

# Ropes on a line embedded in a Grassmannian variety

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## ABSTRACT

Let  $L$  be a line contained in a Grassmannian variety  $G$ . A  $d$ -rope  $C \subset G$  supported on  $L$  is a locally Cohen-Macaulay curve of degree  $d$  with  $C_{\text{red}} = L$  and  $(\mathcal{I}_{L,G})^2 \subset \mathcal{I}_{C,G}$ . We characterize the  $d$ -ropes  $C$  supported on  $L$  and embedded in  $G$ . In some cases we describe also the vector bundles on such a rope  $C$ . Finally, we describe the parameter spaces for ropes embedded in  $G$ .

*Key words:* multiple line, Grassmannian variety, normal sheaf, embeddings.

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## 1. Introduction

Given a smooth curve  $L$ , a multiple structure  $C$  supported on  $L$  is a curve with  $C_{\text{red}} = L$ , where a curve is a locally Cohen-Macaulay scheme of pure dimension 1. Particular multiple structures are the so-called  $d$ -ropes, where a  $d$ -rope is a degree  $d \cdot \deg L$  curve whose ideal sheaf satisfies  $(\mathcal{I}_L)^2 \subset \mathcal{I}_C \subset \mathcal{I}_L$ , i.e. its relative ideal sheaf  $\mathcal{I} = \mathcal{I}_{L,C}$  satisfies  $\mathcal{I}^2 = 0$ .

Geometrically the curve  $C$  is contained between  $L$  and its first infinitesimal neighborhood. It is easy to see that a  $d$ -rope  $C$  of degree  $d \cdot \deg L$  corresponds to a rank  $d-1$  subbundle  $E$  of the normal bundle  $\mathcal{N}_L$  of  $L$  via the exact sequence

$$0 \rightarrow E^* = \mathcal{I}_{L,C} \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_L \rightarrow 0. \quad (1)$$

We say that  $C$  is a rope if it is a  $d$ -rope for some  $d$ .

The definition of rope was given in [5], while 2-ropes (called ribbons) and  $d$ -ropes embedded in the projective space were studied in [1] and [9], respectively.

If the support  $L$  is a line and the rope  $C$  is embedded in  $\mathbb{P}^n$ , the Hilbert function, the homogeneous ideal, the Hartshorne-Rao module, their biliaison classes and their Hilbert schemes were studied in [10] and [11]. Moreover, curves  $C$  contained between two infinitesimal neighborhoods of a line  $L$ , embedded in  $\mathbb{P}^n$ , were studied in [2].

In this work we want to study ropes  $C$  supported on a line  $L$ , both embedded in a Grassmannian variety  $G$ . This fact seems quite natural because the Grassmannian varieties are a generalization of the projective spaces  $\mathbb{P}^n$ .

The plan of the paper is the following.

In Section 2 we give a characterization of the bundles

$$E^* = \mathcal{I}_{L,C} = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1)$$

(see the previous exact sequence (1)) which define  $d$ -rope  $C$  embedded in the Grassmannian variety  $G$ . Furthermore, we deduce some numerical invariants of such embedded ropes (for example we easily prove that  $g_C = -\sum_{i=1}^{d-1} \alpha_i$ , where  $g_C$  is the arithmetic genus of  $C$ ).

In Section 3 we describe the vector bundles  $A$  on a  $d$ -rope  $C$  embedded in  $G$ , supported on a line  $L$ , which satisfy the condition  $A|_L$  is rigid.

Recalling that every Grassmannian variety  $G = G_{r,n}$  can be embedded via Plücker morphism in the projective space  $\mathbb{P}^N$  with  $N = \binom{n}{r} - 1$ , in Section 4 we show that for almost every  $d$ -rope  $C$  embedded in  $G \subset \mathbb{P}^N$ , there exists a  $d'$ -rope  $C' \subset \mathbb{P}^N$  such that  $C$  is the scheme-theoretical intersection of  $C'$  with  $G$ .

Finally, in the last section we study the parameter space for the  $d$ -ropes embedded in  $G$ . Whenever we choose two suitable parameter spaces we describe a flat family of  $d$ -ropes embedded in the same Grassmannian variety  $G$  such that the general element belongs to one of the two parameter spaces and the special one belongs to the other one.

## 2. Characterizations of ropes supported on a line, embedded in the Grassmannian variety

In this section, we consider a line  $L$  contained in a Grassmannian variety  $G$  and we give a characterization of a  $d$ -rope  $C$  supported on a line  $L$  both embedded in  $G$ .

