On the nonlinear Neumann problem involving the critical Sobolev exponent and Hardy potential

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ABSTRACT

In this paper we investigate the solvability of the Neumann problems (1), (12), (16), (32) and (43) involving the critical Sobolev and Hardy exponents. It is assumed that the coefficient Q is a positive and smooth function on $\bar{\Omega}$, μ and λ are real parameters. We examine the common effect of the mean curvature of the boundary $\partial\Omega$, the shape of the graph of the coefficient Q and the singular Hardy potential on the existence and the nonexistence of solutions of these problems.

Key words: Neumann problem, critical Sobolev exponent, singular Hardy potential, least energy solutions, topological linking

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1. Introduction

In this paper we investigate the nonlinear elliptic problem involving the Neumann conditions

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2}u & = Q(x)|u|^{2^*-2}u \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}u(x) & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the coefficient Q is continuous and positive on $\bar{\Omega}$, μ is a real parameter, ν is an outward normal to the boundary $\partial\Omega$ and $2^* = \frac{2N}{N-2}$, $N \geq 3$, is a critical Sobolev exponent. We assume that $0 \in \Omega$ and that Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary. We also study a more general problem (12) (see Section 3) with an additional term λu. In Section 4 we extend this to problem (16), obtained from (12) by replacing μ with $-\mu$, $\mu > 0$. In Sections 7 and 8 the term $\frac{\mu}{|x|^2}u$ is replaced by $\frac{\mu}{|x|^{2^*_\alpha}}|u|^{2^*_\alpha-2}u$, where $2^*_\alpha = \frac{2(N-\alpha)}{N-2}$, $0 < \alpha < 2$ is the critical Hardy-Sobolev exponent.

In recent years the nonlinear Neumann problem involving critical Sobolev exponent has been widely studied in [3], [4], [5], [7], [8], [9]. In these papers the existence of least energy solutions has been established for problem (1), with the singular term $\mu\frac{u}{|x|^2}$ replaced by λu, $\lambda > 0$ and with $Q(x) = 1$ on Ω. Further extensions of these results to the problem with $Q(x) \neq$ constant can be found in [13], [14], [15]. The novelty here is that we consider the Neumann problem involving the singular potential $\frac{1}{|x|^2}$ and the critical Sobolev exponent. Equation (1) with the Dirichlet boundary conditions, has been studied in [1], [16], [23] and [20]. The singular potential $\frac{1}{|x|^2}$ is related to the Hardy inequality. We recall the classical Hardy inequality (known also as the Uncertainty Principle): if $u \in H^1_\circ(\Omega)$, then $\frac{u}{|x|} \in L^2(\Omega)$ and

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le c_N \int_\Omega |\nabla u|^2\,dx, \tag{2}$$

where $c_N = \frac{4}{(N-2)^2}$ and this constant is optimal. It is also known that the constant $\frac{1}{c_N}$ is not achieved. Therefore one can expect an error term on the left side of this inequality. Some estimates of this error term can be found in the papers [2], [12] and [24]. Problem (1) has a variational structure and the underlying Sobolev space for (1) is $H^1(\Omega)$. Since this space contains constant functions, it is clear that this inequality is no longer true in $H^1(\Omega)$. In Section 2 we give a suitable modification of (2) which will be used in this paper. In Sections 3 and 4 we investigate the existence of the least energy solutions. Section 6 is devoted to the case where a parameter λ in interferes with the spectra of $-\Delta + \frac{\mu}{|x|^2}$ and $-\Delta - \frac{\mu}{|x|^2}$. Our approach is based on a min-max principle involving the topological linking [36]. Sections 7 and 8 are devoted to nonlinear Neumann problems involving the critical Hardy-Sobolev exponent. We establish the existence of solutions through the mountain-pass principle.

We recall that a C^1 functional $\phi : X \to \mathbb{R}$ on a Banach space X satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short), if each sequence $\{x_n\} \subset X$ such that $(*)$ $\phi(x_n) \to c$ and $(**)$ $\phi'(x_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence $\{x_n\}$ satisfying $(*)$ and $(**)$ is called a Palais-Smale sequence at level c (a $(PS)_c$ sequence for short).

Throughout this paper we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$ are denoted by $\|\cdot\|$. By $H^1(\Omega)$ we denote a standard Sobolev space on Ω equipped with norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right)dx.$$

2. Palais-Smale condition in the case $\mu > 0$

Throughout this and the next section, we assume that $\mu > 0$. We commence by extending the Hardy inequality to the space $H^1(\Omega)$.

Lemma 2.1. *For every $\delta > 0$ there exists a constant $C = C(\delta, |\Omega|) > 0$ such that*

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le (c_N + \delta)\int_\Omega |\nabla u|^2\,dx + C(\delta,|\Omega|)\int_\Omega u^2\,dx$$

for every $u \in H^1(\Omega)$.

Proof. Let $\rho > 0$ be such that $\overline{B(0,2\rho)} \subset \Omega$. We define a C^1-function ϕ such that $\phi(x) = 1$ on $B(0,\rho)$, $0 \le \phi(x) \le 1$ on Ω and $\phi(x) = 0$ on $\Omega - B(0,\rho)$. It then follows from (2) that

$$\begin{aligned}\int_\Omega \frac{u^2}{|x|^2}\,dx &= \int_\Omega \frac{(\phi u)^2}{|x|^2}\,dx + \int_\Omega \frac{u^2}{|x|^2}\big(1-\phi^2\big)\,dx\\ &\le c_N\int_\Omega |\nabla(\phi u)|^2\,dx + \int_\Omega \frac{u^2(1-\phi^2)}{|x|^2}\,dx.\end{aligned}$$

Applying the Young inequality, we get

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le (c_N+\delta)\int_\Omega |\nabla u|^2\phi^2\,dx + \int_\Omega u^2\big(c_N|\nabla\phi|^2 + \frac{c_N^2}{\delta}|\nabla\phi|^2 + \frac{1-\phi^2}{|x|^2}\big)\,dx$$

and the result follows. □

One can define the best Hardy constant in $H^1(\Omega)$ by

$$S_h = \inf_{u\in H^1(\Omega)} \frac{\int_\Omega(|\nabla u|^2 + u^2)\,dx}{\int_\Omega \frac{u^2}{|x|^2}\,dx}.$$

The constant S_h depends on $|\Omega|$ and tends to 0 as $|\Omega| \to 0$. Since Ω is a bounded domain, it follows from Lemma 2.1 that

$$\|u\|^2 = \int_\Omega \big(|\nabla u|^2 + \frac{u^2}{|x|^2}\big)\,dx$$

defines an equivalent norm on $H^1(\Omega)$.

We set

$$Q_m = \max_{x\in\partial\Omega} Q(x) \quad\text{and}\quad Q_M = \max_{x\in\bar\Omega} Q(x).$$

Let

$$S_\infty = \min\Big(\frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}, \frac{S}{Q_M^{\frac{N-2}{N}}}\Big),$$

where S is the best Sobolev constant defined by

$$S = \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx;\ u \in D^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^*}\, dx\Big\}.$$

Here $D^{1,2}(\mathbb{R}^N)$ denotes the Sobolev space obtained as the completion of $C_\circ^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

To find a solution of (1) we consider the constrained variational problem

$$S_\mu = \inf\{\int_\Omega \big(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2\big)\, dx;\ u \in H^1(\Omega),\ \int_\Omega Q(x)|u|^{2^*}\, dx = 1\}. \tag{3}$$

By Lemma 2.1 and the Sobolev inequality, we see that $0 < S_\mu < \infty$ for every $\mu > 0$. If u is a minimizer for S_μ, then $S_\mu^{\frac{1}{2^*-2}} u$ is a solution of (1).

Proposition 2.2. *If $S_\mu < S_\infty$ for some $\mu > 0$, then S_μ has a minimizer.*

Proof. The proof is standard and relies on P. L. Lions' concentration-compactness principle [25]. Let $\{u_m\}$ be a minimizing sequence for S_μ. Since $\{u_m\}$ is bounded in $H^1(\Omega)$, we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$ and $u_m \to u$ in $L^p(\Omega)$ for $2 \le p < 2^*$. By the concentration-compactness principle, we may assume that

$$|u_m|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} \quad \text{and} \quad |\nabla u_m|^2 \overset{*}{\rightharpoonup} |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j},$$

in the sense of measure, where $\nu_j > 0$, $\mu_j > 0$ are constants and the set J is at most countable. Moreover, we have

$$\text{if } x_j \in \Omega, \text{ then } S\nu_j^{\frac{2}{2^*}} \le \mu_j$$

and

$$\text{if } x_j \in \partial\Omega, \text{ then } S\frac{\nu_j^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} \le \mu_j.$$

The only possible concentration point for $\{\frac{u_m^2}{|x|^2}\}$ is 0. However, if this occurs, then $\{|\nabla u_m|^2\}$ also concentrates at 0. Hence it is sufficient to show that $\mu_j = \nu_j = 0$ for all $j \in J$. We write

$$1 = \int_\Omega Q(x)|u|^{2^*}\, dx + \sum_{j\in J} Q(x_j)\nu_j. \tag{4}$$

We also have

$$
\begin{aligned}
S_\mu &\geq \int_\Omega \big(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2\big)\,dx + \sum_{j\in J}\mu_j \\
&\geq S_\mu\Big(\int_\Omega Q(x)|u|^{2^*}\,dx\Big)^{\frac{2}{2^*}} + \sum_{x_j\in\Omega} S\nu_j^{\frac{2}{2^*}} + \sum_{x_j\in\partial\Omega}\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \\
&\geq S_\mu\Big(\int_\Omega Q(x)|u|^{2^*}\,dx\Big)^{\frac{2}{2^*}} + \sum_{x_j\in\Omega}\frac{S}{Q_M^{\frac{N-2}{N}}}\big(\nu_j Q(x_j)\big)^{\frac{N-2}{N}} \\
&\quad + \sum_{x_j\in\partial\Omega}\frac{S}{Q_m^{\frac{N-2}{N}}\,2^{\frac{2}{N}}}\big(\nu_j Q(x_j)\big)^{\frac{N-2}{N}}.
\end{aligned}
\tag{5}
$$

Since $S_\mu < S_\infty$, we deduce from (5) that $\nu_j = 0$ for every $j \in J$. $\square$

To estimate S_μ, we test the functional

$$
E_{\mu,Q}(u) = \frac{\int_\Omega\big(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2\big)\,dx}{\big(\int_\Omega Q(x)|u|^{2^*}\,dx\big)^{\frac{2}{2^*}}}
\tag{6}
$$

with instantons. We recall that the instanton $U(x) = \frac{d_N}{(1+|x|^2)^{\frac{N-2}{2}}}$, where $d_N > 0$ is a normalizing constant, satisfies the equation

$$
-\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N.
$$

Furthermore, we have $\int_{\mathbb{R}^N} U^{2^*}\,dx = \int_{\mathbb{R}^N}|\nabla U|^2\,dx = S^{\frac{N}{2}}$. We set

$$
U_{\epsilon,y}(x) = \frac{d_N\epsilon^{\frac{N-2}{2}}}{\big(\epsilon^2 + |x-y|^2\big)^{\frac{N-2}{2}}}, \quad y\in\mathbb{R}^N, \quad \epsilon > 0.
$$

We now observe that

$$
\int_\Omega \frac{1}{|x|^2}U_{\epsilon,y}(x)^2\,dx = O(\epsilon^2) \quad \text{if } y \neq 0
\tag{7}
$$

and

$$
c_1 \leq \int_\Omega \frac{1}{|x|^2}U_{\epsilon,0}(x)^2\,dx \leq c_2
\tag{8}
$$

for all $\epsilon > 0$, where $c_1, c_2 > 0$ are constants. We always have

$$
S_\mu \leq S_\infty.
$$

This follows by testing the functional $E_{\mu,Q}$ with $U_{\epsilon,y}$ centered at a point y where either Q_m or Q_M is achieved. However, if Q_M is is achieved only at 0 we cannot use directly $U_{\epsilon,0}$ since by (8) $\int_\Omega \frac{1}{|x|^2} U_{\epsilon,0}\,dx$ is bounded away from 0 for small $\epsilon > 0$. In this case we choose a sequence $y_k \to 0$. Testing $E_{\mu,Q}$ with U_{ϵ,y_k} we obtain the estimate $S_\mu \leq \frac{S}{Q(y_k)^{\frac{N-2}{N}}}$. Letting $y_k \to 0$ the desired estimate follows. By testing the functional $E_{\mu,Q}$ with $u = 1$ we get

$$S_\mu \leq \frac{\mu \int_\Omega \frac{1}{|x|^2}\,dx}{\left(\int_\Omega Q(x)\,dx\right)^{\frac{2}{2^*}}}.$$

Therefore by Proposition 2.2 problem (1) for $\mu < \frac{S_\infty \left(\int_\Omega Q(x)\,dx\right)^{\frac{2}{2^*}}}{\int_\Omega \frac{1}{|x|^2}\,dx}$ has a least energy solution.

