Existence Results for Quasilinear Degenerated Equations Via Strong Convergence of Truncations

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ABSTRACT

In this paper we study the existence of solutions for quasilinear degenerated elliptic operators $A(u) + g(x, u, \nabla u) = f$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ into its dual, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s. The right hand side f is assumed to belong either to $W^{-1,p'}(\Omega, w^*)$ or to $L^1(\Omega)$.

Key words: Weighted Sobolev spaces, Hardy inequality, Quasilinear degenerated elliptic operators, Truncations

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1. Introduction

In this paper, we shall be concerned with the existence of solutions for quasilinear degenerated elliptic equations of the type

$$(\mathcal{P}) \qquad \begin{cases} Au + g(x, u, \nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega), \quad g(x, u, \nabla u)u \in L^1(\Omega), \end{cases}$$

where

$$Au = -\mathrm{div}(a(x, u, \nabla u))$$

is a weighted Leray-Lions operator from the weighted Sobolev space $X = W_0^{1,p}(\Omega, w)$ into $X^* = W^{-1,p'}(\Omega, w^*)$ and where g is a nonlinear lower order term having natural growth $(|g(x, s, \xi)| \leq b(|s|)(c(x) + \sum_{i=1}^{N} w_i |\xi_i|^p))$ which satisfies the sign condition $g(x, s, \xi)s \geq 0$. The right hand side f is assumed to belong to X^* or $L^1(\Omega)$. In the last case, we also assume that $g(x, s, \xi)$ has an "exact natural growth" i.e., $|g(x, s, \xi)| \geq \rho \sum_{i=1}^{N} w_i |\xi_i|^p$. It will turn out that for any solution u, $g(x, u, \nabla u) \in L^1(\Omega)$, but for a general $v \in X$, $g(x, v, \nabla v)$ can be more singular.

Drabek and Nicolosi in [10] proved the existence of bounded solution for the degenerated problem $(\mathcal{P})$ where $g(x, u, \nabla u) = -c_0 |u|^{p-2} u$, more precisely for the problem,

$$Au - c_0 |u|^{p-2} u = f(x, u, \nabla u)$$

with some more general degeneracy, but under some other assumptions on f and $a(x, s, \xi)$. The existence result for the problem $(\mathcal{P})$ (respectively, unilateral problem) where f lies in the dual space $W^{-1,p'}(\Omega, w^*)$ is also studied in [1] (respectively, [2]), namely, the authors obtain the existence results by proving that the positive part u_ε^+ (resp. u_ε^-) of u_ε strongly converges to u^+(resp. u^-), where u_ε is a solution of the approximate problem.

Our first aim of this paper is to prove (in Theorem 3.7) the same existence result as in [1] by using another approach based on the strong convergence of the truncations $T_k(u_\varepsilon)$ in $W_0^{1,p}(\Omega, w)$. Moreover, we assume only the weak integrability condition $\sigma^{1-q'} \in L_{loc}^1(\Omega)$ (see assumption (H_1) below) instead of the stronger one which is $\sigma^{1-q'} \in L^1(\Omega)$ as in [1]. For that, we approximate the term $g(x, s, \xi)$ by some functions involving $\chi_{\Omega_\varepsilon}$ where Ω_ε is a sequence of compacts covering the bounded open set Ω and $\chi_{\Omega_\varepsilon}$ is a characteristic function, i.e., $g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon |g(x, s, \xi)|} \chi_{\Omega_\varepsilon}(x)$.

The second aim of this paper is to prove (in Theorem 3.12) the existence result for the following problem

$$(\tilde{\mathcal{P}}) \qquad \begin{cases} Au + g(x, u, \nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega), \end{cases}$$

where $f \in L^1(\Omega)$, under some added hypothesis (see (35) below).

Note that in the non weighted case Boccardo, Gallouët and Murat in [6] have proved the existence of at least one solution for the problem $(\mathcal{P})$ and $(\tilde{\mathcal{P}})$. Let us point out that another work in this direction can be found in [4] where the right hand side f is assumed to belong to $W^{-1,p'}(\Omega)$ and in [5] with $f \in L^1(\Omega)$.

Our results (Theorem 3.7 and Theorem 3.12) generalize those obtained in [4], [5] and [6], in the weighted case.

The present paper is organized as follows: In Section 2, we give some preliminaries. In the first part of Section 3, we prove some technical lemmas concerning some

convergences in weighted Sobolev spaces and determinate basic assumptions. And in the second part we study the first main result (where $f \in X^*$). In the third part, we study the case where $f \in L^1(\Omega)$. The fifth part is devoted to an example which illustrates our abstract hypotheses. Note that, in the proof of our main general results, many ideas have been adopted from the work of [6].

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 1$), let $1 < p < \infty$, and let $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions; i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Furthermore, we suppose in all our considerations that

$$w_i \in L^1_{loc}(\Omega) \tag{1}$$

and

$$w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega) \tag{2}$$

for any $0 \leq i \leq N$.

We define the weighted space $L^p(\Omega, \gamma)$, where γ is a weight function on Ω, by

$$L^p(\Omega, \gamma) = \{u = u(x),\ u\gamma^{\frac{1}{p}} \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\ dx \right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for all } i = 1, \dots, N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0(x)\ dx + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x)\ dx \right)^{\frac{1}{p}}. \tag{3}$$

Since we shall deal with the Dirichlet problem, we shall use the space

$$X = W_0^{1,p}(\Omega, w)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (3). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|\cdot\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \forall i = 0, \dots, N\}$, and p' is the conjugate of p, i.e. $p' = \frac{p}{p-1}$. For more details, we refer the reader to [9].