Throughout this paper we work over an algebraically closed field  $K$  of any characteristic and we'll use the following notation. Let  $G := G_{r,n}$  be the set of  $r$ -dimensional

linear subspaces of the vector space  $K^n$ . Of course each  $r$ -dimensional linear subspace of  $K^n$  can be view as a  $(r - 1)$ -plane in the corresponding projective space  $\mathbb{P}^{n-1}$ . We have that  $\dim G_{r,n} = r(n - r)$  and it is well known that  $G_{r,n}$  can be embedded via Plücker morphism in the projective space  $\mathbb{P}^N = \text{Proj}(K[x_0, \dots, x_N])$  with  $N = \binom{n}{r} - 1$ .

We refer to [6] for generalities about the Grassmannian variety.

We recall also this well known result that we use in the following. We give a proof for the convenience of the reader.

**Lemma 2.1.** *Let  $L$  be a line contained in  $G$ . Then the normal sheaf of  $L$  restricted to  $G$  is*

$$\mathcal{N}_{L|G} \cong \mathcal{O}_L^{n-2}(1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}.$$

*Proof.* Let  $Q$  ( $S$ , respectively) be the tautological quotient bundle of rank  $r$  (tautological quotient subbundle of rank  $n - r$ , respect.) of the Grassmannian  $G = G_{r,n}$ . We have that the tangent bundles of the Grassmannian variety  $G$  and of the line  $L$  are  $\mathcal{T}G \cong Q \otimes S^*$  (see [6], p. 201) and  $\mathcal{T}L \cong \mathcal{O}_L(2)$ . Moreover,  $Q|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L^{r-1}$  and  $S^*|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L^{n-r-1}$  because  $Q$  and  $S^*$  are spanned,  $\det(Q) \cong \mathcal{O}_G(1)$ , and  $\det(S^*) \cong \mathcal{O}_G(1)$ . Then, we get  $\mathcal{T}G|_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L(1)^{n-2} \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$  and from the exact sequence:

$$0 \rightarrow \mathcal{T}L \rightarrow \mathcal{T}G|_L \rightarrow \mathcal{N}_{L|G} \rightarrow 0 \tag{2}$$

we can compute  $\mathcal{N}_{L|G}$ . □

*Remark 2.2.* The isomorphism  $\mathcal{N}_{L|G} \cong \mathcal{O}_L^{n-2}(1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$  is the key point of our construction and for this reason the construction holds also for every rational smooth variety  $U$  containing  $L$  such that  $\mathcal{N}_{L|U} = \mathcal{O}_L^s(1) \oplus \mathcal{O}_L^t$ , with  $s \geq 1$  and  $t \geq 0$ .

Now, we recall the definition of a  $d$ -rope following [5]. This definition applies for a rope  $C$  not necessarily embedded in a projective space (and supported on an irreducible smooth curve).

**Definition 2.3.** A  $d$ -rope  $C$  is a projective scheme such that:

- (i)  $L = C_{\text{red}}$  is an irreducible smooth curve;
- (ii) the ideal sheaf  $\mathcal{I} = \mathcal{I}_{L,C}$  has  $\mathcal{I}^2 = 0$  and hence is an  $\mathcal{O}_L$ -module;
- (iii)  $\mathcal{I}$  is locally free of rank  $d - 1$  over  $L$ .

In the following the scheme  $L$  will be a line.

As recalled in Section 1, it is easy to see that a  $d$ -rope supported on a line  $L$  corresponds to a rank  $d - 1$  subbundle  $E$  of the normal bundle  $\mathcal{N}_L$  of  $L$  via the exact sequence (1)

$$0 \rightarrow E^* = \mathcal{I}_{L,C} \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_L \rightarrow 0.$$

The subbundle  $E^*$  is the conormal bundle of  $L$  in  $C$ .

**Theorem 2.4.** *A  $d$ -rope  $C$  supported on a line  $L \subset G_{r,n}$  defined by*

$$E^* = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1)$$

*can be embedded into  $G_{r,n}$  if, and only if, either*

- (i)  $d < \dim G_{r,n}$ ;
- (ii)  $\alpha_i \geq 0 \forall i$ ;
- (iii)  $n \geq 2 + |\{i \mid \alpha_i = 0\}|$ ;

*or*

- (i)  $d = \dim G_{r,n}$ ;
- (ii)  $\alpha_i = 0, 1 \forall i$ ;
- (iii)  $|\{i \mid \alpha_i = 0\}| = (r - 1)(n - r - 1)$ , and  $|\{i \mid \alpha_i = 1\}| = n - 2$ .