3. Existence and nonexistence of least energy solutions for (1)

We distinguish two cases: (i) $Q_M \leq 2^{\frac{2}{N-2}} Q_m$ and (ii) $Q_M > 2^{\frac{2}{N-2}} Q_m$. We denote by $H(y)$ the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$. If $Q = 1$ we set $E_\mu = E_{\mu,1}$. It is known that

$$\begin{aligned} &E_\mu(U_{\epsilon,y}) \leq \\ &\leq \frac{S}{2^{\frac{2}{N}}} - \begin{cases} A_N H(y)\epsilon \log\frac{1}{\epsilon} - a_N\mu\epsilon + O(\epsilon) + o(\mu\epsilon), & N=3 \\ A_N H(y)\epsilon - a_N\mu\epsilon^2\log\frac{1}{\epsilon} + O\big(\epsilon^2\log\frac{1}{\epsilon}\big) + o\big(\mu\epsilon^2\log\frac{1}{\epsilon}\big), & N=4 \\ A_N H(y)\epsilon - a_N\mu\epsilon^2 + O(\epsilon^2) + o(\mu\epsilon^2), & N \geq 5, \end{cases} \end{aligned} \tag{9}$$

where $a_N > 0$ is a constant depending on N. The asymptotic estimation (9) has been established in [3] and [31] for E_μ with the singular term $\frac{u^2}{|x|^2}$ replaced by u^2. Since $y \in \partial\Omega$ and $0 \in \Omega$, the proof of (9) is the same as in the nonsingular case.

Theorem 3.1. *Let $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that*

$$|Q(x) - Q(y)| = o(|x-y|) \tag{10}$$

for x near y with $Q(y) = Q_m$ and $H(y) > 0$. Then for every $\mu > 0$ problem (1) has a least energy solution.

Proof. Under our assumptions $S_\infty = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$. Using (9) and (10) we get

$$E_{\mu,Q}\big(U_{\epsilon,y}\big) < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$$

for $\epsilon > 0$ sufficiently small. The result follows from Proposition 2.2. □

We now consider the case $Q_M > 2^{\frac{2}{N-2}} Q_m$. We recall the existence result from [15] for the Neumann problem without the singular term

$$\begin{cases} -\Delta u + \lambda u &= Q(x)|u|^{2^*-2}u \text{ in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) &= 0 \text{ on } \partial\Omega. \end{cases} \tag{11}$$

Theorem 3.2. *Suppose that $Q_M > 2^{\frac{2}{N-2}} Q_m$. Then there exists a constant $\bar{\Lambda} > 0$ such that for $0 < \lambda \le \bar{\Lambda}$ problem* (11) *has a least energy solution and no least energy solution for $\lambda > \bar{\Lambda}$ and moreover*

$$\frac{S}{Q_M^{\frac{N-2}{N}}} = \inf_{u \in H^1(\Omega) - \{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx}{\left(\int_\Omega Q(x)|u|^{2^*} \, dx\right)^{\frac{2}{2^*}}}$$

for $\lambda \ge \bar{\Lambda}$.

Theorem 3.3. *Suppose that $Q_M > 2^{\frac{2}{N-2}} Q_m$. Then there exists a constant $\Lambda_1 > 0$ such that for $0 < \mu < \Lambda_1$ problem* (1) *has a least energy solution and no least energy solution for $\mu > \Lambda_1$ and moreover $S_\mu = \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $\mu \ge \Lambda_1$.*

Proof. Let $r = \inf_{x \in \bar{\Omega} - \{0\}} \frac{1}{|x|^2} > 0$. Then for every $u \in H^1(\Omega)$ we have

$$\int_\Omega \left(|\nabla u|^2 + \mu r u^2\right) dx \le \int_\Omega \left(|\nabla u|^2 + \frac{\mu}{|x|^2} u^2\right) dx.$$

Hence

$$\inf\{\int_\Omega \left(|\nabla u|^2 + \mu r u^2\right) dx;\, u \in H^1(\Omega);\, \int_\Omega Q(x)|u|^{2^*} \, dx = 1\} \le S_\mu$$

and the result follows from Theorem 3.2. □

Theorems 3.1 and 3.3 remain true for the problem

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u + \lambda u &= Q(x)|u|^{2^*-2}u \text{ in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) &= 0 \text{ on } \partial\Omega, \end{cases} \tag{12}$$

where $\mu > 0$ and $\lambda > 0$. We set

$$S_{\mu,\lambda} = \inf\Big\{\int_\Omega \left(|\nabla u|^2 + \frac{\mu}{|x|^2} + \lambda u^2\right) dx;\, u \in H^1(\Omega),\, \int_\Omega Q(x)|u|^{2^*} \, dx = 1\Big\}.$$

Theorem 3.4. *Let $Q_M \le 2^{\frac{2}{N-2}} Q_m$ and suppose that* (10) *holds. Then problem* (12) *has a least energy solution for every $\mu > 0$ and $\lambda > 0$.*

Theorem 3.5. *Let $Q_M > 2^{\frac{2}{N-2}}$. Then there exists $\Lambda > 0$ such that for $\lambda + \mu r < \Lambda$ problem* (12) *has a least energy solution and no least energy solution for $\lambda + \mu r > \Lambda$. Moreover, $S_{\mu,\lambda} = \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $\lambda + \mu r \geq \Lambda$.*

If $Q_M \leq 2^{\frac{2}{N-2}} Q_m$ and Q_m is achieved only at points of $\partial\Omega$ with the negative mean curvature then least energy solutions do not exist for large $\mu > 0$. This follow from the following result [14]:

Theorem 3.6. *Let $N \geq 5$ and $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that* (10) *holds and moreover*

$$\{x \in \partial\Omega;\, H(x) < 0\} \neq \emptyset \quad \textit{and} \quad \{x \in \partial\Omega;\, Q(x) = Q_m\} \subset \{x \in \partial\Omega;\, H(x) < 0\}.$$

Then there exists $\tilde{\lambda} > 0$ such that for $0 < \lambda \leq \tilde{\lambda}$ problem (11) *has a least energy solution and no least energy solution for $\lambda > \tilde{\lambda}$ and*

$$\frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} = \inf_{H^1(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx}{\left(\int_\Omega Q(x)|u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}$$

for $\lambda \geq \tilde{\lambda}$.

Theorem 3.7. *Suppose that the assumptions of Theorem 3.6 hold. Then there exists a constant $\tilde{\Lambda} > 0$ such that for $0 < \mu < \tilde{\Lambda}$ problem* (1) *has a least energy solution and no least energy solution for $\mu > \tilde{\Lambda}$ and moreover $S_\mu = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$ for $\mu \geq \tilde{\Lambda}$.*

We now examine $S_{\mu,\lambda}$ as a function of λ for a fixed $\mu > 0$. It is clear that $S_{\mu,\lambda}$ is continuous and non decreasing. It is also bounded from above by S_∞. Testing $S_{\mu,\lambda}$ with $u = 1$ on Ω, we see that

$$S_{\mu,\lambda} \leq \frac{\mu \int_\Omega \frac{dx}{|x|^2} + \lambda|\Omega|}{\left(\int_\Omega Q(x)\, dx\right)^{\frac{2}{2^*}}}.$$

Hence $\lim_{\lambda\to-\infty} S_{\mu,\lambda} = -\infty$. We show, below in Proposition 3.8, that $S_{\mu,\lambda}$ admits a minimizer for every $\lambda \in \mathbb{R}$ with $S_{\mu,\lambda} \leq 0$. However, these minimizers do not satisfy (12).

Proposition 3.8. (i) *If $S_{\mu,\lambda} < 0$ for some $\mu > 0$ and $\lambda \in \mathbb{R}$, then there exists a minimizer u for $S_{\mu,\lambda}$ which after rescaling $|S_{\mu,\lambda}|^{\frac{1}{2^*-2}} u$ satisfies*

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u + \lambda u &= -Q(x)|u|^{2^*-2} u \quad \textit{in } \Omega, \\ \frac{\partial}{\partial\nu} u(x) &= 0 \quad \textit{on } \partial\Omega. \end{cases} \tag{13}$$

(ii) *There exists a unique* $\lambda_\circ \leq -\mu r$ *such that* $S_{\mu,\lambda_\circ} = 0$*, where* $r = \min_{x\in\bar\Omega-\{0\}} \frac{1}{|x|^2}$*. Moreover,* $-\lambda_\circ$ *is an eigenvalue of the following problem*

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2}u &= \sigma u \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}u(x) &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{14}$$

Proof. Suppose that $S_{\mu,\lambda} < 0$ for some $\lambda \in \mathbb{R}$. Let $\{u_m\}$ be a minimizing sequence for $S_{\mu,\lambda}$, that is,

$$\int_\Omega \left(|\nabla u_m|^2 + \frac{\mu}{|x|^2}u_m^2 + \lambda u_m^2\right) dx \to S_{\mu,\lambda} \text{ and } \int_\Omega Q(x)|u_m|^{2^*}\, dx = 1. \tag{15}$$

Since $\{u_m\}$ is bounded in $H^1(\Omega)$ we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$ and $u_m \to u$ in $L^2(\Omega)$. It follows from (15) that $u \neq 0$. We claim that $\int_\Omega Q(x)|u|^{2^*}\, dx = 1$. In the contrary case there exists $t > 1$ such that

$$t^{2^*}\int_\Omega Q(x)|u|^{2^*}\, dx = 1.$$

Then by the lower semicontinuity of a norm with respect to a weak convergence, we have

$$S_{\mu,\lambda} \leq t^2 \int_\Omega \left(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2 + \lambda u^2\right) dx \leq t^2 S_{\mu,\lambda}.$$

Since $S_{\mu,\lambda} < 0$, we must have $t^2 \leq 1$, which is impossible. Thus $\int_\Omega Q(x)|u|^{2^*}\, dx = 1$ and u is a minimizer. Letting $v = |S_{\mu,\lambda}|^{\frac{1}{2^*-2}}u$, we verify that v is a solution of problem (13). If $S_{\mu,\lambda_\circ} = 0$ for some $\lambda_\circ = \lambda_\circ(\mu) < 0$, then the limit u of a minimizing sequence $\{u_m\}$ must be nonzero. Indeed, if $u = 0$ on Ω, then (15) yields $u_m \to 0$ in $H^1(\Omega)$ which is impossible. If $\int_\Omega Q(x)|u|^{2^*}\, dx < 1$, then a suitable multiple tu for some $t > 1$ is a minimizer. By the continuity of $S_{\mu,\lambda}$ we can find $\delta > 0$ such that $S_{\mu,\lambda} < S_\infty$ for every $\lambda < \lambda_\circ + \delta$. Since $S_{\mu,\lambda}$ is attained for each $\lambda < \lambda_\circ + \delta$, $S_{\mu,\lambda}$ is strictly increasing on this interval. Therefore $S_{\mu,\lambda}$ vanishes only at $\lambda_\circ$. On the other hand considering the Rayleigh quotient for the first eigenvalue we get

$$\inf_{u\in H^1(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2\right) dx}{\int_\Omega u^2\, dx} \geq \mu r.$$