Now we state the following assumption:

Assumption (H_1). *The expression*

$$|||u|||_X = \left(\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{\frac{1}{p}} \tag{4}$$

is a norm defined on X and it is equivalent to the norm (3). There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1_{loc}(\Omega), \tag{5}$$

with $q' = \frac{q}{q-1}$. The Hardy inequality,

$$\left(\int_{\Omega} |u(x)|^q \sigma \, dx \right)^{\frac{1}{q}} \leq c \left(\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{\frac{1}{p}}, \tag{6}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$X \hookrightarrow L^q(\Omega, \sigma), \tag{7}$$

expressed by the inequality (6) is compact.

Note that $(X, |||\cdot|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1. Assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, \infty[\cap [\frac{1}{p-1}, \infty[$ such that $w_i^{-\nu} \in L^1(\Omega)$ for all $i = 1, \ldots, N$ (which is stronger than (2)). Then

$$|||u|||_X = \left(\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{\frac{1}{p}}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it is equivalent to (3). Moreover

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

for all $1 \leq q < p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu + 1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see [9]). Thus the hypotheses (H_1) is satisfied for $\sigma \equiv 1$.

Definition. Let X be a reflexive Banach space. An operator B from X to its dual X^* satisfies property (M) if for any sequence $(u_n) \subset X$ satisfying $u_n \rightharpoonup u$ in X weakly, $B(u_n) \rightharpoonup \chi$ in X^* weakly and $\limsup_{n \to \infty} \langle Bu_n, u_n \rangle \leq \langle \chi, u \rangle$ then one has $\chi = B(u)$.

3. Main results

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined as

$$Au = -\operatorname{div}(a(x, u, \nabla u))$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory vector-function satisfying the following assumptions:

Assumption (H_2).

$$|a_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \quad for\ i = 1, \dots, N, \quad (8)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0,\ for\ all\ \xi \neq \eta \in \mathbb{R}^N, \tag{9}$$

$$a(x, s, \xi) \cdot \xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{10}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants.

Let $g(x, s, \xi)$ be a Carathéodory function satisfying the following assumptions:

Assumption (H_3).

$$g(x, s, \xi)s \geq 0 \tag{11}$$

$$|g(x, s, \xi)| \leq b(|s|)\left(\sum_{i=1}^{N} w_i |\xi_i|^p + c(x)\right), \tag{12}$$

where $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous increasing function and $c(x)$ a positive function which is in $L^1(\Omega)$.

3.1. Some technical lemmas

Let us give and prove the following lemmas which are needed below. Note that lemmas 3.1, 3.2, 3.4 and 3.6 are proved in [1]. Nevertheless, for the sake of completeness, we provide their proofs.

Lemma 3.1. *Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{r,\gamma} \leq c$, $1 < r < \infty$. If $g_n(x) \longrightarrow g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^r(\Omega, \gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Ω.*

Proof. Since $g_n \gamma^{\frac{1}{r}}$ is bounded in $L^r(\Omega)$ and $g_n(x)\gamma^{\frac{1}{r}}(x) \longrightarrow g(x)\gamma^{\frac{1}{r}}(x)$, a.e. in Ω, then by the Lemma 3.2 [13], we have

$$g_n \gamma^{\frac{1}{r}} \rightharpoonup g\gamma^{\frac{1}{r}} \quad in\ L^r(\Omega).$$

Moreover, for all $\varphi \in L^{r'}(\Omega, \gamma^{1-r'})$, we have $\varphi \gamma^{-\frac{1}{r}} \in L^{r'}(\Omega)$. Then

$$\int_\Omega g_n \varphi \, dx \longrightarrow \int_\Omega g\varphi \, dx, \quad \text{i.e. } g_n \rightharpoonup g \text{ in } L^r(\Omega, \gamma). \qquad \square$$

Lemma 3.2. *Assume that (H_1) holds. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitz, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \}, \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases}$$

Remark 3.3. The previous lemma is a generalization of the corresponding in [11] (pp. 151–152) (where $w \equiv 1$ and $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$) and of the corresponding in [3] (where $w_0 \equiv w_1 \equiv \cdots \equiv w_N$ is some weight function and $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$). Also note that the previous lemma implies that functions in $W_0^{1,p}(\Omega, w)$ can be truncated.

Proof of Lemma 3.2. First, note that the proof of the second part of Lemma 3.2 is identical to the corresponding in the non-weighted case (see [11]).

Consider firstly the case $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Let $u \in W_0^{1,p}(\Omega, w)$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$, there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$u_n \longrightarrow u \text{ in } W_0^{1,p}(\Omega, w).$$

Passing to a subsequence, we can assume that

$$u_n \longrightarrow u \text{ a.e. in } \Omega \quad \text{and} \quad \nabla u_n \longrightarrow \nabla u \text{ a.e. in } \Omega.$$

Then,

$$F(u_n) \longrightarrow F(u) \text{ a.e. in } \Omega. \tag{13}$$

On the other hand, from the relation $|F(u_n)|^p w_0 \le \|F'\|_\infty |u_n|^p w_0$ and

$$\left|\frac{\partial F(u_n)}{\partial x_i}\right|^p w_i = \left|F'(u_n)\frac{\partial u_n}{\partial x_i}\right|^p w_i \le M\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i,$$

we deduce that the function $F(u_n)$ remains bounded in $W_0^{1,p}(\Omega, w)$. Thus, going to a further subsequence, we obtain

$$F(u_n) \rightharpoonup v \text{ in } W_0^{1,p}(\Omega, w). \tag{14}$$

Thanks to (13), (14) and (7) we conclude that

$$v = F(u) \in W_0^{1,p}(\Omega, w).$$

We now turn our attention to the general case. Taking convolutions with mollifiers ρ_n in $\mathbb{R}$, we have $F_n = F * \rho_n$, $F_n \in C^1(\mathbb{R})$ and $F'_n \in L^\infty(\mathbb{R})$. Then, by the first case we have $F_n(u) \in W_0^{1,p}(\Omega, w)$. Since $F_n \to F$ uniformly in every compact, we have $F_n(u) \to F(u)$ a.e. in Ω. On the other hand, $(F_n(u))$ is bounded in $W_0^{1,p}(\Omega, w)$, then for a subsequence $F_n(u) \rightharpoonup \bar{v}$ in $W_0^{1,p}(\Omega, w)$ and a.e. in Ω (due to (7)). Hence,

$$\bar{v} = F(u) \in W_0^{1,p}(\Omega, w). \qquad \square$$

The following lemmas follow from the previous lemma.