*Proof.* According to Definition 2.3 and recalling that  $G_{r,n}$  is smooth in a neighborhood of  $L$ , the rope  $C$  can be embedded into  $G_{r,n}$  by  $\mathcal{O}_C(1)$  if, and only if, we can give an injective map

$$E = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(1 - \alpha_i) \rightarrow \mathcal{N}_{L|G} = \mathcal{O}_L^{n-2}(1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \tag{3}$$

which does not drop rank in codimension 1, that is to say, it provides an embedding of vector bundles.

This is possible if, and only if,  $(1 - \alpha_i) \leq 1, \forall i = 1, \dots, d - 1$ , and either  $rk(E) < rk(\mathcal{N}_{L|G})$  i. e.  $\dim G > d$ , and at most  $n - 2$  integers  $\alpha_i - 1$  are equal to  $-1$ , or  $rk(E) = rk(\mathcal{N}_{L|G})$ , i. e.  $\dim G = d$ , and  $E \cong \mathcal{N}_{L|G}$ .

From the embedding of vector bundles (3), the exact sequence (2), and the fact that  $E$  is a free  $\mathcal{O}_L$ -module, we get that there exists a surjective morphism  $\mathcal{T}G|_L \rightarrow E$  whose kernel is  $\mathcal{T}\hat{E}C$ , and so the claim follows. □

*Remark 2.5.* For every  $x \in L = C_{\text{red}}$ ,  $\mathcal{T}C_x$  has dimension  $\dim G - d + 1$ .

**Corollary 2.6.** *If  $C$  is a  $d$ -rope embedded into a Grassmannian variety  $G$  and supported on a line  $L \subset G$  then  $\mathcal{O}_C$  is an  $\mathcal{O}_L$ -module and the sequence (1) splits as sequence of  $\mathcal{O}_L$ -modules.*

*Proof.* If we apply  $\text{Hom}(\mathcal{O}_L, -)$  to the sequence (1), we get

$$\dots \rightarrow \text{Hom}(\mathcal{O}_L, \mathcal{O}_C) \rightarrow \text{Hom}(\mathcal{O}_L, \mathcal{O}_L) \rightarrow \text{Ext}^1(\mathcal{O}_L, E^*) \rightarrow \dots$$

By [7], Ch. III, Proposition 6.3(c),  $\text{Ext}^1(\mathcal{O}_L, E^*) \cong H^1(L, E^*) = 0$  because of previous Theorem 2.4, and so we have the surjectivity of the map  $\text{Hom}(\mathcal{O}_L, \mathcal{O}_C) \rightarrow \text{Hom}(\mathcal{O}_L, \mathcal{O}_L)$ . Hence, there exists a map  $\psi : \mathcal{O}_L \rightarrow \mathcal{O}_C$  which lifts the identity  $id_L : \mathcal{O}_L \rightarrow \mathcal{O}_L$ , that is to say,  $\mathcal{O}_C$  is an  $\mathcal{O}_L$ -module and the sequence (1) splits as sequence of  $\mathcal{O}_L$ -modules. □

*Remark 2.7.* (i) Previous Corollary 2.6 proves that the split ropes supported on a line, which are the simplest possible abstract ropes, are the only one that can be embedded in a Grassmannian variety.

(ii) The existence of a retraction of the map  $\mathcal{O}_C \rightarrow \mathcal{O}_L$  of the sequence (1) can be proved directly using a projection argument, as in [3], Lemma 2.6

*Remark 2.8.* If the conditions of Theorem 2.4 hold, then the map

$$E \rightarrow \mathcal{N}_{L|G}$$

induces a surjective map:

$$\mathcal{N}_{L|G}^* \rightarrow E^*$$

and so we can deduce the exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{N}_{L|G}^* \rightarrow E^* \rightarrow 0$$

which can be also written as

$$0 \rightarrow \bigoplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \xrightarrow{\varphi_B} \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \xrightarrow{\varphi_A} \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1) \rightarrow 0 \quad (4)$$

Now, we can deduce some numerical invariants for the rope  $C$  embedded in  $G_{r,n}$ .

**Proposition 2.9.** *Let  $C$  be a  $d$ -rope supported on a line and embedded in the Grassmannian variety  $G = G_{r,n}$ . We have:*

(i)  $\sum_{j=1}^{\dim G-d} \beta_j = \sum_{i=1}^{d-1} \alpha_i + n - 1 - \dim G;$

(ii)  $g_C = -\sum_{i=1}^{d-1} \alpha_i.$

*Proof.* (i) Computing the Hilbert polynomials from (4), we get:

$$\sum_{j=1}^{\dim G-d} \binom{z - \beta_j}{1} - (n - 2) \binom{z}{1} - (r - 1)(n - r - 1) \binom{z + 1}{1} + \sum_{i=1}^{d-1} \binom{z + \alpha_i}{1} = 0.$$

Now, an easy computation gives the first claim.