Hence $\lambda_\circ \leq -\mu r$. □

4. Problem (12) with $\mu < 0$

It is convenient to write problem (12) with $\mu < 0$ in the following way

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2}u + \lambda u &= Q(x)|u|^{2^*-2}u \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}u(x) &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $\mu > 0$ and $\lambda \in \mathbb{R}$. To find solutions of (16) we consider the constrained minimization problem

$$S_{-\mu,\lambda} = \inf_{u \in H^1(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \frac{\mu}{|x|^2} u^2 + \lambda u^2\right) dx}{\left(\int_\Omega Q(x)|u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}.$$

First we consider the case where $S_{-\mu,\lambda} > 0$. To examine the concentration phenomena of minimizing sequences we need the following quantity

$$S_{-\mu} = \inf_{D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\mu}{|x|^2} u^2\right) dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}.$$

It is known [18] that if $0 < \mu < \bar{\mu} = \frac{(N-2)^2}{4} = \frac{1}{c_N}$, then

$$S_{-\mu} = \inf_{u \in H^1_\circ(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}, \tag{17}$$

which means that the inf over $H^1_\circ(\Omega)$ is independent of Ω. If $\Omega = \mathbb{R}^N$ the constant $S_{-\mu}$ is attained by a family of functions (see [29])

$$U^\mu_\epsilon(x) = \frac{k_N \epsilon^{\sqrt{\bar{\mu}-\mu}}}{\left(\epsilon^{\frac{\gamma-\gamma'}{\sqrt{\bar{\mu}}}} |x|^{\frac{\gamma'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\gamma}{\sqrt{\bar{\mu}}}}\right)^{\sqrt{\bar{\mu}}}}, \quad \epsilon > 0,$$

where $k_N > 0$ is a normalizing constant and $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu}-\mu}$ and $\gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu}$. We also have

$$\int_{\mathbb{R}^N} |\nabla U^\mu_\epsilon(x)|^2\, dx - \mu \int_{\mathbb{R}^N} \frac{\left(U^\mu_\epsilon(x)\right)^2}{|x|^2}\, dx = \int_{\mathbb{R}^N} \left(U_\epsilon(x)^\mu\right)^{2^*} dx = S^{\frac{N}{2}}_{-\mu}.$$

We obviously have $S_{-\mu} < S$ for $\mu < \bar{\mu}$ and $\lim_{\mu \to 0} S_{-\mu} = S$. On the other hand the constant in (17) is not attained if Ω is a bounded star-shaped domain containing the origin (see [21]).

Let $S_{-\mu,\lambda} > 0$. Suppose that $\{u_m\}$ is a minimizing sequence for $S_{-\mu,\lambda}$. Since $\{u_m\}$ is bounded in $H^1(\Omega)$, we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*}(\Omega)$ and $u_m \to u$ in $L^2(\Omega)$. It follows from the concentration-compactness principle that

$$|u_n|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_\circ \delta_\circ$$

and

$$|\nabla u_m|^2 - \mu \frac{u_m^2}{|x|^2} \overset{*}{\rightharpoonup} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_\circ \delta_\circ,$$

where μ_j, $\mu_\circ$, ν_j and $\nu_\circ$ are positive constant and J is at most countable set. The sequence $\{\frac{u_m^2}{|x|^2}\}$ can only concentrate at 0. Hence $\mu_\circ\delta_\circ$ is a joint effect of the concentration of $\{|\nabla u_m|^2\}$ and $\{\frac{u_m^2}{|x|^2}\}$ at 0. Moreover, we have

$$S\nu_j^{\frac{2}{2^*}} \le \mu_j \text{ if } x_j \ne 0 \text{ and } x_j \in \Omega,$$
$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \le \mu_j \text{ if } x_j \in \partial\Omega$$

and

$$\bar{S}_\infty = \min\left(\frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}, \frac{S}{Q_M^{\frac{N-2}{N}}}, \frac{S_{-\mu}}{Q(0)^{\frac{N-2}{N}}}\right).$$

Proposition 4.1. *Suppose that*

$$0 < S_{-\mu,\lambda} < \bar{S}_\infty$$

for some $0 < \mu < \bar{\mu}$ *and* $\lambda \in \mathbb{R}$. *Then problem* (16) *has a least energy solution.*

The proof is similar to that of Proposition 2.2 and is omitted.

To apply Proposition 4.1 we must ensure the existence of λ and $\mu \in (0, \bar{\mu})$ such that $0 < S_{-\mu,\lambda} < \bar{S}_\infty$. It follows from Lemma 2.1 that for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\begin{aligned}\big(1 - (c_N + \delta)\mu\big)\int_\Omega |\nabla u|^2\,dx + \big(\lambda - C(\delta)\mu\big)\int_\Omega u^2\,dx \le \\ \le \int_\Omega \Big(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2 + \lambda u^2\Big)\,dx \quad (18)\end{aligned}$$

for every $u \in H^1(\Omega)$. Choosing $\delta > 0$ so that $(1-(c_N+\delta)\mu) > 0$ and then taking $\lambda > C(\delta)$ we can guarantee $S_{-\mu,\lambda}$ to be positive. Testing $S_{-\mu,\lambda}$ with constant functions we deduce that $\lim_{\lambda\to-\infty} S_{-\mu,\lambda} = -\infty$ for each $\mu \in (0, \frac{1}{c_N})$. We are now in a position to formulate the existence results for problem (16).

Theorem 4.2. (i) *Suppose that* $\bar{S}_\infty = \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}$. *If* (10) *holds, then for every* $0 < \mu < \bar{\mu}$, *there exists* $\bar{\lambda} = \bar{\lambda}(\mu) > 0$ *such that for every* $0 < \mu < \bar{\mu}$ *and* $\bar{\lambda} < \lambda$ *problem* (16) *has a least energy solution.*

(ii) *Suppose that* $\bar{S}_\infty = \frac{S}{Q_M^{\frac{N-2}{N}}}$. *Then there exists* $\Lambda > 0$ *such that for every* $0 < \lambda_\circ < \Lambda$ *there exists* $0 < \mu_\circ < \bar{\mu}$ *such that for* $0 < \mu < \mu_\circ$ *and* $\lambda_\circ < \lambda < \Lambda$ *problem* (16) *has a least energy solution.*

(iii) *Suppose that* $\bar{S}_\infty = \frac{S_{-\mu}}{Q(0)^{\frac{N-2}{N}}}$ *Then there exists* $\lambda^* > 0$ *such that for every* $0 < \lambda_* < \lambda^*$ *there exists* $0 < \mu_\circ < \bar{\mu}$ *such that for every* $\lambda_* < \lambda < \lambda^*$ *and* $0 < \mu < \mu_\circ$ *problem* (16) *has a least energy solution.*

Proof. (i) The estimate (18) shows that for every $0 < \mu < \bar{\mu}$ there exists $\bar{\lambda}(\mu)$ such that $S_{-\mu,\lambda} > 0$ for $\lambda > \bar{\lambda}$. We now apply the asymptotic estimate (9) to verify that

$$S_{-\mu,\lambda} < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$$

for $\lambda > \bar{\lambda}$.

(ii) First, we observe that

$$S_{-\mu,\lambda} \leq S_{0,\lambda} \tag{19}$$

for every $0 < \mu < \bar{\mu}$ and $\lambda \in \mathbb{R}$. According to Theorem 3.2 there exists $\Lambda > 0$ such that $S_{0,\lambda} < \frac{S}{Q_M^{\frac{N-2}{N}}}$ for every $0 < \lambda < \Lambda$ and $S_{0,\lambda} = \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $\lambda \geq \Lambda$. Let $\lambda_\circ < \Lambda$ be given. Then using (18) we may choose $0 < \mu_\circ < \bar{\mu}$ such that $S_{-\mu,\lambda} > 0$ for every $0 < \mu < \mu_\circ$ and $\lambda_\circ < \lambda < \Lambda$. The result follows from Proposition 4.1.

(iii) In this case using (19) and the fact that $\lim_{\lambda \to 0} S_{0,\lambda} = 0$ we can find $\lambda^* > 0$ such that

$$S_{-\mu,\lambda} < \frac{S_{-\mu}}{Q(0)^{\frac{N-2}{N}}}$$

for every $\lambda < \lambda^*$. It then follows from (18) that given $\lambda_* < \lambda^*$ we can choose $0 < \mu_\circ \leq \bar{\mu}$ so that $0 < S_{-\mu,\lambda}$ for $0 < \mu \leq \mu_\circ$ and $\lambda_* \leq \lambda \leq \lambda^*$ and the result follows. □

We now establish a result for problem (16) which analogous to Proposition 3.8.

Proposition 4.3. (i) *Let* $S_{-\mu,\lambda} < 0$ *for some* $\lambda \in \mathbb{R}$. *Then there exists a minimizer* u *and* $|S_{-\mu,\lambda}|^{\frac{1}{2^*-2}} u$ *is a solution of problem*

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2} u + \lambda u & = -Q(x)|u|^{2^*-2} u \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{20}$$

(ii) *There exists a* $\lambda_\circ \geq \mu r$ *such that* $S_{-\mu,\lambda_\circ} = 0$ *and* $-\lambda_\circ$ *is an eigenvalue of the problem*

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2} u & = \sigma u \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{21}$$

Proof. (i) Let $\{u_m\}$ be a minimizing sequence for $S_{-\mu,\lambda} < 0$, that is,

$$\int_\Omega \left(|\nabla u_m|^2 - \frac{\mu}{|x|^2} u_m^2 + \lambda u_m^2 \right) dx = S_{-\mu,\lambda} + o(1) \text{ and } \int_\Omega Q(x)|u_m|^{2^*} dx = 1$$

for every m. We may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*}(\Omega)$ and $u_m \to u$ on $L^2(\Omega)$. By Lemma 2.1 for every $\delta > 0$, with $\mu(c_N + \delta) < 1$, there exists $C(\delta) > 0$ such that

$$(1 - (c_N + \delta)\mu) \int_\Omega |\nabla u_m|^2 \, dx + (\lambda - C(\delta)\mu) \int_\Omega u_m^2 \, dx \leq S_{-\mu,\lambda} + o(1).$$

This yields $u \neq 0$. We now show that $\int_\Omega Q(x)|u|^{2^*} \, dx = 1$. In the contrary case we have $\int_\Omega Q(x)|tu|^{2^*} \, dx = 1$ for some $t > 1$. On the other hand by the Lieb-Brézis lemma [10], letting $v_m = u_m - u$, we get

$$\int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v_m|^2 \, dx - \mu \int_\Omega \frac{u^2}{|x|^2} \, dx - \mu \int_\Omega \frac{v_m^2}{|x|^2} \, dx \\ + \lambda \int_\Omega u^2 \, dx + \lambda \int_\Omega v_m^2 \, dx = S_{-\mu,\lambda} + o(1).$$

Let $0 < \delta$ and $(c_N + \delta)\mu < 1$. Applying Lemma 2.1 we get

$$\big(1 - (c_N + \delta)\mu\big) \int_\Omega |\nabla v_m|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx - \mu \int_\Omega \frac{u^2}{|x|^2} \, dx \\ + (\lambda - \mu C(\delta)) \int_\Omega v_m^2 \, dx + \lambda \int_\Omega u^2 \, dx \leq S_{-\mu,\lambda} + o(1).$$

Since $v_m \to 0$ in $L^2(\Omega)$, we deduce from the above inequality that

$$\int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + \lambda u^2\Big) \, dx \leq S_{-\mu,\lambda}.$$

From this we derive that

$$S_{-\mu,\lambda} \leq t^2 \int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + \lambda u^2\Big) \, dx \leq S_{-\mu,\lambda} t^2.$$

Since $S_{-\mu,\lambda} < 0$, we see that $t^2 \leq 1$ which is impossible. Therefore u is a minimizer.

