretic union of d real hyperplanes. Then it is clear that X has exactly d pseudo-hyperplanes, when counted with multiplicities.

Conversely, suppose that X has exactly d pseudo-hyperplanes, when counted with multiplicities. We show that X is a scheme-theoretic union of real hyperplanes.

Clearly, one may assume that X is reduced. Let C be a pseudo-hyperplane of X. Since $\dim(C) = n-1$, there is a smooth point P of X that belongs to C. We show that the projective tangent space $T_P X$ of X at P is contained in X. It will follow that X is the scheme-theoretic union of real hyperplanes.

Let L be a real projective line in $T_P X$ passing through P. We show that L is contained in X. Suppose that $L \not\subseteq X$. Then the intersection product $L \cdot X$ contains P with multiplicity ≥ 2. Moreover, $L(\mathbb{R})$ intersects each of the $d-1$ pseudo-hyperplanes C' of X that are distinct from C. It follows that $\deg(L \cdot X) \geq 2 + (d-1) = d+1$, contradiction. □

From Propositions 2.1, 2.2 and 2.3 one deduces the following consequence.

Corollary 2.4. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface. Then X has exactly one pseudo-hyperplane.*

3. Real cubic hypersurfaces

Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface. Then, by Corollary 2.4 above, X has exactly one pseudo-hyperplane. Let C be the pseudo-hyperplane of X. Let $\mathcal{L}$ be the set of real linear subspaces L of $\mathbb{P}^n$ of dimension $n-2$ that are contained in X and that satisfy $L(\mathbb{R}) \subseteq C$. Note that the last condition on L is superfluous if C is entirely contained in the smooth locus of X. To put it otherwise, if all points of C are smooth points of X then $\mathcal{L}$ is nothing but the set of real linear subspaces of $\mathbb{P}^n$ of dimension $n-2$ that are contained in X.

The set $\mathcal{L}$ is well understood. If $n = 2$, the set $\mathcal{L}$ is equal to the pseudo-line of X. If $n = 3$, the set $\mathcal{L}$ is finite if X is smooth or if X is singular with isolated rational singularities [3, p. 66]. More generally, for arbitrary $n \geq 2$, let $X \subseteq \mathbb{P}^n$ have *rational singularities in codimension* ≥ 2, i.e., the singular locus of X has codimension ≥ 2 and any general section of X by a real 3-dimensional linear subspace of $\mathbb{P}^n$ has only rational singularities. Then $\mathcal{L}$ is finite. This follows easily from [3].

Let Z be the subset of $\mathcal{L} \times \mathcal{L}$ consisting of all pairs (L, L) such that there is either no real hyperplane H with $H \cdot X \geq 2L$, or there are several such hyperplanes. Equivalently, Z is the subset of the diagonal Δ of $\mathcal{L} \times \mathcal{L}$ whose complement in Δ consists of all pairs (L, L) such that there is exactly 1 real hyperplane H in $\mathbb{P}^n$ with $H \cdot X \geq 2L$.

Proposition 3.1. *Suppose that C is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$. There is a unique partial composition law*

$$\circ \colon \mathcal{L} \times \mathcal{L} \setminus Z \longrightarrow \mathcal{L}$$

determined by $L'' = L \circ L'$ if and only if there is a real hyperplane H in $\mathbb{P}^n$ such that $H \cdot X = L + L' + L''$.

Proof. Let $L, L' \in \mathcal{L}$ with $(L, L') \notin Z$. The homology classes $[L(\mathbb{R})]$ and $[L'(\mathbb{R})]$ are nonzero in $H_{n-2}(C, \mathbb{Z}/2\mathbb{Z})$. Since C is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$, the intersection product $[L(\mathbb{R})] \cdot [L'(\mathbb{R})]$ is nonzero. It follows that the linear subspaces L and L' intersect in a real linear subspace of $\mathbb{P}^n$ of dimension $\geq n-3$. If $L \neq L'$, the dimension of the intersection is equal to $n-3$. Hence, if $L \neq L'$, there is a unique real hyperplane H in $\mathbb{P}^n$ such that $H \cdot X \geq L + L'$. If $L = L'$ then there is also a unique real hyperplane H in $\mathbb{P}^n$ such that $H \cdot X \geq L + L'$ since $(L, L') \notin Z$.