(ii) We have that  $\chi(\mathcal{O}_L(z)) = z + 1$  and  $\chi(E^*(z)) = \sum_{i=1}^{d-1} (z + \alpha_i) = (d - 1)z + \sum_{i=1}^{d-1} \alpha_i$ . Then, we obtain  $\chi(\mathcal{O}_C(z)) = dz + 1 + \sum_{i=1}^{d-1} \alpha_i$  which gives the genus.  $\square$

*Remark 2.10.* In [10] the authors studied  $d$ -ropes supported on a line, embedded in  $\mathbb{P}^n$ . In particular they consider the exact sequence

$$0 \rightarrow \bigoplus_{j=1}^{n-d} \mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_{\mathbb{P}^1}^{n-1}(-1) \xrightarrow{\varphi_A} \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1) \rightarrow 0$$

similar to the sequence (4), which defines a  $d$ -rope  $C$  in  $\mathbb{P}^n$ . Using this sequence, they deduce that  $\sum_{j=1}^{n-d} \beta_j = \sum_{i=1}^{d-1} \alpha_i$  and  $g_C = -\sum_{i=1}^{d-1} \alpha_i$  (see Lemma 2.8 and Proposition 2.9 in [10]).

We observe that both for ropes in  $\mathbb{P}^n$  and for ropes in  $G_{r,n}$  the genus depends on the twist of the sheaf  $E$ . Moreover, the relations between the shifts  $\alpha_i$  and  $\beta_j$  are different and depend on the Grassmannian variety where the rope is embedded.

### 3. Vector bundles on ropes on the Grassmannian variety

In this section we want to describe the vector bundles  $A$  on a  $d$ -rope  $C$  supported on a line  $L$ , both embedded in  $G_{r,n}$ , under some constrains on  $C$ .

In the previous section we characterized a  $d$ -rope  $C$ , supported on  $L$  and embedded in  $G$ , using the sheaf  $E^* = \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1)$ .

We need the following definitions

**Definition 3.1.** We say that  $C$  is semipositive if  $\alpha_i \geq 1, \forall i = 1, \dots, d - 1$ .

**Definition 3.2.** We say that a sheaf  $\oplus_{i=1}^m \mathcal{O}_L(a_i)$ , with  $a_1 \geq a_2 \geq \dots \geq a_m$ , is rigid if  $a_m \geq a_1 - 1$ .

For the semipositive ropes on  $G$  we have:

**Proposition 3.3.** *Let  $A$  be a vector bundle on a semipositive  $d$ -rope  $C$  supported on a line  $L$  embedded in a Grassmannian variety  $G$ . If  $A|_L \cong \oplus_{i \in I} \mathcal{O}_L(a_i)$  is rigid then  $A \cong \oplus_{i \in I} \mathcal{O}_C(a_i)$ .*

*Proof.* We set  $B = \oplus_{i \in I} \mathcal{O}_C(a_i)$ . We have that  $B|_L \cong A|_L$  or, more precisely, there is an isomorphism:

$$\psi : A|_L \rightarrow B|_L.$$

We have that  $\mathcal{H}om(A, B)|_L \cong \mathcal{H}om(\oplus_{i \in I} \mathcal{O}_L(a_i), \oplus_{j \in I} \mathcal{O}_L(a_j)) \cong \oplus_{i, j \in I} \mathcal{O}_L(a_j - a_i)$  and for the rigidity, we have that  $-1 \leq a_j - a_i \leq 1$ .

If we tensorize the exact sequence (1)

$$0 \rightarrow E^* \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_L \rightarrow 0$$

by  $\mathcal{H}om(A, B)$  and if we write the associated cohomology sequence, we obtain:

$$\begin{aligned} 0 \rightarrow H^0(E^* \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B)) \rightarrow H^0(\mathcal{O}_C \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B)) \rightarrow \\ \rightarrow H^0(\mathcal{O}_L \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B)) \rightarrow H^1(E^* \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B)) \rightarrow \dots \end{aligned}$$

In fact  $E^* \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B) \cong \oplus_{i=1}^m \mathcal{O}_L(\alpha_i - 1) \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B)$  and this is a sum of line bundles twisted by  $\omega_i \geq -1$ .