(ii) In a similar manner we show that if $S_{-\mu,\lambda_\circ} = 0$ for some $\lambda_\circ$, then there exists a minimizer u satisfying (21). By the continuity of $S_{-\mu,\lambda}$ there exists $\delta > 0$ such that $S_{-\mu,\lambda} < \bar{S}_\infty$ for $\lambda \in (-\infty, \lambda_\circ + \delta)$. Since for every $\lambda \in (-\infty, \lambda_\circ + \delta)$ $S_{-\mu,\lambda}$ is achieved, $S_{-\mu,\lambda}$ is strictly increasing and $\lambda_\circ$ is unique. □

5. Eigenvalue problems

We consider two eigenvalue problems (14) and (21). We begin by proving the existence of the first eigenvalues denoted by λ_1^μ and $\lambda_1^{-\mu}$, respectively.

Proposition 5.1. (i) *For every $\mu > 0$ there exists the first (smallest) eigenvalue λ_1^μ of problem* (14) *which satisfies $\lambda_1^\mu \geq \mu r$.*

(ii) *For every* $0 < \mu < \frac{1}{c_N}$ *there exists the first (smallest) eigenvalue* $\lambda_1^{-\mu}$ *of problem* (21) *which satisfies* $\lambda_1^{-\mu} \le -\mu r$.

Proof. (ii) We define $\lambda_1^{-\mu}$ by

$$\lambda_1^{-\mu} = \inf_{u \in H^1(\Omega)-\{0\}} \frac{\int_\Omega \big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\big)\, dx}{\int_\Omega u^2\, dx}.$$

Using Lemma 2.1 we verify that $\lambda_1^{-\mu} > -\infty$. It is clear that $\lambda_1^{-\mu} \le -\mu r$. Let $\{u_m\}$ be a minimizing sequence for $\lambda_1^{-\mu}$. Then

$$\int_\Omega \big(|\nabla u_m|^2 - \frac{\mu}{|x|^2} u_m^2\big)\, dx = \lambda_1^{-\mu} + o(1) \ \text{ and } \ \int_\Omega |u_m|^2\, dx = 1. \tag{22}$$

With the aid of Lemma 2.1 we show that $\{u_m\}$ is bounded in $H^1(\Omega)$. We may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $u_m \to u$ in $L^2(\Omega)$. Letting $v_m = u_m - u$ we have

$$\int_\Omega |\nabla u_m|^2\, dx = \int_\Omega |\nabla v_m|^2\, dx + \int_\Omega |\nabla u|^2\, dx + o(1),$$
$$\int_\Omega \frac{u_m^2}{|x|^2}\, dx = \int_\Omega \frac{v_m^2}{|x|^2}\, dx + \int_\Omega \frac{u^2}{|x|^2}\, dx + o(1)$$

and

$$\int_\Omega u_m^2\, dx = 1 + o(1).$$

Substituting these relations into (22) we get

$$\int_\Omega |\nabla v_m|^2\, dx + \int_\Omega |\nabla u|^2\, dx - \mu \int_\Omega \frac{v_m^2}{|x|^2}\, dx - \mu \int_\Omega \frac{u^2}{|x|^2}\, dx = \lambda_1^{-\mu} + o(1). \tag{23}$$

We fix $\delta > 0$ so that $\mu + \delta < \frac{1}{c_N}$. Then $\lambda_1^{-(\mu+\delta)} > -\infty$. Since

$$\lambda_1^{-(\mu+\delta)} \int_\Omega v_m^2\, dx \le \int_\Omega |\nabla v_m|^2\, dx - (\mu+\delta) \int_\Omega \frac{v_m^2}{|x|^2}\, dx,$$

we get from (23) that

$$\lambda_1^{-(\mu+\delta)} \int_\Omega v_m^2\, dx + \delta \int_\Omega \frac{v_m^2}{|x|^2}\, dx + \int_\Omega |\nabla u|^2\, dx - \mu \int_\Omega \frac{u^2}{|x|^2}\, dx \le \lambda_1^{-\mu} + o(1).$$

From this we deduce that $\delta \int_\Omega \frac{v_m^2}{|x|^2}\, dx = o(1)$, so $\int_\Omega \frac{u_m^2}{|x|^2}\, dx \to \int_\Omega \frac{u^2}{|x|^2}\, dx$. Therefore by (23) we have

$$\int_\Omega \big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\big)\, dx \le \lambda_1^{-\mu}.$$

Since $\int_\Omega u^2\, dx = 1$, u is a minimizer for $\lambda_1^{-\mu}$.

The proof of (i) is similar and is omitted. □

The proof of (ii) strongly relies on the fact that $0 < \mu < \frac{1}{c_N}$. We were unable to show whether $\lambda_1^{-\mu}$ with $\mu = \frac{1}{c_N}$ is attained or not. It is known that the constant $\lambda_1^{-\mu}$ defined on $H^1_\circ(\Omega) - \{0\}$ is not attained for $\mu = \frac{1}{c_N}$ (see [6], [30]). Also, in the case of $H^1_\circ(\Omega)$, $\lambda_1^{-\mu}$ is positive. Since $H^1(\Omega)$ contains constant functions, $\lambda_1^{-\mu}$ in our case is negative.

By Lemma 2.1 $-\Delta - \frac{\mu}{|x|^2} + C$ with the Neumann boundary conditions, $0 < \mu < \frac{1}{c_N}$ and $C > 0$ sufficiently large is a positive definite and self-adjoint operator. Therefore its spectrum $\sigma_{-\mu}$ is discrete and consists of an increasing sequence of eigenvalues $\{\lambda_j^{-\mu,C}\}$ converging to infinity as $j \to \infty$. Eigenvalues $\{\lambda_j^{-\mu}\}$ of the operator $-\Delta - \frac{\mu}{|x|^2}$ are given by $\lambda_j^{-\mu} = \lambda_j^{-\mu,C} - C$. Eigenfunctions of $-\Delta - \frac{\mu}{|x|^2}$ can be characterized by usual Rayleigh quotients. In particular, if ϕ is an eigenfunction corresponding to the smallest eigenvalue $\lambda_1^{-\mu}$, then $|\phi|$ is also an eigenfunction of $\lambda_1^{-\mu}$. Consequently we may assume that $\phi \geq 0$ on Ω. Applying Theorem 8.19 (the strong maximum principle for weak solutions) in [22] we can choose $\phi > 0$ a. e. on Ω. It is easy to show that the eigenvalue $\lambda_1^{-\mu}$ is simple.

Similarly, the spectrum σ_μ of $-\Delta + \frac{\mu}{|x|^2}$ is also discrete, each eigenvalue λ_k^μ and has a finite multiplicity. The smallest eigenvalue λ_1^μ is simple. Moreover $\lambda_k^\mu \to \infty$ as $k \to \infty$.

6. Topological linking

A min-max principle based on a topological linking will be used to investigate the existence of solutions of problems (1) and (16) in the cases where a parameter λ interferes with the spectrum σ_μ and $\sigma_{-\mu}$. We rewrite both problems in the following way

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u &= \lambda u + Q(x)|u|^{2^*-2}u \ \text{ in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) &= 0 \ \text{ on } \partial\Omega. \end{cases} \tag{24}$$

and

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2} u &= \lambda u + Q(x)|u|^{2^*-2}u \ \text{ in } \Omega, \\ \frac{\partial}{\partial \nu} u(x) &= 0 \ \text{ on } \partial\Omega, \end{cases} \tag{25}$$

where $\mu > 0$ for problem (24) and $0 < \mu < \frac{1}{c_N} = \bar{\mu}$ for problem (25). The range for a parameter λ will be given later. Solutions of problems (24) and (25) will be obtained as critical points of the variational functionals

$$I_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + \mu\frac{u^2}{|x|^2} - \lambda u^2\big)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx$$

and

$$I_{-\mu,\lambda}(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2\big)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx.$$

Lemma 6.1. *For every $c \in \mathbb{R}$, $(PS)_c$ sequences for both functionals $I_{-\mu,\lambda}$ and $I_{\mu,\lambda}$ are bounded in $H^1(\Omega)$.*

Proof. First, we consider the functional $I_{-\mu,\lambda}$. We assume that $0 < \mu < \frac{1}{c_N}$. Let $\{u_m\}$ be a $(PS)_c$ sequence. Arguing by contradiction assume that $\|u_m\| \to \infty$. We set $v_m = \frac{u_m}{\|u_m\|}$. Since $\{u_m\}$ is bounded in $H^1(\Omega)$ we may assume that $v_m \rightharpoonup v$ in $H^1(\Omega)$ and $v_m \to v$ in $L^p(\Omega)$ for every $2 \le p < 2^*$. Then

$$\frac{1}{\|u_m\|^{2^*-2}} \int_\Omega \big(\nabla v_m \nabla \phi - \mu \frac{v_m \phi}{|x|^2} - \lambda v_m \phi\big)\, dx = \int_\Omega Q(x)|v_m|^{2^*-2} v_m \phi \, dx + o(1).$$

We deduce from this that

$$\int_\Omega Q(x)|v|^{2^*-2} v\phi \, dx = 0$$

for every $\phi \in H^1(\Omega)$. This yields $v = 0$ a.e. on Ω. Since $\{u_m\}$ is a $(PS)_c$ sequence, we get

$$\frac{1}{2} \int_\Omega \big(|\nabla v_m|^2 - \mu \frac{v_m^2}{|x|^2} - \lambda v_m^2\big)\, dx - \frac{1}{2^*} \|u_m\|^{2^*-2} \int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0$$

and

$$\int_\Omega \big(|\nabla v_m|^2 - \mu \frac{v_m^2}{|x|^2} - \lambda v_m^2\big)\, dx - \|u_m\|^{2^*-2} \int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0$$

as $m \to \infty$. Since $v_m \to 0$ in $L^2(\Omega)$, these two relations yield that

$$\lim_{m\to\infty} \int_\Omega \left(|\nabla v_m|^2 - \mu \frac{v_m^2}{|x|^2}\right) dx = 0 \tag{26}$$

and

$$\lim_{m\to\infty} \|u_m\|^{2^*-2} \int_\Omega Q(x)|v_m|^{2^*}\, dx = 0.$$

We now apply Lemma 2.1 with $\delta > 0$ chosen so that $\mu(c_N + \delta) < 1$. Thus

$$\int_\Omega \left(|\nabla v_m|^2 - \mu \frac{v_m^2}{|x|^2}\right) dx \ge \big(1 - \mu(c_N + \delta)\big) \int_\Omega |\nabla v_m|^2 \, dx - C(\delta)\mu \int_\Omega v_m^2 \, dx \tag{27}$$

for some constant $C(\delta) > 0$. Since $v_m \to 0$ in $L^2(\Omega)$, we deduce from (26) and (27) that $\lim_{m\to\infty} \int_\Omega |\nabla v_m|^2 \, dx = 0$, which is a contradiction.