Now, $H \cdot X$ is a real cubic hypersurface in the real projective space H. It has at least 2 pseudo-hyperplanes, when counted with multiplicities. From Propositions 2.1 and 2.3 it follows that there is a unique real linear subspace L'' of $\mathbb{P}^n$ of dimension $n-2$ such that $H \cdot X = L + L' + L''$. Since Since $[H(\mathbb{R})] \cdot [C] \neq 0$ and $[L(\mathbb{R})] + [L'(\mathbb{R})] = 0$ in $H_{n-2}(C(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, one has $L''(\mathbb{R}) \subseteq C$, i.e., $L'' \in \mathcal{L}$. □

It will be convenient, as in the case of cubic curves, to have an element $O \in \mathcal{L}$ such that there exist a unique real hyperplane H_0 in $\mathbb{P}^n$ with $H_0 \cdot X = 3O$. Therefore, we consider the following conditions on X:

(i) X is smooth in codimension 1,

(ii) C is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$, and

(iii) there is a real hyperplane H_0 in $\mathbb{P}^n$ such that $H_0 \cdot X = 3O$ in $\mathrm{Div}(X)$.

There are lots of real cubic hypersurfaces satisfying conditions (i), (ii) and (iii): smooth real cubic curves in $\mathbb{P}^2$ satisfy the conditions (i) and, whenever an irreducible real cubic hypersurface in $\mathbb{P}^n$ satisfies the conditions, then a projective cone over it in $\mathbb{P}^{n+1}$ also satisfies the conditions (i), (ii) and (iii). And these are not the only ones [3].

Note, however, that a real cubic hypersurface X satisfying conditions (i), (ii) and (iii) is necessarily singular if $n \geq 3$. Indeed, after a change of coordinates, one may assume that H_0 is given by the equation $X_0 = 0$, and that O is the linear subspace of $\mathbb{P}^n$ defined by the equations $X_0 = 0$ and $X_1 = 0$. Then, X is defined by a homogeneous polynomial of the form $X_1^3 + X_0 F$, where F is a real quadratic form in $X_0, \ldots, X_n$. The closed subscheme of X defined by the equations $X_0 = 0$, $X_1 = 0$ and $F = 0$ is contained in the singular locus of X. If $n \geq 3$ then this closed subscheme is nonempty. Therefore, X is singular if $n \geq 3$.

Lemma 3.2. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i), (ii) *and* (iii) *above. Then $O \in \mathcal{L}$ and $(O, O) \notin Z$.*

Proof. Since $H_0 \cdot X = 3O$, O is a real linear subspace of $\mathbb{P}^n$ of dimension $n-2$. Since $n - 2 \geq 0$, the set of real points $O(\mathbb{R})$ of O is nonempty. Since $O(\mathbb{R}) \subseteq X(\mathbb{R})$ and $O(\mathbb{R})$ is irreducible, there is an irreducible global real analytic branch C' of $X(\mathbb{R})$ such that $O(\mathbb{R}) \subseteq C'$. Since X is smooth in codimension 1, O is not contained in the singular locus of X. It follows that $O(\mathbb{R})$ contains a smooth point of X. In particular, C' is a real analytic variety of dimension $n-1$. Suppose that $[C'] = 0$ then also $[H_0(\mathbb{R})] \cdot [C'] = [O(\mathbb{R})] = 0$. But $[O(\mathbb{R})] \neq 0$, contradiction. Therefore, $[C'] \neq 0$, i.e., C' is a pseudo-hyperplane of X. It follows from Corollary 2.4 that $C' = C$ and $O \in \mathcal{L}$.