Lemma 3.4. *Assume that (H_1) holds. Let $u \in W_0^{1,p}(\Omega, w)$ and let $T_k(u)$, $k \in \mathbb{R}^+$, the usual truncation. Then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$T_k(u) \to u \text{ strongly in } W_0^{1,p}(\Omega, w).$$

Lemma 3.5. *Assume that (H_1) holds. Let (u_n) be a sequence of $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$. Then, $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, w)$.*

Proof. Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$ and by (7) we have for a subsequence $u_n \to u$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω. On the other hand,

$$|||T_k(u_n)|||_X^p = \sum_{i=1}^N \int_\Omega \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p w_i = \sum_{i=1}^N \int_\Omega \left| T'_k(u_n) \frac{\partial u_n}{\partial x_i} \right|^p w_i$$

$$\leq \sum_{i=1}^N \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i = |||u_n|||_X^p.$$

Then $(T_k(u_n))$ is bounded in $W_0^{1,p}(\Omega, w)$. Hence, using (7), we have $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, w)$. $\qquad \square$

The following lemma generalizes to the weighted case the analogous Lemma 5 in [7]. For that, we use the method of [7] and [13] which gives the strong convergence of u_n.

Lemma 3.6. *Assume that (H_1) and (H_2) are satisfied, and let (u_n) be a sequence in $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\, dx \to 0. \tag{15}$$

Then, $u_n \to u$ in $W_0^{1,p}(\Omega, w)$.

Proof. Let $D_n = [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)$. Then, by (9), D_n is a positive function and by (15), $D_n \to 0$ in $L^1(\Omega)$.

Extracting a subsequence, still denoted by u_n, and using (7) we can write

$$u_n \to u \text{ a.e. in } \Omega, \qquad D_n \to 0 \text{ a.e. in } \Omega.$$

Then, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega\backslash B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|k(x)| < \infty$, $w_i(x) > 0$ and $u_n(x) \rightarrow u(x)$, $D_n(x) \rightarrow 0$.

We set $\xi_n = \nabla u_n(x)$ and $\xi = \nabla u(x)$. Then

$$
\begin{aligned}
D_n(x) &= [a(x, u_n, \xi_n) - a(x, u_n, \xi)](\xi_n - \xi) \\
&\geq \alpha \sum_{i=1}^{N} w_i |\xi_n^i|^p + \alpha \sum_{i=1}^{N} w_i |\xi^i|^p \\
&\quad - \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[k(x) + \sigma^{\frac{1}{p'}} |u_n|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\xi_n^j|^{p-1} \right] |\xi^i| \\
&\quad - \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[k(x) + \sigma^{\frac{1}{p'}} |u_n|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\xi^j|^{p-1} \right] |\xi_n^i|
\end{aligned}
$$

i.e.,

$$
D_n(x) \geq \alpha \sum_{i=1}^{N} w_i |\xi_n^i|^p - c_x \left[1 + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\xi_n^j|^{p-1} + \sum_{i=1}^{N} w_i^{\frac{1}{p}} |\xi_n^i| \right] \tag{16}
$$

where c_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \rightarrow u(x)$, we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then, by a standard argument $|\xi_n|$ is bounded uniformly with respect to n. Indeed, (16) becomes

$$
D_n(x) \geq \sum_{i=1}^{N} |\xi_n^i|^p \left(\alpha w_i - \frac{c_x}{N |\xi_n^i|^p} - \frac{c_x w_i^{\frac{1}{p'}}}{|\xi_n^i|} - \frac{c_x w_i^{\frac{1}{p}}}{|\xi_n^i|^{p-1}} \right).
$$

If $|\xi_n| \rightarrow \infty$ (for a subsequence), there exists at least one i_0 such that $|\xi_n^{i_0}| \rightarrow \infty$, which implies that $D_n(x) \rightarrow \infty$ which gives a contradiction.

Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by the continuity of a with respect to the two last variables we obtain

$$
(a(x, u(x), \xi^*) - a(x, u(x), \xi))(\xi^* - \xi) = 0.
$$

In view of (9) we have $\xi^* = \xi$. The uniqueness of the cluster point implies

$$
\nabla u_n(x) \rightarrow \nabla u(x) \text{ a.e. in } \Omega.
$$

Since the sequence $a(x, u_n, \nabla u_n)$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$ and $a(x, u_n, \nabla u_n) \rightarrow a(x, u, \nabla u)$ a.e. in Ω, Lemma 3.1 implies

$$
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*) \text{ and a.e. in } \Omega.
$$

We set $\bar{y}_n = a(x, u_n, \nabla u_n)\nabla u_n$ and $\bar{y} = a(x, u, \nabla u)\nabla u$. As in the proof of Lemma 5 in [7] we can write $\bar{y}_n \to \bar{y}$ in $L^1(\Omega)$. By (10), we have

$$\alpha \sum_{i=1}^{N} w_i \left| \frac{\partial u_n}{\partial x_i} \right|^p \leq a(x, u_n, \nabla u_n)\nabla u_n.$$

Let $z_n = \sum_{i=1}^{N} w_i |\frac{\partial u_n}{\partial x_i}|^p$, $z = \sum_{i=1}^{N} w_i |\frac{\partial u}{\partial x_i}|^p$, $y_n = \frac{\bar{y}_n}{\alpha}$ and $y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's lemma we obtain

$$\int_\Omega 2y\, dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\, dx,$$

i.e.