We now consider the functional $I_{\mu,\lambda}$ with $\mu > 0$. If $\{u_m\}$ is a $(PS)_c$ sequence of this functional, then

$$\frac{1}{N} \int_\Omega Q(x)|u_m|^{2^*}\, dx = I_{\mu,\lambda}(u_m) - \frac{1}{2}\langle I'_{\mu,\lambda}(u_m), u_m\rangle = c + o(\|u_m\|) + o(1).$$

Hence

$$\int_\Omega Q(x)|u_m|^{2^*}\,dx \le C_1 + C_2\|u_m\|$$

and

$$\int_\Omega u_m^2\,dx \le C_1 + C_2\|u_m\|$$

for some constants $C_1, C_2 > 0$ and all m. These two estimates combined with the inequality

$$\begin{aligned}\frac{1}{2}\int_\Omega \left(|\nabla u_m|^2 + \mu r u_m^2\right) dx &\le \frac{1}{2}\int_\Omega \left(|\nabla u_m|^2 + \mu \frac{u_m^2}{|x|^2}\right) dx\\ &= c + \frac{\lambda}{2}\int_\Omega u_m^2\,dx + \frac{1}{2^*}\int_\Omega Q(x)|u_m|^{2^*}\,dx + o(1)\end{aligned}$$

imply that the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. □

To proceed further we set

$$S_{\infty,h} = \min\left(\frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}\right) \quad \text{and} \quad \bar{S}_{\infty,h} = \min\left(S_{\infty,h}, \frac{S_{-\mu}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}\right).$$

Proposition 6.2. (i) *Let $0 < \mu < \frac{1}{c_N}$ and $\lambda \in \mathbb{R}$. Then $I_{-\mu,\lambda}$ satisfies the $(PS)_c$ condition for $c < \bar{S}_{\infty,h}$.*

(ii) *Let $\mu > 0$ and $\lambda \in \mathbb{R}$. Then $I_{\mu,\lambda}$ satisfies $(PS)_c$ condition for $c < S_{\infty,h}$.*

The proof is straightforward application of the concentration-compactness principle and is omitted.

We are now in a position to establish the existence results through a min-max principle based on a topological linking. First we consider problem (24). We assume that

$$\lambda_{k-1}^\mu < \lambda < \lambda_k^\mu \text{ for some } k. \tag{28}$$

Let $E_\mu^- =$ span $\{e_1^\mu, \dots, e_l^\mu\}$, where $e_1^\mu, \dots, e_l^\mu$ are all eigenfunctions corresponding to eigenvalues $\lambda_1^\mu \dots, \lambda_{k-1}^\mu$. We have the orthogonal decomposition

$$H^1(\Omega) = E_\mu^- \oplus E_\mu^+.$$

Let $w \in E_\mu^+ - \{0\}$ and define a set

$$M^\mu = \{u \in H^1(\Omega);\ u = v + sw,\ v \in E_\mu^-,\ s \ge 0,\ \|u\| \le R\}.$$

Proposition 6.3. *There exists $\alpha > 0$, $\rho > 0$ and $R > \rho$ (R depending on w) such that*

$$I_{\mu,\lambda}(u) \geq \alpha \ \text{ for all } u \in E_\mu^+ \cap \partial B(0,\rho)$$

and

$$I_{\mu,\lambda}(u) \leq 0 \ \text{ for } u \in \partial M^\mu.$$

The proof is standard and is omitted.

We now define

$$Z_\epsilon = E_\mu^- \oplus \mathbb{R}U_{\epsilon,y} = E_\mu^- \oplus \mathbb{R}U_{\epsilon,y}^+,$$

where $U_{\epsilon,y}^+$ denotes the projection of $U_{\epsilon,y}$ onto E_μ^+. From now on we use $U_{\epsilon,y}^+$ in the definition of M^μ.

Theorem 6.4. *Suppose that $N \geq 5$.*

(i) *Let $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that* (10) *and* (28) *hold. Then problem* (24) *has a solution.*

(ii) *Let $Q_M > 2^{\frac{2}{N-2}} Q_m$. Suppose that $Q_M = Q(y)$ with $y \neq 0$, $Q \in C^2(B(0,\rho))$ for some ball $B(0,\rho) \subset \Omega$ and $D_{ij}Q(y) = 0$, $i, j = 1, \ldots, N$. Then for every μ there exists an integer $k(\mu) \geq 1$ such that for $\lambda_{j-1} < \lambda < \lambda_j$ with $j \geq k(\mu)$, problem* (24) *has a solution.*

Proof. (i) We follow, with some modifications, the argument on pp. 52–53 in [36]. For $u \neq 0$ we have

$$\max_{t \geq 0} I_{\mu,\lambda}(tu) = \frac{1}{N} \frac{\left(\int_\Omega \left(|\nabla u|^2 + \mu \frac{u^2}{|x|^2} - \lambda u^2\right) dx\right)^{\frac{N}{2}}}{\left(\int_\Omega Q(x)|u|^{2^*}\, dx\right)^{\frac{N-2}{2}}}$$

whenever the integral in the numerator is positive and the maximum is 0 otherwise. In what follows we always denote by C_i positive constants independent of ϵ. It is sufficient to show that

$$m_\epsilon^\mu = \sup_{u \in Z_\epsilon, \|u\|_{2^*,Q} = 1} \int_\Omega \left(|\nabla u|^2 + \mu \frac{u^2}{|x|^2} - \lambda u^2\right) dx < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}.$$

This obviously implies that

$$\sup_{u \in M} I_{\mu,\lambda}(u) < \frac{S^{\frac{N}{2}}}{2N Q_m^{\frac{N-2}{2}}}.$$

If $u \in Z_\epsilon$ and $\|u\|_{2^*,Q} = 1$, then

$$u = u^- + sU_{\epsilon,y} = \left(u^- + sU_{\epsilon,y}^-\right) + sU_{\epsilon,y}^+,$$

where $U^-_{\epsilon,y}$ and $U^+_{\epsilon,y}$ denote the projections of $U_{\epsilon,y}$ on E^-_μ and E^+_μ, respectively. We now observe that

$$\int_\Omega \big(|\nabla U^-_{\epsilon,y}|^2 + \frac{\mu}{|x|^2}(U^-_{\epsilon,y})^2 - \lambda(U^-_{\epsilon,y})^2\big)\,dx \le 0,$$

so

$$\int_\Omega \big(|\nabla U^-_{\epsilon,y}|^2 + \frac{\mu}{|x|^2}(U^-_{\epsilon,y})^2\big)\,dx \le \int_\Omega \lambda(U^-_{\epsilon,y})^2\,dx \le \lambda\int_\Omega U^2_{\epsilon,y}\,dx = O(\epsilon^2).$$

By the Sobolev inequality, we deduce

$$\|U^-_{\epsilon,y}\|^2_{2^*} \le C_1\int_\Omega \big(|\nabla U^-_{\epsilon,y}|^2 + \mu r\big(U^-_{\epsilon,y}\big)^2\big)\,dx \le C_1\int_\Omega \big(|\nabla U^-_{\epsilon,y}|^2 + \frac{\mu}{|x|^2}\big(U^-_{\epsilon,y}\big)^2\big)\,dx \to 0$$

as $\epsilon \to 0$. Therefore there exists a constant $C_2 > 0$ such that $0 < s \le C_2$ and $\|u^-\|_{2^*} \le C_2$. It follows from the convexity of $\|\cdot\|^{2^*}_{2^*,Q}$ that

$$\begin{aligned} 1 = \|u\|^{2^*}_{2^*,Q} &\ge \|sU_{\epsilon,y}\|^{2^*}_{2^*,Q} + 2^*\int_\Omega Qu^-\big(sU_{\epsilon,y}\big)^{2^*-1}\,dx \\ &\ge \|sU_{\epsilon,y}\|^{2^*}_{2^*,Q} - C_3\|U_{\epsilon,y}\|^{2^*-1}_{2^*-1}\|u^-\|_2. \end{aligned}$$

Hence

$$\|sU_{\epsilon,y}\|^{2^*}_{2^*,Q} \le C_4\epsilon^{\frac{N-2}{2}} + 1.$$

Since all norms in E^-_μ are equivalent, we see that

$$\begin{aligned} &\int_\Omega \big(\nabla u^-\nabla U_{\epsilon,y} + \frac{\mu}{|x|^2}u^-U_{\epsilon,y}\big)\,dx \le \\ &\qquad \le C_5\big(\|\nabla U_{\epsilon,y}\|_1 + \|\frac{1}{|\cdot|^2}U_{\epsilon,y}\|_1\big)\|u^-\|_2 = O\big(\epsilon^{\frac{N-2}{2}}\big)\|u^-\|_2. \end{aligned} \tag{29}$$

It follows from (10) that

$$\|U_{\epsilon,y}\|^{2^*}_{2^*Q} = Q_m\int_\Omega U^{2^*}_{\epsilon,y}\,dx + o(\epsilon).$$

With the aid of (29) we obtain

$$
\begin{aligned}
&\int_\Omega \big(|\nabla u|^2 + \frac{\mu}{|x|^2}u^2 - \lambda u^2\big)\,dx \\
&\qquad \le (\lambda^\mu_{k-1} - \lambda)\int_\Omega |u^-|^2\,dx + O\big(\epsilon^{\frac{N-2}{2}}\big)\|u^-\|_2 \\
&\qquad + s^2\int_\Omega \big(|\nabla U_{\epsilon,y}|^2 + \mu\frac{U^2_{\epsilon,y}}{|x|^2} - \lambda U^2_{\epsilon,y}\big)\,dx \\
&\qquad = -(\lambda - \lambda^\mu_{k-1})\|u^-\|_2^2 + O\big(\epsilon^{\frac{N-2}{2}}\big)\|u^-\|_2 \\
&\qquad + s^2\int_\Omega \big(|\nabla U_{\epsilon,y}|^2 + \mu\frac{U^2_{\epsilon,y}}{|x|^2} - \lambda U^2_{\epsilon,y}\big)\,dx \\
&\qquad = -(\lambda - \lambda^\mu_{k-1})\|u^-\|_2^2 + O\big(\epsilon^{\frac{N-2}{2}}\big)\|u^-\|_2 \\
&\qquad + \frac{\int_\Omega\big(|\nabla U_{\epsilon,y}|^2 + \mu\frac{U^2_{\epsilon,y}}{|x|^2} - \lambda U^2_{\epsilon,y}\big)\,dx}{\big(\int_\Omega Q(x)U^{2^*}_{\epsilon,y}\,dx\big)^{\frac{2}{2^*}}}\Big(s^{2^*}\int_\Omega Q(x)U^{2^*}_{\epsilon,y}\,dx\Big)^{\frac{2}{2^*}}.
\end{aligned}
\tag{30}
$$

We now take into account the estimate (9), in order to estimate the ratio term on the right hand side of (30). We then have

$$
\begin{aligned}
m^\mu_\epsilon &\le -(\lambda - \lambda^\mu_{k-1})\|u^-\|_2^2 + O\big(\epsilon^{\frac{N-2}{2}}\|u_-\|_2\big) \\
&+ \Bigg(\frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}} - \epsilon A_N Q_m^{\frac{N-2}{N}}H(y) + o(\epsilon)\Bigg)\big(1 + C_4\epsilon^{\frac{N-2}{2}}\big)
\end{aligned}
$$

for some constant $A_N > 0$ and the result follows.