Since X is smooth in codimension 1, the hyperplane H_0 is the unique real hyperplane satisfying $H_0 \cdot X \geq 2O$. Hence, $(O, O) \notin Z$. □

From now on, suppose that $X \subseteq \mathbb{P}^n$ is an irreducible real cubic hypersurface satisfying conditions (i), (ii) and (iii) above. Define a partial composition law $\oplus$ on $\mathcal{L}$,

$$\oplus\colon \mathcal{L} \times \mathcal{L} \setminus Z \longrightarrow \mathcal{L}$$

by $L \oplus L' = O \circ (L \circ L')$ for all $(L, L') \in \mathcal{L}^2 \setminus Z$. Note that this is well defined by Lemma 3.2. Define also a map

$$\ominus\colon \mathcal{L} \longrightarrow \mathcal{L}$$

by $\ominus L = O \circ L$ for all $L \in \mathcal{L}$. Note again that this well defined.

Let $\mathrm{Pic}(X)$ be the Picard group of X. Since X is smooth in codimension 1, the group $\mathrm{Pic}(X)$ is the group of linear equivalence classes of divisors on X [2]. Define a map

$$\varphi\colon \mathcal{L} \longrightarrow \mathrm{Pic}(X)$$

by $\varphi(L) = \mathrm{cl}(L - O)$, for all $L \in \mathcal{L}$, where cl denotes the linear equivalence class.

Theorem 3.3. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i), (ii) *and* (iii) *above. Then the map φ is injective. Moreover, for all $(L, L') \in \mathcal{L}^2 \setminus Z$ one has*

$$\varphi(L \oplus L') = \varphi(L) + \varphi(L').$$

And, for all $L \in \mathcal{L}$ one has

$$\varphi(\ominus L) = -\varphi(L).$$

Proof. Let $L, L' \in \mathcal{L}$ such that $\varphi(L) = \varphi(L')$. Then the invertible sheaves $\mathcal{O}(L)$ and $\mathcal{O}(L')$ on X are isomorphic. Let $P \subseteq \mathbb{P}^n$ be a general real linear subspace of dimension 2. Then, $E = P \cap X$ is a smooth real cubic curve, $P \cap L$ and $P \cap L'$ are real points of E, and the invertible sheaves $\mathcal{O}(P \cap L)$ and $\mathcal{O}(P \cap L')$ on E are isomorphic. It follows (cf. [5]) that $P \cap L = P \cap L'$. Since P is general, one has $L = L'$. This proves that φ is injective.

Let $L \in \mathcal{L}$. By Proposition 3.1, there is a real hyperplane H of $\mathbb{P}^n$ such that

$$H \cdot X = O + L + \ominus L.$$

Then

$$\mathrm{div}\left(\frac{H}{H_0}\right) = (O + L + \ominus L) - 3O = (L - O) + (\ominus L - O).$$

It follows that $\varphi(\ominus L) = -\varphi(L)$.

Similarly, if $(L, L') \in \mathcal{L}^2 \setminus Z$, then $\varphi(L \oplus L') = \varphi(L) + \varphi(L')$. □

Corollary 3.4. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i)*,* (ii) *and* (iii) *above. Suppose that for each $L \in \mathcal{L}$ there is a real hyperplane H in $\mathbb{P}^n$ such that $H \cdot X \geq 2L$. Then $(\mathcal{L}, \oplus, \ominus, O)$ is an abelian group and the map φ is an isomorphism from $\mathcal{L}$ onto a subgroup of* $\mathrm{Pic}(X)$*.*

If $n = 2$, then X is a smooth real cubic curve, C is the pseudo-line of X, the set $\mathcal{L}$ is equal to C, and $Z = \emptyset$. Therefore, Corollary 3.4 reconstructs the classical group structure on C [5]. This is not surprising since we used in the proof of Theorem 3.3 the classical fact that the map φ is injective if $n = 2$. More generally, if $X \subseteq \mathbb{P}^n$ is a real projective cone over a nonsingular real cubic curve E, then there is an obvious bijection between $\mathcal{L}$ and the real pseudoline of E, and, again, $Z = \emptyset$. Therefore, $\mathcal{L}$ is a group that is isomorphic to the group structure on the pseudo-line of E. More interesting cases are the cases where X has rational singularities in codimension ≥ 2.