$$0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx,$$

hence,

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\, dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx \leq 0.$$

This implies

$$\nabla u_n \to \nabla u \ \text{ in } \prod_{i=1}^{N} L^p(\Omega, w_i),$$

which with (4) completes the present proof. $\qquad\square$

3.2. Case where $f \in W^{-1,p'}(\Omega, w^*)$

In this subsection we assume that

$$f \in W^{-1,p'}(\Omega, w^*). \tag{17}$$

Consider the nonlinear problem with Dirichlet boundary conditions,

$$(\mathcal{P}) \qquad \begin{cases} Au + g(x, u, \nabla u) = f \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega), \quad ug(x, u, \nabla u) \in L^1(\Omega). \end{cases}$$

Our main result is then the following:

Theorem 3.7. *Under the assumptions (H_1)–(H_3) and (17), there exists a solution of $(\mathcal{P})$.*

Remarks 3.8. (i) The previous result is also proved in [1] by using another approach based on the strong convergence both of positive and negative parts of the solution u_ε of the approximate problem (see also [4] in non weighted case).

(ii) Theorem 3.7, generalizes to weighted case the analogous statement in [6].

(iii) Note that in [1] the authors have assumed that $\sigma^{1-q'} \in L^1(\Omega)$ which is stronger than (5).

Proof of Theorem 3.7. STEP (1) (The approximate problem and priori estimates) Let Ω_ε be a sequence of compact subsets of Ω such that Ω_ε is increasing to Ω as $\varepsilon \to 0$.

We consider the sequence of approximate equations,

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = f \\ u_\varepsilon \in W_0^{1,p}(\Omega, w) \end{cases}$$

where

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon|g(x, s, \xi)|} \chi_{\Omega_\varepsilon}(x).$$

and where $\chi_{\Omega_\varepsilon}$ is the characteristic function of Ω_ε.

Note that $g_\varepsilon(x, s, \xi)$ satisfies the following condition

$$g_\varepsilon(x, s, \xi)s \geq 0, \quad |g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}.$$

We define the operator $G_\varepsilon : \; X \longrightarrow X^*$ by

$$\langle G_\varepsilon u, v \rangle = \int_\Omega g_\varepsilon(x, u, \nabla u) v \, dx.$$

Thanks to Hölder's inequality, we have for all $u \in X$ and $v \in X$

$$\begin{aligned} \left| \int_\Omega g_\varepsilon(x, u, \nabla u) v \, dx \right| &\leq \left(\int_\Omega |g_\varepsilon(x, u, \nabla u)|^{q'} \sigma^{-\frac{q'}{q}} \, dx \right)^{\frac{1}{q'}} \left(\int_\Omega |v|^q \sigma \, dx \right)^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon} \left(\int_{\Omega_\varepsilon} \sigma^{1-q'} \, dx \right)^{\frac{1}{q'}} \|v\|_{q,\sigma} \\ &\leq c_\varepsilon \||v|\|_X. \end{aligned} \qquad (18)$$

For the above inequality we have used (5) and (7).

Lemma 3.9. *The operator $A + G_\varepsilon : \; X \longrightarrow X^*$ is bounded, coercive, hemicontinuous and satisfies property (M).*

This lemma will be proved below.

In view of Lemma 3.9, Problem $(\mathcal{P}_\varepsilon)$ has a solution by a classical result (cf. Theorem 2.1 and Remark 2.1 in Chapter 2 of [12]). Since g_ε verifies the sign condition, using (10) we obtain

$$\alpha \sum_{i=1}^N \int_\Omega w_i \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p \leq \langle f, u_\varepsilon \rangle,$$

i.e. $\alpha |||u_\varepsilon|||^p \leq \|f\|_{X^*} |||u_\varepsilon|||$, then

$$|||u_\varepsilon||| \leq \beta_1, \tag{19}$$

where β_1 is some positive constant.

STEP (2) (Strong convergence of $T_k(u_\varepsilon)$)   Note that many ideas in this step and step (3) have been adapted from the one used in [6].

Thanks to (19) and (7), we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega, w) \text{ and } u_\varepsilon \longrightarrow u \text{ a.e. in } \Omega. \tag{20}$$

Let $k > 0$, by Lemma 3.5 we have

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega, w) \text{ as } \varepsilon \to 0.$$

Our objective is to prove that

$$T_k(u_\varepsilon) \longrightarrow T_k(u) \text{ strongly in } W_0^{1,p}(\Omega, w) \text{ as } \varepsilon \to 0. \tag{21}$$

Fix k, and make the notation $z_\varepsilon = T_k(u_\varepsilon) - T_k(u)$. We use as a test function in $(\mathcal{P}_\varepsilon)$ $v_\varepsilon = \varphi_\lambda(z_\varepsilon)$ where $\varphi_\lambda(s) = s e^{\lambda s^2}$. Since v_ε is bounded in X and converges to zero a.e. in Ω and using (7), we have $v_\varepsilon \rightharpoonup 0$ in X as $\varepsilon \to 0$, then

$$\langle f, v_\varepsilon \rangle \longrightarrow 0. \tag{22}$$

This implies that

$$\eta_1(\varepsilon) = \langle Au_\varepsilon, v_\varepsilon \rangle + \langle G_\varepsilon u_\varepsilon, v_\varepsilon \rangle = \langle f, v_\varepsilon \rangle \longrightarrow 0 \text{ as } \varepsilon \to 0. \tag{23}$$

Since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) v_\varepsilon \geq 0$ in the subset $\{ x \in \Omega, \ |u_\varepsilon(x)| \geq k \}$, statement (23) yields

$$\langle Au_\varepsilon, v_\varepsilon \rangle + \int_{\{|u_\varepsilon| \leq k\}} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) v_\varepsilon \, dx \leq \eta_1(\varepsilon). \tag{24}$$