(ii) The only change is in the estimating the ratio term on the right-hand side of (30). First we observe that

$$
\int_\Omega \frac{1}{|x|^2}U^2_{\epsilon,y}\,dx = O(\epsilon^2) \quad \text{and} \quad \int_\Omega U^2_{\epsilon,y}\,dx \ge c_1\epsilon^2
$$

for some $c_1 > 0$ independent of $\epsilon > 0$. Moreover, we have

$$
\int_\Omega Q(x)U^{2^*}_{\epsilon,y}\,dx = Q_M\int_\Omega U^{2^*}_{\epsilon,y}\,dx + o(\epsilon^2).
$$

Since $\lim_{j\to\infty}\lambda^\mu_j = \infty$, we can find an integer $k(\mu)$ such for $\lambda^\mu_{j-1} < \lambda < \lambda_j$ and $j \ge k(\mu)$ the term $\lambda\int_\Omega U^2_{\epsilon,y}\,dx$ dominates $\mu\int_\Omega \frac{1}{|x|^2}U^2_{\epsilon,y}\,dx$ and the result follows. $\square$

We now consider problem (25). We use similar notations as in the case (24). By $\{(e^{-\mu}_j\}$, $j = 1, 2, \ldots$, we denote the sequence of eigenfunctions corresponding to eigenvalues $\{\lambda^{-\mu}_j\}$, $j = 1, 2, \ldots$. We assume that a parameter λ satisfies

$$
\lambda^{-\mu}_{k-1} < \lambda < \lambda^{-\mu}_k \quad \text{for some} \quad k. \tag{31}
$$

We set $E^-_{-\mu} = \operatorname{span}\{e_1^{-\mu}, \dots, e_l^{-\mu}\}$, where $e_1^{-\mu}, \dots, e_l^{-\mu}$ are eigenfunctions corresponding the eigenvalues $\lambda_1^{-\mu}, \dots, \lambda_{k-1}^{-\mu}$. We have the orthogonal decomposition $H^1(\Omega) = E^-_{-\mu} \oplus E^+_{-\mu}$. Let $w \in E^-_{-\mu} - \{0\}$ and define a set

$$M^{-\mu} = \{u \in H^1(\Omega);\ u = v + sw,\ v \in E^-_{-\mu},\ s \geq 0, \text{ and } \|u\| \leq R\}.$$

Proposition 6.5. *Suppose that (31) holds. Then there exists $\alpha > 0$, $\rho > 0$ and $R > \rho$ (depending on w) such that*

$$I_{-\mu,\lambda}(u) \geq \alpha \ \text{ for every } u \in E^+_{-\mu} \cap \partial B(0,\rho)$$

and

$$I_{-\mu,\lambda}(u) \leq 0 \ \text{ for every } u \in \partial M^{-\mu}.$$

Theorem 6.6. *Suppose that $N \geq 5$.*

(i) *Let $\bar{S}_{\infty,h} = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$. Suppose that for some $y \in \partial\Omega$ with $H(y) > 0$ and $Q_m = Q(y)$ we have*

$$|Q(y) - Q(x)| = o(|x-y|) \ \text{ for } x \text{ close to } y.$$

If λ satisfies (31), then problem (25) has a solution.

(ii) *Let $\bar{S}_{\infty,h} = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$. Suppose that $Q(y) = Q_M$ with $y \neq 0$, $Q \in C^2(B(0,\rho))$ for some ball $B(0,\rho) \subset \Omega$ and $D_{ij}Q(0) = 0$, $i,j = 1,\dots,N$. If $\lambda > 0$ and satisfies (31), then problem (25) has a solution.*

(iii) *Let $\bar{S}_{\infty,h} = \frac{S_{-\mu}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}$ and $Q \in C^2(B(0,\rho))$. Suppose that (31) holds and $\mu < \bar{\mu} - 1$ and $\lambda > 0$ and moreover $D_iQ(0) = 0$ and $D_{ij}Q(0) = 0$, $i,j = 1,\dots,N$. Then problem (25) has a solution.*

Proof. The proof of (i) and (ii) is similar to the proof Theorem 24. In both cases we use $w = U^+_{\epsilon,y}$ in the definition of $M^{-\mu}$.

To show that $\sup_{u \in M^{-\mu}} I_{-\mu,\lambda}(u) < \frac{S_{-\mu}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}$ we take $w = U^\mu_\epsilon$ in the definition of $M^{-\mu}$. By straightforward calculations we verify that

$$a\epsilon^2 \leq \int_\Omega \left(U^\mu_\epsilon\right)^2 dx \leq b\epsilon^2$$

for some constants $a > 0$ and $b > 0$ independent of ϵ. Moreover, if $\mu < \bar{\mu} - 1$, then

$$\int_\Omega \left(|\nabla U^\mu_\epsilon|^2 - \frac{\mu}{|x|^2}\left(U^\mu_\epsilon\right)\right) dx = \int_{\mathbb{R}^N} \left(|\nabla U^\mu_\epsilon|^2 - \frac{\mu}{|x|^2}\left(U^\mu_\epsilon\right)\right) dx + o(\epsilon^2)$$

and

$$\int_\Omega \left(U_\epsilon^\mu\right)^{2^*} dx = \int_{\mathbb{R}^N} \left(U_\epsilon^\mu\right)^{2^*} dx + o(\epsilon^2).$$

These estimates allow to derive the following inequality

$$\frac{\int_\Omega \left(|\nabla U_\epsilon^\mu|^2 - \frac{\mu}{|x|^2}\left(U_\epsilon^\mu\right)^2 - \lambda\left(U_\epsilon^\mu\right)^2\right) dx}{\left(\int_\Omega Q(x)\left(U_\epsilon^\mu\right)^{2^*} dx\right)^{\frac{2}{2^*}}} < \frac{S_{-\mu}^{\frac{N}{2}}}{Q(0)^{\frac{N-2}{N}}} - \lambda \bar{a}\epsilon^2$$

for some constant $\bar{a} > 0$. This obviously implies the desired estimate form above of $I_{-\mu,\lambda}$ on the set $Z^{-\mu}$. □

7. Critical Hardy-Sobolev nonlinearity

In this section we are concerned with the existence of solutions of the following problem

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^\alpha}|u|^{2_\alpha^*-2}u + \lambda u & = Q(x)|u|^{q-2}u \ \text{ in } \Omega, \\ \frac{\partial}{\partial \nu}u(x) & = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{32}$$

We assume that $\lambda > 0$ and $\mu > 0$. For $0 \leq \alpha < 2$, $2_\alpha^* = 2\frac{N-\alpha}{N-2}$ is the limiting exponent for the Hardy-Sobolev embedding $H_\circ^1(\Omega) \to L^{2_\alpha^*}(\Omega, |x|^{-\alpha})$ and $2 < q \leq 2^*$. It is known that that $H_\circ^1(\Omega)$ is continuously embedded into $L^{2_\alpha^*}(\Omega, |x|^{-\alpha})$. If $2 \leq p < 2_\alpha^*$, then $H_\circ^1(\Omega)$ is compactly embedded into $L^p(\Omega, |x|^{-\alpha})$. Let

$$S_\alpha = \inf_{u \in H_\circ^1(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2_\alpha^*}}{|x|^\alpha}\, dx\right)^{\frac{2}{2_\alpha^*}}}. \tag{33}$$

The constant S_α is independent of Ω and is not achieved if $\Omega \neq \mathbb{R}^N$. If $\alpha = 0$, then $S_\circ = S$. For every $\epsilon > 0$, the family of functions

$$u_\epsilon(x) = \frac{\epsilon^{\frac{N-2}{2}}\left((N-2)(N-\alpha)\right)^{\frac{N-2}{2(2-\alpha)}}}{\left(\epsilon^{2-\alpha} + |x|^{2-\alpha}\right)^{\frac{N-2}{2-\alpha}}}$$

satisfies the equation

$$-\Delta u_\epsilon = |x|^{-\alpha}u_\epsilon^{2_\alpha^*} \ \text{ in } \mathbb{R}^N$$

and is a minimizer for

$$S_\alpha = \inf_{u \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^N} |u|^{2_\alpha^*}|x|^{-\alpha}\, dx\right)^{\frac{2}{2_\alpha^*}}}.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla u_\epsilon(x)|^2\, dx = \int_{\mathbb{R}^N} u_\epsilon(x)^{2_\alpha^*}|x|^{-\alpha}\, dx = S_\alpha^{\frac{N-\alpha}{N-2}}.$$

It follows from (33) that

$$S_\alpha\Big(\int_\Omega \frac{|u|^{2^*_\alpha}}{|x|^\alpha}\Big)^{\frac{2}{2^*_\alpha}}\,dx \le \int_\Omega |\nabla u|^2\,dx$$

for every $u \in H^1_\circ(\Omega)$. It is clear that this inequality is no longer true in $H^1(\Omega)$. As in Section 2 we formulate the following modification of this inequality in $H^1(\Omega)$.

Lemma 7.1. *For every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that*

$$\Big(\int_\Omega \frac{|u|^{2^*_\alpha}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}} \le \big(S_\alpha^{-1} + \delta\big)\int_\Omega |\nabla u|^2\,dx + C(\delta)\left[\int_\Omega u^2\,dx + \Big(\int_\Omega |u|^{2^*_\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}}\right].$$

Proof. Let ϕ be a function defined in the proof of Lemma 2.1. Then

$$\begin{aligned}\Big(\int_\Omega \frac{|u|^{2^*_\alpha}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}} &\le \Big(\int_\Omega \frac{|u\phi|^{2^*_\alpha}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}} + \Big(\int_\Omega \frac{|u|^{2^*_\alpha}\big(1-\phi^{2^*_\alpha}\big)}{|x|^\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}} \\ &\le S_\alpha^{-1}\int_\Omega \big(|\nabla u|^2\phi^2 + 2u\nabla u\phi\nabla\phi + u^2|\nabla\phi|^2\big)\,dx + C_1\Big(\int_\Omega |u|^{2^*_\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}}. \qquad (34)\end{aligned}$$

for some constant $C_1 > 0$. An application of the Young inequality completes the proof. □

Solutions of (32) will be sought as critical points of the functional

$$J_{\alpha,-\mu}(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + \lambda u^2\big)\,dx - \frac{\mu}{2^*_\alpha}\int_\Omega \frac{|u|^{2^*_\alpha}}{|x|^\alpha}\,dx - \frac{1}{q}\int_\Omega Q(x)|u|^q\,dx.$$

It follows from Lemma 7.1 that the functional $J_{\alpha,-\mu}(u)$ is well defined for $u \in H^1(\Omega)$. It is easy to verify that $J_{\alpha,-\mu}$ is C^1 and

$$\langle J'_{\alpha,-\mu}(u), \phi\rangle = \int_\Omega \big(\nabla u\nabla\phi + \lambda u\phi\big)\,dx - \mu\int_\Omega \frac{|u|^{2^*_\alpha-2}u}{|x|^\alpha}\phi\,dx - \int_\Omega Q(x)|u|^{q-2}u\phi\,dx$$

for every $\phi \in H^1(\Omega)$.