Then, the isomorphism  $\psi \in H^0(\mathcal{O}_L \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B))$  can be lifted to a morphism  $\psi' \in H^0(\mathcal{O}_C \otimes_{\mathcal{O}_L} \mathcal{H}om(A, B))$ . By Nakayama's Lemma the morphism  $\psi'$  is an isomorphism, too. □

*Remark 3.4.* Let  $C$  be a semipositive  $d$ -rope supported on a line  $L$  both embedded in a smooth rational variety  $U$  (see also Remark 2.2) with  $\mathcal{O}_U(1)|_L = \mathcal{O}_L(1)$ . If  $A$  is a vector bundle on  $C$  such that  $A|_L \cong \mathcal{O}_L(a)^m$ , then  $A \cong \mathcal{O}_C(a)^m$ . In fact, the same proof as in Proposition 3.3 works in this case too, pointing out that  $\mathcal{H}om(A, B) \cong \mathcal{O}_L^{r^2}$ .

### 4. A lifting problem

In this section we want to show that a  $d$ -rope  $C$  supported on a line  $L$ , both embedded in the Grassmannian  $G = G_{r,n} \subseteq \mathbb{P}^N$ , with  $N = \binom{n}{r} - 1$ , can be lifted to a  $d'$ -rope  $C'$  of  $\mathbb{P}^N$  such that  $C$  is the scheme-theoretical intersection of  $C'$  and  $G$ .

**Lemma 4.1.** *Let  $\mathcal{N}_{G, \mathbb{P}^N}$  be the normal sheaf of the Grassmannian  $G$  embedded via Plücker morphism in  $\mathbb{P}^N$ . Then*

$$(\mathcal{N}_{G, \mathbb{P}^N})|_L \cong \mathcal{O}_L^{m_1}(1) \oplus \mathcal{O}_L^{m_2}(2)$$

where  $m_1 = N - 2 \dim G + n - 1$  and  $m_2 = \dim G - n + 1$ .

*Proof.* The Grassmannian variety  $G$  is scheme-theoretically cut out by quadrics in  $\mathbb{P}^N$ . Then  $\mathcal{I}_{G, \mathbb{P}^N}(2)$  is spanned and so

$$\left[ \left( \frac{\mathcal{I}_{G, \mathbb{P}^N}}{\mathcal{I}_{G, \mathbb{P}^N}^2} \right) (2) \right] |_L \cong (\mathcal{N}_{G, \mathbb{P}^N}^*(2))|_L \cong \oplus \mathcal{O}_L(a_i)$$

with  $a_i \geq 0, \forall i$ , i. e.  $(\mathcal{N}_{G, \mathbb{P}^N})|_L \cong \oplus \mathcal{O}_L(b_i)$ , with  $b_i \leq 2$ . Let us consider the exact sequence:

$$0 \rightarrow \mathcal{N}_{L,G} \rightarrow \mathcal{N}_{L, \mathbb{P}^N} \rightarrow (\mathcal{N}_{G, \mathbb{P}^N})|_L \rightarrow 0$$

where  $\mathcal{N}_{L,G} \cong \mathcal{O}_L^p \oplus \mathcal{O}_L^q(1)$  with  $p = (r-1)(n-r-1), q = n-2$  and  $\mathcal{N}_{L, \mathbb{P}^N} \cong \mathcal{O}_L^{N-1}(1)$ .

We deduce that  $(\mathcal{N}_{G, \mathbb{P}^N})|_L$  is ample and then  $b_i \geq 1$  and  $(\mathcal{N}_{G, \mathbb{P}^N})|_L \cong \mathcal{O}_L^{m_1}(1) \oplus \mathcal{O}_L^{m_2}(2)$  with  $m_1 + m_2 = N - \dim G$ , by rank argument. Moreover, comparing the first Chern classes, we get  $m_1 + 2m_2 = N - n + 1$ . A simply calculation gives the claim.  $\square$

*Remark 4.2.* We can prove the same result if we consider a scheme  $U \subset \mathbb{P}^N$  with  $L \subset U$  such that  $\mathcal{N}_{L,U} \cong \mathcal{O}_L^\alpha \oplus \mathcal{O}_L^\beta(1)$ . In this case we have that  $(\mathcal{N}_{U, \mathbb{P}^N})|_L \cong \mathcal{O}_L(c_i)$ .

For example we can take any homogeneous smooth variety as  $U$ .