We study each term in the left hand side of (24). We have

$$\begin{aligned}
\langle Au_\varepsilon, v_\varepsilon \rangle &= \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla(T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
&= \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla(T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
&\quad - \int_{\{|u_\varepsilon|>k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) \varphi_\lambda'(z_\varepsilon) \, dx \\
&= \int_\Omega (a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u))) \nabla(T_k(u_\varepsilon) \\
&\quad - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx + \eta_2(\varepsilon),
\end{aligned} \tag{25}$$

where

$$\eta_2(\varepsilon) = \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u)) \nabla(T_k(u_\varepsilon) - T_k(u)) \varphi'_\lambda(z_\varepsilon) \, dx$$

$$- \int_{\{|u_\varepsilon|>k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) \varphi'_\lambda(z_\varepsilon) \, dx$$

which converges to 0 as $\varepsilon \longrightarrow 0$.

On the other hand,

$$\left| \int_{\{|u_\varepsilon| \le k\}} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) v_\varepsilon \, dx \right| \le \int_{\{|u_\varepsilon| \le k\}} b(k) \left[c(x) + \sum_{i=1}^N \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p w_i \right] |v_\varepsilon| \, dx$$

$$\le b(k) \int_{\{|u_\varepsilon| \le k\}} c(x) |\varphi_\lambda(z_\varepsilon)| \, dx$$

$$+ \frac{b(k)}{\alpha} \int_{\{|u_\varepsilon| \le k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon |\varphi_\lambda(z_\varepsilon)| \, dx \tag{26}$$

$$= \eta_3(\varepsilon) + \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) |\varphi_\lambda(z_\varepsilon)| \, dx$$

$$= \frac{b(k)}{\alpha} \int_\Omega (a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u))) \nabla(T_k(u_\varepsilon)$$

$$- T_k(u)) |\varphi_\lambda(z_\varepsilon)| \, dx + \eta_4(\varepsilon)$$

where

$$\eta_3(\varepsilon) = b(k) \int_{\{|u_\varepsilon| \le k\}} c(x) |\varphi_\lambda(z_\varepsilon)| \, dx \to 0 \quad \text{as } \varepsilon \to 0$$

and

$$\eta_4(\varepsilon) = \eta_3(\varepsilon) + \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u)) \nabla(T_k(u_\varepsilon) - T_k(u)) |\varphi_\lambda(z_\varepsilon)| \, dx +$$

$$\frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u) |\varphi_\lambda(z_\varepsilon)| \, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Note that, when $\lambda \ge \left(\frac{b(k)}{2\alpha} \right)^2$ we have

$$\varphi'_\lambda(s) - \frac{b(k)}{\alpha} |\varphi(s)| \ge \frac{1}{2}.$$

Combining this with (24), (25) and (26) we obtain

$$\int_\Omega \big(a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u)) \big) \nabla(T_k(u_\varepsilon) - T_k(u)) \, dx \le$$

$$\le \eta_5(\varepsilon) = 2(\eta_1(\varepsilon) - \eta_2(\varepsilon) + \eta_4(\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0.$$

Finally, Lemma 3.6 implies (21).

STEP (3) (Passing to the limit)   In virtue of (21) we have for a subsequence

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \Omega,$$

which with (20) yields

$$\begin{cases} a(x, u_\varepsilon, \nabla u_\varepsilon) \to a(x, u, \nabla u) & \text{a.e. in } \Omega, \\ g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to g(x, u, \nabla u) & \text{a.e. in } \Omega \\ g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \to g(x, u, \nabla u) u & \text{a.e. in } \Omega. \end{cases} \tag{27}$$

On the other hand, thanks to (8) and (19), we have that $a(x, u_\varepsilon, \nabla u_\varepsilon)$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$. Then, by Lemma 3.1, we obtain

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*). \tag{28}$$

It remains to prove that

$$g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{29}$$

By (27), applying Vitali's theorem it suffices to prove that $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$ is uniformly equi-integrable. Indeed, multiplying $(\mathcal{P}_\varepsilon)$ by u_ε and thanks to (10), (11) and (19), we obtain

$$0 \le \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx \le \tilde{\beta}, \tag{30}$$

where $\tilde{\beta}$ is some positive constant.

For any measurable subset E of Ω and any $m > 0$, we have

$$\int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx = \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx + \int_{E \cap Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx$$

where

$$X_m^\varepsilon = \{ x \in \Omega, \; |u_\varepsilon(x)| \le m \}, \qquad Y_m^\varepsilon = \{ x \in \Omega, \; |u_\varepsilon(x)| > m \} \tag{31}$$

From these expressions, (12) and (30), we have

$$\begin{aligned} \int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx &\le \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla T_m(u_\varepsilon))| \, dx \\ &\quad + \frac{1}{m} \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx \\ &\le b(m) \int_E \left(\sum_{i=1}^{N} w_i \left| \frac{\partial T_m(u_\varepsilon)}{\partial x_i} \right|^p + c(x) \right) dx + \tilde{\beta} \frac{1}{m}. \end{aligned} \tag{32}$$

Since the sequence $(\nabla T_m(u_\varepsilon))$ converges strongly in $\prod_{i=1}^{N} L^p(\Omega, w_i)$, the above inequality implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$. From (28) and (29), we can pass to the limit in

$$\langle Au_\varepsilon, v \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) v = \langle f, v \rangle$$

and we obtain,

$$\langle Au, v \rangle + \int_\Omega g(x, u, \nabla u) v = \langle f, v \rangle \quad \text{for any } v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \qquad (33)$$

Moreover, since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \geq 0$ a.e. in Ω, by (27), (30) and Fatou's lemma we have

$$g(x, u, \nabla u) u \in L^1(\Omega). \qquad (34)$$

This concludes the proof of Theorem 3.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.10. Note that the statement of (33) holds true for $v = u$, i.e.,

$$\langle Au, u \rangle + \int_\Omega g(x, u, \nabla u) u = \langle f, u \rangle.$$