We set

$$S_{\infty,\alpha,-\mu} = \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{N(Q(0)+\mu)^{\frac{N-2}{2}}}, \frac{(2-\alpha)\mu S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\big(Q(0)+\mu\big)^{\frac{N-2}{2-\alpha}}}\right).$$

Proposition 7.2. (i) *If $q = 2^*$, then the functional $J_{\alpha,-\mu}$ satisfies the $(PS)_c$ condition for $c < S_{\infty,\alpha,-\mu}$.*

(ii) *If* $2 < q < 2^*$, *then the functional* $J_{\alpha,-\mu}$ *satisfies the* $(PS)_c$ *condition for*

$$c < \frac{(2-\alpha)}{2(N-\alpha)} \frac{S^{\frac{N}{2}}}{(Q(0)+\mu)^{\frac{N-2}{2-\alpha}}}.$$

Proof. (i) Let $\{u_m\}$ be a $(PS)_c$ sequence. Then

$$\begin{aligned} J_{\alpha,-\mu}(u_m) &- \frac{1}{2}\langle J'_{\alpha,-\mu}(u_m), u_m\rangle \\ &= \frac{(2-\alpha)\mu}{2(N-\alpha)} \int_\Omega \frac{|u_m|^{2^*_\alpha}}{|x|^\alpha}\, dx + \frac{1}{N}\int_\Omega Q(x)|u_m|^{2^*}\, dx \leq c + 1 + o(\|u_m\|) \end{aligned}$$

for $m \geq m_\circ$. This combined with the fact that $J_{\alpha,-\mu}(u_m) \to c$ implies that the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. Since $\{u_m\}$ is bounded in $H^1(\Omega)$ we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*}(\Omega)$, $L^{2^*_\alpha}(\Omega, |x|^{-\alpha})$ and $u_m \to u$ in $L^p(\Omega)$ for $2 \leq p < \infty$. On the other hand by the concentration - compactness principle we have

$$|u_n|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} + \nu_\circ \delta_\circ$$

and

$$|\nabla u_m|^2 \overset{*}{\rightharpoonup} |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j} + \mu_\circ \delta_\circ.$$

The sequence $\{\frac{|u_m|^{2^*_\alpha}}{|x|^\alpha}\}$ can only concentrate at 0, so we have

$$\frac{|u_m|^{2^*_\alpha}}{|x|^\alpha} \overset{*}{\rightharpoonup} \frac{|u|^{2^*_\alpha}}{|x|^\alpha} + \bar{\nu}_\circ.$$

Using a family of test functions concentrating at x_j (or at 0), we derive the following relations

$$\mu_j = Q(x_j)\nu_j \quad \text{for } x_j \neq 0 \tag{35}$$

and

$$\mu_\circ = Q(0)\nu_\circ + \mu\bar{\nu}_\circ. \tag{36}$$

We now show that all coefficients ν_j and $\bar{\nu}_\circ$ vanish. If $\nu_j > 0$ for some $x_j \in \Omega$, then by (35) and the fact that $S\nu_j^{\frac{2}{2^*}} \leq \mu_j$ we get that

$$\nu_j \geq \frac{S^{\frac{N}{2}}}{Q(x_j)^{\frac{N}{2}}}. \tag{37}$$

If $x_j \in \partial\Omega$, then $\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \leq \mu_j$ and

$$\nu_j \geq \frac{S^{\frac{N}{2}}}{2Q(x_j)^{\frac{N}{2}}}. \tag{38}$$

Assuming that $\nu_j > 0$ for $x_j \in \Omega$, then

$$c = \lim_{m\to\infty} \left[J_{\alpha,-\mu}(u_m) - \frac{1}{2}\langle J_{\alpha,-\mu}(u_m), u_m\rangle \right] \geq \frac{1}{N} Q(x_j)\nu_j \geq \frac{S^{\frac{N}{2}}}{NQ_m^{\frac{N-2}{2}}}$$

and we get a contradiction. In a similar manner we show that $\nu_j = 0$ if $x_j \in \partial\Omega$. We now distinguish two cases: (a) $\nu_\circ < \bar{\nu}_\circ$ and (b) $\bar{\nu}_\circ \leq \nu_\circ$. If (a) occurs, then $\mu_\circ \leq \big(Q(0) + \mu\big)\bar{\nu}_\circ$ and by (33) we get

$$\bar{\nu}_\circ \geq \frac{S_\alpha^{\frac{N-2}{2-\alpha}}}{\big(Q(0)+\mu\big)^{\frac{N-\alpha}{2-\alpha}}}.$$

Since

$$\begin{aligned} c &> \lim_{m\to\infty} \left[J_{\alpha,-\mu}(u_m) - \frac{1}{2}\langle J_{\alpha,-\mu}(u_m), u_m\rangle \right] \\ &\geq \frac{(2-\alpha)\mu S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\big(Q(0)+\mu\big)^{\frac{N-\alpha}{2-\alpha}}} \end{aligned}$$

we have arrived at a contradiction. If (b) prevails, then $\mu_\circ \leq \big(Q(0) + \mu\big)\nu_\circ$ and consequently

$$\nu_\circ > \frac{S^{\frac{N}{2}}}{\big(Q(0)+\mu\big)^{\frac{N-2}{2}}}.$$

Again, as in the previous case, we get a contradiction. □

To obtain critical points of $J_{\alpha,-\mu}$ we apply the mountain-pass principle. First we check that the functional $J_{\alpha,-\mu}$ has a mountain-pass geometry. It follows from Lemma 7.1 and Sobolev inequalities that

$$\begin{aligned} J_{\alpha,-\mu}(u) \geq{}& \frac{\min(1,\lambda)}{2} \int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx \\ &- \frac{\mu}{2_\alpha^*} \left[\big(S_\alpha^{-1} + \delta\big) \int_\Omega \big(|\nabla u|^2 + \lambda u^2\big)\,dx + \tilde{C}(\delta) \int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx \right]^{\frac{2_\alpha^*}{2}} \\ &- C_1 \left(\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx \right)^{\frac{q}{2}} \end{aligned}$$

for some constants $\tilde{C}(\delta) > 0$ and $C_1 > 0$. Since $2_\alpha^* > 2$, we can choose constants $\rho > 0$ and $\alpha > 0$ so that

$$J_{\alpha,-\mu}(u) \geq \alpha \quad \text{for } \|u\| = \rho.$$

For every $v \neq 0$ in $H^1(\Omega)$ we have $J_{\alpha,-\mu}(tv) < 0$ and $\|tv\| > \rho$ for sufficiently large $t > 0$. We now define the mountain-pass level

$$d_{\alpha,-\mu} = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J_{\alpha,-\mu}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\Omega));\ \gamma(0) = (0),\ \gamma(1) = v\}$$

with $\|v\| > \rho$ and $J_{\alpha,-\mu}(v) < 0$.

First we establish an existence result in the subcritical case.

Theorem 7.3. *Suppose that $2 < q < 2^*$ if $N \geq 4$ and $4 < q < 6$ if $N = 3$. Then for every $\lambda >$ and $\mu > 0$ problem (32) admits a solution.*

Proof. According to Proposition 7.2 we must show that

$$d_{\alpha,-\mu} < \frac{(2-\alpha)S^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}}. \tag{39}$$

We take $v = u_\epsilon$ in the definition of the mountain-pass level. Since $\lim_{t\to\infty} J_{\alpha,-\mu}(tu_\epsilon) = -\infty$, there exists $t_\epsilon > 0$ such that

$$J_{\alpha,-\mu}(t_\epsilon, u_\epsilon) = \max_{t\geq 0} J_{\alpha,-\mu}(tu_\epsilon)$$

and

$$t_\epsilon\|u_\epsilon\|^2 - \mu t_\epsilon^{2^*_\alpha-1}\int_\Omega \frac{u_\epsilon^{2^*_\alpha}}{|x|^\alpha}\,dx = t_\epsilon^{2^*_\alpha-1}\int_\Omega Q(x)u_\epsilon^{2^*-1}\,dx.$$

Hence

$$t_\epsilon \leq \left(\frac{\|u_\epsilon\|^2}{\mu\int_\Omega \frac{u_\epsilon^{2^*_\alpha}}{|x|^\alpha}\,dx}\right)^{\frac{1}{2^*_\alpha-2}}$$

and

$$J_{\alpha,-\mu}(t_\epsilon u_\epsilon) \leq \frac{(2-\alpha)}{2(N-\alpha)}\frac{\left(\|u_\epsilon\|^2\right)^{\frac{2^*_\alpha}{2^*_\alpha-2}}}{\left(\mu\int_\Omega \frac{u_\epsilon^{2^*_\alpha}}{|x|^\alpha}\,dx\right)^{\frac{2}{2^*_\alpha-2}}} - \frac{t_\epsilon^q}{q}\int_\Omega Q(x)u_\epsilon^q\,dx. \tag{40}$$

Since

$$\int_\Omega \left(|\nabla u_\epsilon|^2 + \lambda u_\epsilon^2\right)dx = S_\alpha^{\frac{N-\alpha}{2-\alpha}} + O\left(\epsilon^{N-\alpha}\right) + \lambda O\left(\epsilon^2\right)$$

and

$$\int_\Omega \frac{u_\epsilon^{2^*_\alpha}}{|x|^\alpha}\,dx = S_\alpha^{\frac{N-\alpha}{2-\alpha}} + O\left(\epsilon^{N-\alpha}\right)$$

we deduce from (40) that

$$J_{\alpha,-\mu}(t_\epsilon u_\epsilon) \le \frac{2-\alpha}{2(N-\alpha)} \frac{S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{\mu^{\frac{N-2}{2-\alpha}}} + \lambda O\big(\epsilon^2\big) - t_\epsilon^q \int_\Omega Q(x) u_\epsilon^q \, dx.$$

We now verify that

$$\int_\Omega Q(x) u_\epsilon^q \, dx \ge b\epsilon^{N-q\frac{N-2}{2}}$$

provided $\frac{N}{N-2} < q$. This condition is satisfied if $2 < q$ for $N \ge 4$ and $3 < q$ for $N = 3$. In both cases we have $N - q\frac{N-2}{2} < 2$. Hence

$$\max_{t\ge 0} J_{\alpha,-\mu}(tu_\epsilon) < \frac{(2-\alpha)}{2(N-\alpha)} \frac{S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{\mu^{\frac{N-2}{2-\alpha}}}$$

for $\epsilon > 0$ sufficiently small. This completes the proof of (39) and the result follows from the mountain-pass theorem. □

The assertion of Proposition 7.2 becomes more transparent if $Q(0) = 0$ (we assume that $Q(x) > 0$ for $x \ne 0$).

Proposition 7.4. *Let $q = 2^*$ and $Q(0) = 0$. Then $J_{\alpha,-\mu}$ satisfies the $(PS)_c$ condition for*

$$c < \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{(2-\alpha)S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}} \right).$$

We set

$$S^\circ_{\infty,\alpha,-\mu} = \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{(2-\alpha)S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}} \right).$$

We now consider two cases:

(i) there exists a constant $\mu_\circ = \mu_\circ(N, Q_m, Q_M) > 0$ such that

$$S^\circ_{\infty,\alpha,-\mu} = \frac{(2-\alpha)S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}} \tag{41}$$

for $\mu \ge \mu_\circ$.

(ii) there exists a constant $\mu_1 = \mu_1(N, Q_M, Q_m)$ such that

$$\min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}} \right) = S^\circ_{\infty,\alpha,-\mu} \tag{42}$$

for $0 < \mu \le \mu_1$.