Now, we consider  $L \subset G \subset \mathbb{P}^N$ . In Section 2 we showed that we can construct a rope supported on  $L$  contained in  $G$ , using the exact sequence (4):

$$0 \rightarrow \oplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \xrightarrow{\varphi_A} \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1) \rightarrow 0$$

We can also consider the dualized exact sequence of normal sheaves written in the proof of Lemma 4.1:

$$0 \rightarrow (\mathcal{N}_{G, \mathbb{P}^N}^*)|_L \rightarrow \mathcal{N}_{L, \mathbb{P}^N}^* \rightarrow \mathcal{N}_{L,G}^* \rightarrow 0 \tag{5}$$

Pointing out that  $\mathcal{N}_{L,G}^* \cong \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$ , we construct the following diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & (\mathcal{N}_{G,\mathbb{P}^N}^*)|_L & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{N}_{L,\mathbb{P}^N}^* & & \\
 & & & \nearrow \varphi'_B & \downarrow \rho & & \\
 0 \rightarrow & \oplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) & \xrightarrow{\varphi_B} & \mathcal{N}_{L,G}^* & \xrightarrow{\varphi_A} & \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1) & \rightarrow 0 \\
 & & & \downarrow & & & \\
 & & & 0 & & & 
 \end{array}$$

**Theorem 4.3.** *With the notation as above, suppose that  $\beta_j \geq 0$  for all  $j$ . We can lift the morphism  $\varphi_B$  to a morphism  $\varphi'_B : \oplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) \rightarrow \mathcal{N}_{L,\mathbb{P}^N}^*$  (not uniquely) which gives a  $d'$ -rope  $C' \subset \mathbb{P}^N$ , supported on  $L$ , with  $d' = N - \dim G + d$ .*

*We have that  $C$  is the scheme-theoretical intersection of  $C'$  with  $G$ .*

*Proof.* Applying  $\text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), -)$  to the sequence (5) we get:

$$\begin{aligned}
 0 \rightarrow & \text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), (\mathcal{N}_{G,\mathbb{P}^N}^*)|_L) \rightarrow \\
 & \rightarrow \text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), (\mathcal{N}_{L,\mathbb{P}^N}^*)|_L) \rightarrow \text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), \mathcal{N}_{L,G}^*) \rightarrow 0.
 \end{aligned}$$

If we write the associated cohomology sequence we have that

$$H^1(\mathcal{H}om(\oplus_j \mathcal{O}_L(-\beta_j - 1), \mathcal{N}_{G,\mathbb{P}^N}^*|_L)) = 0$$

because  $\beta_j \geq 0$  and then the morphism  $\varphi_B \in \text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), \mathcal{N}_{L,G}^*)$  can be lifted (not uniquely) to a morphism  $\varphi'_B \in \text{Hom}(\oplus_j \mathcal{O}_L(-\beta_j - 1), \mathcal{N}_{L,\mathbb{P}^N}^*)$  which gives the  $d'$ -rope  $C'$ . The commutativity of the diagram ( $\rho\varphi'_B = \varphi_B$ ) assures that  $C = C' \cap G$ , while  $d'$  can be computed as  $N - rk(\varphi'_B) = N - \dim G + d$  (cf. [10], Remark 2.5 (i)).  $\square$

*Remarks 4.4.* (i) (Geometrical meaning) In some sense, given a rope  $C$  on  $G$  we can find the directions transverse to  $G$  obtaining a rope in  $\mathbb{P}^N$ .

(ii) We can prove the proposition replacing  $G$  with a scheme  $U$  satisfying the conditions introduced in Remark 4.2 and with the extra assumption  $\beta_j \geq c - 1$  where  $c = \max_j \{c_j\}$ .

(iii) The lifted ropes  $C'$  are completely studied in [10].



### 5. Families of ropes on the Grassmannian variety

Whenever we want to construct a rope  $C$  on  $G$  we start with a sheaf

$$E^* = \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1)$$

with  $\alpha_i \geq 0$  and we fix a surjective morphism

$$\varphi_A : \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \rightarrow \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1).$$

As shown in Section 2 we naturally get the sequence (4):

$$0 \rightarrow \oplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \xrightarrow{\varphi_A} \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1) \rightarrow 0$$

To fix notation, suppose that in (4)  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{d-1}$  and  $\beta_{\dim G-d} \geq \dots \geq \beta_2 \geq \beta_1 > 0$ .

The decreasing sequence of integers  $\underline{\alpha} = (\alpha_1, \dots, \alpha_{d-1})$  is the splitting type of  $E^*(1)$  and analogously, the decreasing sequence of integers  $\underline{\beta} = (-\beta_1, \dots, -\beta_{\dim G-d})$  is the splitting type of  $\ker \varphi_A(1)$ .