Indeed, putting $v = T_k(u)$ in (33) and using Lemma 3.4, we have

$$\langle Au - f, T_k(u) \rangle \to \langle Au - f, u \rangle.$$

On the other hand, using Lebesgue's dominated convergence theorem, since

$$|g(x, u, \nabla u) T_k(u)| \leq |g(x, u, \nabla u)||u| \in L^1(\Omega) \quad \text{(due to (34))}$$

and

$$g(x, u, \nabla u) T_k(u) \to g(x, u, \nabla u) u \quad \text{a.e. in } \Omega.$$

we conclude that

$$g(x, u, \nabla u) T_k(u) \to g(x, u, \nabla u) u \text{ in } L^1(\Omega).$$

Proof of Lemma 3.9. We set $B_\varepsilon = A + G_\varepsilon$. Using (8) and Hölder's inequality we can show that A is bounded [8]. Thanks to (18) we have B_ε bounded. The coercivity follows from (10) and (11). To show that B_ε is hemicontinuous, let $t \to t_0$, and prove that

$$\langle B_\varepsilon(u + tv), \tilde{w} \rangle \to \langle B_\varepsilon(u + t_0 v), \tilde{w} \rangle \text{ as } t \to t_0 \text{ for all } u, v, \tilde{w} \in X.$$

Since for a.e. $x \in \Omega$, $a_i(x, u + tv, \nabla(u + tv)) \to a_i(x, u + t_0 v, \nabla(u + t_0 v))$ as $t \to t_0$, thanks to the growth condition (8), Lemma 3.1 implies

$$a_i(x, u + tv, \nabla(u + tv)) \rightharpoonup a_i(x, u + t_0 v, \nabla(u + t_0 v)) \text{ in } L^{p'}(\Omega, w_i^{1-p'}) \quad \text{as } t \to t_0.$$

Finally for all $\tilde{w} \in X$,

$$\langle A(u + tv), \tilde{w} \rangle \longrightarrow \langle A(u + t_0 v), \tilde{w} \rangle \text{ as } t \to t_0.$$

On the other hand, $g_\varepsilon(x, u + tv, \nabla(u + tv)) \to g_\varepsilon(x, u + t_0 v, \nabla(u + t_0 v))$ as $t \to t_0$ for a.e. in Ω. Also $(g_\varepsilon(x, u + tv + \nabla(u + tv)))_t$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$ because

$$\int_\Omega |g_\varepsilon(x, u + tv, \nabla(u + tv))|^{q'} \sigma^{1-q'} \leq \left(\frac{1}{\varepsilon}\right)^{q'} \int_{\Omega_\varepsilon} \sigma^{1-q'} \leq c_\varepsilon.$$

Then, Lemma 3.1 gives

$$g_\varepsilon(x, u + tv, \nabla(u + tv)) \rightharpoonup g_\varepsilon(x, u + t_0 v, \nabla(u + t_0 v)) \text{ in } L^{q'}(\Omega, \sigma^{1-q'}) \text{ as } t \to t_0.$$

Since $\tilde{w} \in L^q(\Omega, \sigma)$ for all $\tilde{w} \in X$,

$$\langle G_\varepsilon(u + tv), \tilde{w} \rangle \longrightarrow \langle G_\varepsilon(u + t_0 v), \tilde{w} \rangle \text{ as } t \longrightarrow t_0.$$

Next we show that B_ε satisfies property (M); i.e. for a sequence u_j in X satisfying

(i) $u_i \rightharpoonup u$ in X,

(ii) $B_\varepsilon u_j \rightharpoonup \chi$ in X^* and

(iii) $\limsup_{j \to \infty} \langle B_\varepsilon u_j, u_j - u \rangle \leq 0$,

we have $\chi = B_\varepsilon u$. Indeed, by Hölder's inequality and (7),

$$\int_\Omega g_\varepsilon(x, u_j, \nabla u_j)(u_j - u) \leq \left(\int_\Omega |g_\varepsilon(x, u_j, \nabla u_j)|^{q'} \sigma^{\frac{-q'}{q}} \, dx\right)^{\frac{1}{q'}} \left(\int_\Omega |u_j - u|^q \sigma \, dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\varepsilon} \left(\int_{\Omega_\varepsilon} \sigma^{\frac{-q'}{q}} \, dx\right)^{\frac{1}{q'}} \|u_j - u\|_{q,\sigma} \to 0 \qquad \text{as } j \to \infty,$$

i.e., $\langle G_\varepsilon u_j, u_j - u \rangle \to 0$ as $j \to \infty$. Combining the last convergence with (iii), we obtain

$$\limsup_{j \to \infty} \langle A u_j, u_j - u \rangle \leq 0.$$

And by the pseudo-monotonicity of A (see Proposition 1 [8]), we have $A u_j \rightharpoonup A u$ in X^* and $\lim_{j \to \infty} \langle A u_j, u_j - u \rangle = 0$. On the other hand,

$$0 = \lim_{j \to \infty} \int_\Omega a(x, u_j, \nabla u_j) \nabla(u_j - u) \, dx$$

$$= \lim_{j \to \infty} \int_\Omega (a(x, u_j, \nabla u_j) - a(x, u_j, \nabla u)) \nabla(u_j - u) \, dx$$

$$+ \int_\Omega a(x, u_j, \nabla u) \nabla(u_j - u) \, dx.$$

The last integral in the right hand tends to zero since $a(x, u_j, \nabla u) \to a(x, u, \nabla u)$ in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ as $j \to \infty$. Hence, by Lemma 3.6 we have $\nabla u_j \to \nabla u$ a.e. in Ω. Then

$$g_\varepsilon(x, u_j, \nabla u_j) \to g_\varepsilon(x, u, \nabla u) \text{ a.e. in } \Omega \quad \text{as } j \to \infty.$$