Theorem 7.5. *Let $q = 2^*$ and $Q(0) = 0$.*

(i) *Suppose that (41) holds for $\mu \geq \mu_\circ$. Moreover we assume that Q is C^2 in small ball around 0 and that the Hessian $\{D_{ij}Q(0)\}$ is positive definite. Then there exists $\bar{\lambda} = \bar{\lambda}(\mu) > 0$ such that problem (32) has a solution for $\mu \geq \mu_\circ$ and $0 < \lambda \leq \bar{\lambda}$.*

(ii) *Suppose that (42) holds for $0 < \mu \leq \mu_1$ and let $\alpha < 1$. If $Q_M > 2^{\frac{2}{N-2}} Q_m$, $|Q(x) - Q(y)| = o(|x-y|)$ for x near y with $Q(y) = Q_M$, then problem (32) has a solution for $0 < \mu \leq \mu_1$ and $\lambda > 0$.*

(iii) *Suppose that (42) holds for $0 < \mu \leq \mu_1$. If $Q_M \leq 2^{\frac{2}{N-2}}$, $|Q(x) - Q(y)| = o(|x-y|)$ for x near y with $Q(y) = Q_m$ and $H(y) > 0$, then problem (32) has a solution for $\lambda > 0$ and $0 < \mu \leq \mu_1$.*

Proof. (i) We proceed as in the proof of Theorem 7.3. There exists $t_\epsilon > 0$ such that

$$J_{\alpha,-\mu}(t_\epsilon u) = \max_{0 \leq t} J_{\alpha,-\mu}(t u_\epsilon) \leq \frac{(2-\alpha)}{2(N-\alpha)} \frac{\left(\|u_\epsilon\|\right)^{\frac{2^*_\alpha}{2^*_\alpha - 2}}}{\left(\mu \int_\Omega \frac{u_\epsilon^{2_\alpha *}}{|x|^\alpha}\, dx\right)^{\frac{2}{2^*_\alpha - 2}}} - t_\epsilon^{2^*} \int_\Omega Q(x) u_\epsilon^{2^*}\, dx.$$

Since $\{D_{ij}Q(0)\}$ is positive definite, we see that

$$J_{\alpha,-\mu}(t_\epsilon, u_\epsilon) \leq \frac{(2-\alpha) S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}} + \lambda O\big(\epsilon^2\big) - c\epsilon^2$$

for some constant $c > 0$. We can now find a constant $\bar{\lambda} = \bar{\lambda}(\mu)$ such that for $0 < \lambda \leq \bar{\lambda}$ and $\mu > \mu_\circ$

$$J_{\alpha,-\mu}(t_\epsilon, u_\epsilon) < \frac{(2-\alpha) S_\alpha^{\frac{N-\alpha}{2-\alpha}}}{2(N-\alpha)\mu^{\frac{N-2}{2-\alpha}}}.$$

(ii) We take $v = U_{\epsilon,y}$ with $Q(y) = Q_M$ in the definition of the mountain-pass level. First we observe that

$$\int_\Omega \frac{U_{\epsilon,y}^{2^*_\alpha}}{|x|^\alpha}\, dx \geq b\epsilon^\alpha$$

for some constant $b > 0$. We then have

$$\begin{aligned} J_{\alpha,-\mu}(\bar{t}_\epsilon U_{\epsilon,y}) &\leq \frac{1}{N} \frac{\left(\int_\Omega (|\nabla U_{\epsilon,y}|^2 + \lambda U_{\epsilon,y}^2)\, dx\right)^{\frac{2^*}{2^*-2}}}{\left(\int_\Omega Q(x) U_{\epsilon,y}^{2^*}\, dx\right)^{\frac{2}{2^*-2}}} - \frac{\mu}{2^*_\alpha} \int_\Omega \frac{U_{\epsilon,y}^{2^*_\alpha}}{|x|^\alpha}\, dx \\ &= \frac{\left(S^{\frac{N}{2}} + \lambda O(\epsilon^2)\right)^{\frac{2}{2^*-2}}}{\left(Q_M S^{\frac{N}{2}} + o(\epsilon)\right)^{\frac{2}{2^*-2}}} - \mu b \epsilon^\alpha. \end{aligned}$$

Since $\alpha < 1$ the result follows.

(iii) The proof is similar to the part (ii) and we use the asymptotic estimates (9). $\square$

If $Q(x) > 0$ on $\bar{\Omega}$, then it is rather difficult to verify that the mountain-pass level is strictly below $S_{\infty,\alpha,-\mu}$. A rudimentary estimate of the mountain-pass level can be obtained with the use of a constant test function. First, we observe that there exists $t_\circ > 0$ such that

$$J_{\alpha,-\mu}(t_\circ) = \max_{0\le t} J_{\alpha,-\mu}(t) \le \min\Bigg(\frac{(2-\alpha)(\lambda|\Omega|)^{\frac{2^*_\alpha}{2^*_\alpha - 2}}}{2(N-2)\big(\mu\int_\Omega \frac{dx}{|x|^\alpha}\big)^{\frac{2}{2^*_\alpha}}} - \frac{t_\circ^{2^*}}{2^*}\int_\Omega Q(x)\,dx, \frac{1}{N}\frac{(\lambda|\Omega|)^{\frac{2^*}{2^*-2}}}{\big(\int_\Omega Q(x)\,dx\big)^{\frac{2}{2^*-2}}} - \frac{t_\circ^{2^*_\alpha}}{2^*_\alpha}\mu\int_\Omega \frac{dx}{|x|^\alpha}\Bigg).$$

Using this inequality, for a given interval $[\delta, A]$, $\delta > 0$, we can find $\lambda_\circ > 0$ such that for $0 < \lambda \le \lambda_\circ$ an $\delta \le \mu \le A$, we have $J_{\alpha,-\mu}(t_\circ) < S_{\infty,\alpha,-\mu}$.

Conversely, given an interval $(0, \Lambda]$ we can find a constant $B > 0$, such that for $0 < \lambda < \Lambda$ and $\mu > B$, we have $J_{\alpha,-\mu}(t_\circ) < S_{\infty,\alpha,-\mu}$. In both cases we obtain the existence of mountain-pass solutions.

8. Critical Hardy-Sobolev nonlinearity, case $\mu > 0$

We now consider the following modification of problem (32)

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^\alpha}|u|^{2^*_\alpha - 2}u + \lambda u & = Q(x)|u|^{q-2}u \text{ in } \Omega, \\ \frac{\partial}{\partial\nu}u(x) & = 0 \text{ on } \partial\Omega \end{cases} \tag{43}$$

with $q \le 2^*$. A variational functional for problem (43) is given by

$$J_{\alpha,\mu}(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + \lambda u^2\big)\,dx + \frac{\mu}{2^*_\alpha}\int_\Omega \frac{|u|^{2^*_\alpha}}{|x|^\alpha}\,dx - \frac{1}{q}\int_\Omega Q(x)|u|^q\,dx.$$

Proposition 8.1. (i) *Let $q = 2^*$. The functional $J_{\alpha,\mu}$ satisfies the $(PS)_c$ condition for*

$$c < \min\Bigg(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\Bigg).$$

(ii) *If $2 < q < 2^*$, then $J_{\alpha,\mu}$ satisfies the $(PS)_c$ condition for every c.*

Proof. (i) First we show that a $(PS)_c$ sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. Indeed, we have

$$J_{\alpha,\mu}(u_m) - \frac{1}{2^*_\alpha}\langle J'_{\alpha,\mu}(u_m), u_m\rangle = \Big(\frac{1}{2} - \frac{1}{2^*_\alpha}\Big)\int_\Omega \big(|\nabla u_m|^2 + \lambda u_m^2\big)\,dx$$
$$+ \Big(\frac{1}{2^*_\alpha} - \frac{1}{2^*}\Big)\int_\Omega Q(x)|u|^{2^*}\,dx \le c + 1 + o(\|u_m\|)$$

for $m \ge m_\circ$. This obviously implies that the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. We can also assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*}(\Omega)$, $L^{2^*_\alpha}(\Omega, |x|^{-\alpha})$. We also have

$$|u_m|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j} \quad\text{and}\quad |\nabla u_m|^2 \overset{*}{\rightharpoonup} |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j},$$

in the sense of measure. A possible concentration point of $\{\frac{|u_m|^{2^*_\alpha}}{|x|^\alpha}\}$ is 0. Using a family of functions concentrating at x_j or 0 we derive that

$$Q(x_j)\nu_j = \mu_j \ \text{ if } \ x_j \ne 0 \ \text{ and } \ Q(0)\nu_\circ = \mu_\circ + \mu\bar{\nu}_\circ.$$

As in the proof of Proposition 2.2 we show that $\nu_j = 0$ for every j and the result follows.

Since $\{u_m\}$ is bounded in $H^1(\Omega)$ we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $u_m \to u$ in $Lk^q(\Omega)$. Writing for $n > m$

$$\langle J'_{\alpha,\mu}(u_m) - J'_{\alpha,\mu}(u_n), u_m - u_n\rangle = \int_\Omega \big(|\nabla(u_n - u_m)|^2 + \lambda(u_n - u_m)^2\big)\,dx$$
$$+ \mu\int_\Omega \frac{(|u_m|^{2^*_\alpha-2}u_m - |u_n|^{2^*_\alpha-2}u_n)(u_m - u_n)}{|x|^\alpha}\,dx$$
$$= \int_\Omega Q(x)\big(|u_m|^{2^*-2}u_m - |u_n|^{2^*-2}u_n\big)(u_n - u_m)\,dx,$$

we deduce that $\{u_m\}$ satisfies the Cauchy condition and the result follows. □

As a consequence of Proposition 8.1 we can formulate the following existence result.

Proposition 8.2. (i) *Let $q = 2^*$, $1 < \alpha < 2$ and $Q_M \le 2^{\frac{2}{N-2}}Q_m$. Suppose that*

$$|Q(y) - Q(x)| = o(|x - y|)$$

for x near y with $Q(y) = Q_M$ and $H(y) > 0$. Then for every $\lambda > 0$ and $\mu > 0$ problem (43) has a solution.

(ii) *Let $q = 2^*$ and $Q_M > 2^{\frac{2}{N-2}}$. Then there exists $\lambda^* > 0$ and $\mu^* > 0$ such that for $0 < \lambda \le \lambda^*$ and $0 < \mu \le \mu^*$ problem (43) has a solution.*

(iii) *Let $2_\alpha^* < q < 2^*$. Then there exist $\bar{\lambda} > 0$ and $\bar{\mu} > 0$ such that for $0 < \lambda \leq \bar{\lambda}$ and $0 < \mu \leq \bar{\mu}$ problem (43) has a solution.*

(iv) *Let $2 < q < 2_\alpha^*$. Then for every $0 < a < b$ there exists $\tilde{\lambda} > 0$ such that for every $a \leq \mu \leq b$ and $0 < \lambda \leq \tilde{\lambda}$ problem (43) has a solution (this solution is a global minimizer).*

Proof. To prove (i), (ii) and (iii) we use the mountain-pass theorem. Part (i) follows by testing $J_{\alpha,\mu}$ with $U_{\epsilon,y}$ and applying the asymptotic estimates (9). Parts (ii) and (iii) follow by testing $J_{\alpha,\mu}$ with a constant function.

(iv) First we show that $J_{\alpha,\mu}$ is bounded from below on $H^1(\Omega)$. It follows from the Young inequality that for every $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\begin{aligned}\int_\Omega Q(x)|u|^q\,dx &\leq Q_M\Big(\int_\Omega \frac{|u|^{2_\alpha^*}}{|x|^\alpha}\,dx\Big)^{\frac{q}{2_\alpha^*}}\Big(\int_\Omega |x|^{\frac{q\alpha}{2_\alpha^*-q}}\,dx\Big)^{\frac{2_\alpha^*-q}{2_\alpha^*}}\\ &\leq \delta\int_\Omega \frac{|u|^{2_\alpha^*}}{|x|^\alpha}\,dx + C(\delta).\end{aligned}$$

Selecting $\delta < \frac{a}{2_{\alpha^*}}$ we get

$$J_{\alpha,\mu}(u) \geq \frac{1}{2}\int_\Omega \big(|\nabla u|^2\big)\,dx + \lambda u^2 + \big(\frac{\mu}{2_\alpha^*} - \delta\big)\int_\Omega \frac{|u|^{2_\alpha^*}}{|x|^\alpha}\,dx - \frac{C(\delta)}{2^*}.$$

This shows that for every $a \leq \mu \leq b$ and $\lambda > 0$ $J_{\alpha,\mu}$ is bounded from below on $H^1(\Omega)$. For $t > 0$ we have

$$J_{\alpha,\mu}(t) = \frac{\lambda t^2|\Omega|}{2} + t^q\Big(\frac{\mu t^{2_\alpha^*-q}}{2_\alpha^*}\int_\Omega \frac{dx}{|x|^\alpha} - \frac{1}{2^*}\int_\Omega Q(x)\,dx\Big).$$

First we choose t so small that

$$\frac{\mu t^{2_\alpha^*-q}}{2_\alpha^*}\int_\Omega \frac{dx}{|x|^\alpha} - \frac{1}{2^*}\int_\Omega Q(x)\,dx < 0$$

for $a \leq \mu \leq b$. Then we choose $\lambda_* > 0$ small so that $J_{\alpha,\mu}(t) < 0$. Hence $\inf_{u\in H^1(\Omega)} J_{\alpha,\mu}(u) < 0$. The existence of a global minimizer follows from the Ekeland variational principle. □

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