**Definition 5.1.** We say that the sequence of integers  $\underline{\alpha} = (\alpha_1, \dots, \alpha_{d-1})$  is admissible if  $E^* = \oplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_i - 1)$  satisfies the hypotheses of Theorem 2.4.

The sequence of integers  $\underline{\beta} = (-\beta_1, \dots, -\beta_{\dim G-d})$  is admissible if there exists an admissible sequence  $\underline{\alpha}$  such that in the exact sequence (4)  $\ker \varphi_A$  has splitting type  $\underline{\beta}$ .

The pair  $(\underline{\alpha}, \underline{\beta})$  is admissible if  $\underline{\alpha}$  is admissible and  $\ker \varphi_A$  has splitting type  $\underline{\beta}$ .

If the rope  $C$  on  $G$  is associated to the sequence (4) we say that  $C$  has  $\alpha$ -type  $\underline{\alpha}$  and  $\beta$ -type  $\underline{\beta}$ .

We define the degree of  $\underline{\alpha}$  ( $\underline{\beta}$  respect.) as  $\deg \underline{\alpha} = \sum_i \alpha_i$  ( $\deg \underline{\beta} = -\sum_i \beta_i$ , respect.). Now, we define the following partial order between splitting types of the same degree.

**Definition 5.2.** Let  $\underline{\alpha}_1 = (\alpha_{1,1}, \dots, \alpha_{1,d-1})$  and  $\underline{\alpha}_2 = (\alpha_{2,1}, \dots, \alpha_{2,d-1})$  be two  $\alpha$ -types of the same degree. We put:  $\underline{\alpha}_1 \geq \underline{\alpha}_2$  if  $\alpha_{1,1} + \dots + \alpha_{1,j} \leq \alpha_{2,1} + \dots + \alpha_{2,j}$  for  $1 \leq j \leq d-1$ .

The analogous definition holds for the  $\beta$ -types.

We can also define a partial order between the admissible pairs.

**Definition 5.3.** Let  $(\underline{\alpha}_1, \underline{\beta}_1)$  and  $(\underline{\alpha}_2, \underline{\beta}_2)$  be two admissible pairs. We say that  $(\underline{\alpha}_1, \underline{\beta}_1) \geq (\underline{\alpha}_2, \underline{\beta}_2)$  if  $\underline{\alpha}_1 \geq \underline{\alpha}_2$  and  $\underline{\beta}_1 \geq \underline{\beta}_2$  according with Definition 5.2.

Now, let  $\Gamma_{\underline{\alpha}}$  be the set of all ropes  $C$  with admissible  $\alpha$ -type  $\underline{\alpha}$ ,  $\Delta_{\underline{\beta}}$  the set of all ropes  $C$  with admissible  $\beta$ -type  $\underline{\beta}$  and let  $\Omega_{(\underline{\alpha}, \underline{\beta})}$  be the non-empty set of all ropes  $C$  with  $\alpha$ -type  $\underline{\alpha}$  and  $\beta$ -type  $\underline{\beta}$ , with  $(\underline{\alpha}, \underline{\beta})$  admissible pair.

For ropes  $C$  in  $G_{r,n}$  we can state a result analogous to Theorem 1 in [2] and we can prove it with similar arguments.

**Theorem 5.4.** *Let  $(\underline{\alpha}_1, \underline{\beta}_1)$  and  $(\underline{\alpha}_2, \underline{\beta}_2)$ , be two admissible pairs with  $\deg \underline{\alpha}_1 = \deg \underline{\alpha}_2$ ,  $\deg \underline{\beta}_1 = \deg \underline{\beta}_2$ ,  $(\underline{\alpha}_1, \underline{\beta}_1) \geq (\underline{\alpha}_2, \underline{\beta}_2)$  and with the extra assumption  $\alpha_{1,d-1} \geq 1$  and  $\alpha_{2,d-1} \geq 1$ . Let  $C \in \Omega_{(\underline{\alpha}_2, \underline{\beta}_2)}$ . Then there exists a flat family of ropes in  $G$  parameterized by a non empty open subset of an affine line whose special member is  $C$  and whose general member is an element of  $\Omega_{(\underline{\alpha}_1, \underline{\beta}_1)}$ .*

We need some preliminary results.