Let $\mathbb{Z}[\mathcal{L}]$ be the free abelian group generated by the elements of $\mathcal{L}$. Let H be the subgroup of $\mathbb{Z}[\mathcal{L}]$ generated by the elements

$$L \oplus L' - L - L',$$

for $(L, L') \in \mathcal{L}^2 \setminus Z$, and the elements

$$\ominus L + L,$$

for $L \in \mathcal{L}$, and the element O. Let G be the quotient group $\mathbb{Z}[\mathcal{L}]/H$.

Proposition 3.5. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i)*,* (ii) *and* (iii) *above. Then*

$$G = \mathcal{L} \cup \{\, mL \mid (L, L) \in Z \text{ and } m \geq 2 \,\}.$$

Proof. Let R be the right-hand side of the equation. Let g be an element of G. We may assume that $g = \sum_{i=1}^{\ell} L_i$, where $L_i \in \mathcal{L}$ for $i = 1, \ldots, \ell$. We show that one can reduce ℓ successively to get in the end $g \in R$.

If $\ell \leq 1$ then we are done. Suppose therefore that $\ell \geq 2$. If $(L_{\ell-1}, L_\ell) \notin Z$ then put $L'_{\ell-1} = L_{\ell-1} \oplus L_\ell$. One has $g = \sum_{i=1}^{\ell-1} L'_i$, where $L'_i = L_i$ for $i = 1, \ldots, \ell - 2$. Continuing in this way, one has in the end either $g \in \mathcal{L}$ or $g = mL$ for some $L \in \mathcal{L}$ with $(L, L) \in Z$ and $m \geq 2$, i.e., $g \in R$. □

Corollary 3.6. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i)*,* (ii) *and* (iii) *above. Suppose that X has rational singularities in codimension ≥ 2. Then* $\mathrm{rank}(G) \leq 1$.

Proof. Since X has rational singularities in codimension ≥ 2, the set $\mathcal{L}$ is finite [3]. By Proposition 3.5, the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes G$ is a union of finitely many 1-dimensional subspaces. Hence, $\dim(\mathbb{Q} \otimes G) \leq 1$. Since G is a $\mathbb{Z}$-module of finite type, $\mathrm{rank}(G) \leq 1$. □

Corollary 3.7. *Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions* (i), (ii) *and* (iii) *above. Suppose that X has rational singularities in codimension ≥ 2. Then the map $\varphi\colon \mathcal{L} \longrightarrow \operatorname{Pic}(X)$ induces a morphism*

$$\psi\colon G \longrightarrow \operatorname{Pic}(X).$$

The image of ψ is a subgroup of $\operatorname{Pic}(X)$ *of rank ≤ 1.*

Let $X \subseteq \mathbb{P}^n$ be an irreducible real cubic hypersurface satisfying conditions (i), (ii) and (iii) above, and having rational singularities in codimension ≥ 2. One of the following three conditions hold:

(i) $\psi(G) = \varphi(\mathcal{L})$,

(ii) $\psi(G) \neq \varphi(\mathcal{L})$ and $\psi(G)$ is finite, or

(iii) $\psi(G)$ is not finite.

The first case occurs when, for each $L \in \mathcal{L}$, there is a real hyperplane H in $\mathbb{P}^n$ such that $H\cdot X \geq 2L$ (see Proposition 3.5). Explicit examples of real cubic hypersurfaces X having this property can be easily constructed using [3, p. 66]. It would be interesting to construct real cubic hypersurfaces X for which one of the other conditions hold. It would also be interesting to determine the group $\psi(G)$ explicitly in each of the above three cases.

Acknowledgements. I am grateful to Louis Mahé for our discussions on cubic hypersurfaces and group laws.

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