And since

$$\left| g_\varepsilon(x, u_j, \nabla u_j) \sigma^{\frac{1-q'}{q'}} \right| \le \frac{1}{\varepsilon} \sigma^{\frac{1-q'}{q'}} \chi_{\Omega_\varepsilon} \in L^{q'}(\Omega) \quad \text{(due to (5))},$$

by Lebesgue's dominated convergence theorem, we obtain

$$g_\varepsilon(x, u_j, \nabla u_j) \to g_\varepsilon(x, u, \nabla u) \text{ in } L^{q'}(\Omega, \sigma^{1-q'}) \text{ as } j \to \infty,$$

which with (7) implies

$$\int_\Omega g_\varepsilon(x, u_j, \nabla u_j) v \, dx \to \int_\Omega g_\varepsilon(x, u, \nabla u) v \, dx \quad \text{as } j \to \infty, \text{ for all } v \in X,$$

i.e., $G_\varepsilon u_j \rightharpoonup G_\varepsilon u$ in X^*. Finally,

$$B_\varepsilon u_j = A u_j + G_\varepsilon u_j \rightharpoonup A u + G_\varepsilon u = B_\varepsilon u = \chi. \qquad \square$$

Remark 3.11. The assumption (5) appears necessary in order to prove the boundedness of G_ε in $W_0^{1,p}(\Omega, w)$. Thus, when $g \equiv 0$, we don't need to assume (5).

3.3. The case where $f \in L^1(\Omega)$

In this subsection we assume that

$$\begin{cases} f \in L^1(\Omega). \\ \text{There exists } \rho_1 > 0 \text{ and } \rho_2 > 0 \text{ such that} \\ \quad \text{for } |s| \ge \rho_1, \ |g(x, s, \xi)| \ge \rho_2 \sum_{i=1}^{N} w_i |\xi_i|^p. \end{cases} \tag{35}$$

We replace (H_1) by the following assumption

Assumption (H_1').
$$(H_1) \text{ with } \sigma \in L^1(\Omega).$$

Consider the nonlinear problem with Dirichlet boundary conditions

$$(\tilde{P}) \qquad \begin{cases} A u + g(x, u, \nabla u) = f \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega). \end{cases}$$

In this case we have the following existence theorem:

Theorem 3.12. *Under the assumptions (H_1'), (H_2), (H_3) and (35), there exists at least one solution of $(\tilde{\mathcal{P}})$.*

Remark 3.13. Under the assumption (17), $ug(x, u, \nabla u)$ belongs to $L^1(\Omega)$, that is not the case in general when we assume the hypothesis (35) (cf. Remark 3 [5]).

Remark 3.14. Theorem 3.12 generalizes to the weighted case the analogous statement in [5] and [6].

Proof of Theorem 3.12. Let f_ε be a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$ and $\|f_\varepsilon\|_{L^1(\Omega)} \leq c_1$ for some constant c_1.

Now, consider the following approximate problem

$$(\tilde{\mathcal{P}}_\varepsilon) \qquad \begin{cases} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = f_\varepsilon \\ u_\varepsilon \in W_0^{1,p}(\Omega, w). \end{cases}$$

with g_ε is defined as in the problem $(\mathcal{P}_\varepsilon)$. The existence of the solution u_ε of this problem is verified as in the problem $(\mathcal{P}_\varepsilon)$.

Note that the steps of the proof of Theorem 3.12 are similar to those of Theorem 3.7, assuming that the following assertions are verified:

Assertion 1 (Estimate (19)**).** *There exist a constant c such that*

$$\||u_\varepsilon|\|_X \leq c,$$

where u_ε is a solution of $(\tilde{\mathcal{P}}_\varepsilon)$.

Assertion 2 (Convergence (22)**).** *For $v_\varepsilon = \varphi_\lambda(T_k(u_\varepsilon) - T_k(u))$ where $\varphi_\lambda(s) = se^{\lambda s^2}$, we have*

$$\int_\Omega f_\varepsilon v_\varepsilon \to 0 \ \text{ as } \varepsilon \to 0$$

Assertion 3 (Equi-integrability of $\mathbf{g}_\varepsilon(\mathbf{x}, \mathbf{u}_\varepsilon, \nabla\mathbf{u}_\varepsilon)$). *The sequence $(g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))_\varepsilon$ is uniformly equi-integrable in Ω.*

By applying the assertions described above we deduce the result as in the case when $f \in X^*$. $\qquad\qquad\square$

Proof of the assertion 1. Multiplying $(\tilde{\mathcal{P}}_\varepsilon)$ by $T_k(u_\varepsilon) \in X$ and since

$$\int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) \geq 0,$$

we obtain

$$\int_\Omega a(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) \leq kc_1.$$

In view of (10), we have

$$\sum_{i=1}^N \int_\Omega w_i \left| \frac{\partial T_k(u_\varepsilon)}{\partial x_i} \right|^p dx \leq \frac{k}{\alpha} c_1,$$

i.e.

$$\||T_k(u_\varepsilon)|\|^p \leq c_2. \tag{36}$$

On the other hand, we have

$$k \int_{\{|u_\varepsilon|>k\}} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \leq \int_\Omega |f_\varepsilon||T_k(u_\varepsilon)| \, dx \leq kc_1. \tag{37}$$

Then, by (35), (36), (37) and for $k > \rho_1$, we obtain

$$\||u_\varepsilon|\|_X^p = \sum_{i=1}^N \int_{\{|u_\varepsilon|>k|\}} w_i \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^p dx + \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_\varepsilon)}{\partial x_i}\right|^p dx$$

$$\leq \frac{1}{\rho_2} \int_{\{|u_\varepsilon|>k\}} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| + c_3 + c_2 \leq c_4,$$

where c_i, $i = 1, 2, \ldots$, are various positive constants. Then,

$$\||u_\varepsilon|\| \leq c. \qquad \qquad \square$$

Proof of the assertion 2. Since v_ε converges to zero weakly $*$ in $L^\infty(\Omega)$. and f_ε converges strongly to f in $L^1(\Omega)$. Then,

$$\int_\Omega f_\varepsilon v_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \qquad \qquad \square$$

Proof of the assertion 3. For any measurable subset E of Ω and any $m > 0$, we have, as in (32),

$$\int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx = \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx + \int_{E \cap Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx$$

$$\leq b(m) \int_E \left(\sum_{i=1}^N \omega_i \left|\frac{\partial T_m(u_\varepsilon)}{\partial x_i}\right|^p + c(x)\right) dx + \int_{Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx \tag{38}$$

For fixed m, the first integral of the right hand side of (38) is small uniformly in ε when the measure of E is small (due to $\nabla T_m(u_\varepsilon)$ converges strongly in $\Pi_{i=1}^N L^p(\Omega, w_i)$).

We now discuss the behaviour of the second integral of the right hand side of (38). We use in $(\tilde{\mathcal{P}}_\varepsilon)$ the test function $\psi_m(u_\varepsilon)$, where for $m > 1$

$$\begin{cases} \psi_m(s) = 0 & \text{if } |s| \leq m - 1, \\ \psi_m(s) = 1 & \text{if } s \geq m, \\ \psi_m(s) = -1 & \text{if } s \leq -m, \\ \psi_m'(s) = 1 & \text{if } m - 1 \leq |s| \leq m. \end{cases}$$

This yields

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \psi_m'(u_\varepsilon)\, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \psi_m(u_\varepsilon)\, dx = \int_\Omega f_\varepsilon \psi_m(u_\varepsilon)\, dx,$$

and thus

$$\int_{\{|u_\varepsilon| > m\}} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\, dx \leq \int_{\{|u_\varepsilon| > m-1\}} |f_\varepsilon|\, dx.$$

From the condition $\sigma \in L^1(\Omega)$, it is easy to verify that $|\{\, x \in \Omega,\ |u_\varepsilon| > m - 1\,\}| \to$ 0 uniformly in ε when $m \to +\infty$, and since $f_\varepsilon \to f$ strongly in $L^1(\Omega)$, we have $\int_{\{|u_\varepsilon| > m-1\}} |f_\varepsilon|\, dx$ is small uniformly in ε when $m \to \infty$, which implies that the second term of the right hand side of (38) is small uniformly in ε and in E, when m is sufficiently large. This completes the proof of the uniform equi-integrability of $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$. $\qquad\square$

Remark 3.15. The hypothesis $\sigma \in L^1(\Omega)$ in (H_1') appears in order to prove that $|\{x \in \Omega, |u_\varepsilon| > m - 1\}| \to 0$ uniformly in ε.

3.4. Example

Some ideas of this example come from [8]. Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 1$), satisfying the cone condition. Let us consider the Carathéodory functions:

$$a_i(x, s, \xi) = w_i |\xi_i|^{p-1} \operatorname{sgn}(\xi_i) \text{ for } i = 1, \dots, N$$

$$g(x, s, \xi) = \rho s |s|^r \sum_{i=1}^N w_i |\xi_i|^p, \qquad \rho > 0, \quad r > 0$$

where $w_i(x)$ ($i = 0, 1, \dots, N$) are given weight functions, strictly positive almost everywhere in Ω. We shall assume that the weight functions satisfy, $w_i(x) = w(x)$, $x \in \Omega$, for all $i = 0, \dots, N$. Then, we can consider the Hardy inequality (6) in the form,

$$\left(\int_\Omega |u(x)|^q \sigma(x)\, dx \right)^{\frac{1}{q}} \leq c \left(\int_\Omega |\nabla u(x)|^p w \right)^{\frac{1}{p}}.$$

It is easy to show that the $a_i(x, s, \xi)$ are Carathéodory functions satisfying the growth condition (8) and the coercivity (10). Also the Carathéodory function $g(x, s, \xi)$ satisfies the conditions (11), (12) and (35) with $|s| \geq \rho_1 = 1$ and $\rho_2 = \rho > 0$. On the other hand, the monotonicity condition is satisfied, in fact,

$$\sum_{i=1}^N \big(a_i(x, s, \xi) - a_i(x, s, \hat{\xi})\big)(\xi_i - \hat{\xi}_i)$$

$$= w(x) \sum_{i=1}^N \big(|\xi_i|^{p-1} \operatorname{sgn}\xi_i - |\hat{\xi}_i|^{p-1} \operatorname{sgn}\hat{\xi}_i\big)(\xi_i - \hat{\xi}_i) > 0$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^N$ with $\xi \neq \hat{\xi}$, since $w > 0$ a.e. in Ω. In particular, let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \qquad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\left(\int_\Omega |u(x)|^q \, d^\mu(x) \, dx \right)^{\frac{1}{q}} \leq c \left(\int_\Omega |\nabla u(x)|^p \, d^\lambda(x) \, dx \right)^{\frac{1}{p}}.$$

The corresponding imbedding is compact if:

(i) For, $1 < p \leq q < \infty$,

$$\lambda < p - 1, \qquad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \qquad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0. \qquad (39)$$

(ii) For, $1 \leq q < p < \infty$,

$$\lambda < p - 1, \qquad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0. \qquad (40)$$

Remark 3.16. Conditions (39) or (40) are sufficient for the compact imbedding (7) to hold (see for example [8, Example 1], [9, Example 1.5, p. 34], and [14, theorems 19.17 and 19.22]).

Finally, the hypotheses of Theorem 3.7 (resp. Theorem 3.12) are satisfied, therefore the problem $(\mathcal{P})$ (resp. $(\tilde{\mathcal{P}})$) has at least one solution.

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