**Lemma 5.5.** *Let  $\underline{\alpha}_1$  and  $\underline{\alpha}_2$  be two admissible  $\alpha$ -types with  $\deg \underline{\alpha}_1 = \deg \underline{\alpha}_2$  and  $\underline{\alpha}_1 \geq \underline{\alpha}_2$ . Let  $C \in \Gamma_{\underline{\alpha}_2}$ . Then there exists a flat family of ropes in  $G$ , (parameterized by a non-empty open subset of an affine line) whose special member is  $C$  and whose general member is an element of  $\Gamma_{\underline{\alpha}_1}$ .*

*Proof.* We set  $\underline{\alpha}_1 = (\alpha_{1,1}, \dots, \alpha_{1,d-1})$  and  $\underline{\alpha}_2 = (\alpha_{2,1}, \dots, \alpha_{2,d-1})$ . We observe that  $\alpha_{1,i} \geq 0, \alpha_{2,i} \geq 0$  for all  $i = 1, \dots, d - 1$  for the admissibility of the  $\alpha$ -types.

Because of the inequality  $\underline{\alpha}_1 \geq \underline{\alpha}_2$  it is well known that there exists a flat family of rank  $d - 1$  vector bundles on  $L$  (the support of the rope  $C$ ) parameterized by an open subset  $T$  of an affine line  $\mathbb{A}^1$  whose special member is  $A_2 \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_{2,i} - 1)$  and whose general member is isomorphic to  $A_1 \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_{1,i} - 1)$ .

A surjective morphism  $f \in H^0(L, \mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_2))$  induces a rope  $C$  in  $G$  given by the sequence:

$$0 \rightarrow \bigoplus_{j=1}^{\dim G-d} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \xrightarrow{f} A_2 \rightarrow 0.$$

Since  $\deg \underline{\alpha}_1 = \deg \underline{\alpha}_2$ , we have that

$$h^0(L, \mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_2)) = h^0(L, \mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_1)).$$

Applying [8], it is easy to check the existence of a vector bundle  $E$  on  $T$  with  $rk(E) = h^0(L, \mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_1))$  which is a universal parameter space for the family of the homomorphisms parameterized by  $T$ , that is to say,  $\forall P \in T$  the fibre  $E_P \cong H^0(L, \mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_1))$ .

Hence, we can find a rational path in the total space  $E$  joining the element representing  $f$  to a surjection  $g : \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)} \rightarrow A_1$ , and so the claim holds. □

We can state the same result for the  $\underline{\beta}$ -types.

**Lemma 5.6.** *Let  $\underline{\beta}_1$  and  $\underline{\beta}_2$  be two admissible  $\beta$ -types for with  $\deg \underline{\beta}_1 = \deg \underline{\beta}_2$  and  $\underline{\beta}_1 \geq \underline{\beta}_2$ . Let us take a rope  $C \in \Delta_{\underline{\beta}_2}$ . Then there exists a flat family of ropes in  $G$ , (parameterized by a non-empty open subset of an affine line) whose special member is  $C$  and whose general member is an element of  $\Delta_{\underline{\beta}_1}$ .*

*Proof of Theorem 5.4.* Set

$$A_1 = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_{1,i} - 1), \quad A_2 = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(\alpha_{2,i} - 1)$$

and

$$B_1 = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(-\beta_{1,i} - 1), \quad B_2 = \bigoplus_{i=1}^{d-1} \mathcal{O}_L(-\beta_{2,i} - 1).$$

Let  $C$  be a rope corresponding to a surjective morphism  $f \in H^0(\mathcal{H}om(\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}, A_2))$ . The rope  $C$  is defined by an extension of  $A_2$  by  $B_2$  with middle term isomorphic to  $\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$  (which is a rigid bundle). Since there exists such an extension, by semicontinuity the general extension of  $A_2$  and  $B_2$  has middle term isomorphic to  $\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$ .

As in the proof of Lemma 5.5, we have that there exists a flat family of pairs of vector bundles on  $L$ , parameterized by an open subset  $T$  of the affine line with  $(A_2, B_2)$  as special fiber and  $(A_1, B_1)$  as general fibre.

Since the pairs  $(\underline{\alpha}_i, \underline{\beta}_i)$  are admissible, for  $i = 1, 2$ , and  $\alpha_{i,d-1} \geq 1$ , for  $i = 1, 2$  then  $h^0(L, \mathcal{H}om(A_1, B_1)) = h^0(L, \mathcal{H}om(A_2, B_2)) = 0$  and so, using Riemann-Roch Theorem we get  $h^1(L, \mathcal{H}om(A_1, B_1)) = h^1(L, \mathcal{H}om(A_2, B_2))$ .

This implies that there exists a vector bundle  $E$  on  $T$  with

$$rk(E) = h^1(L, \mathcal{H}om(A_1, B_1)) = h^1(L, \mathcal{H}om(A_2, B_2))$$

such that  $\forall P \in T$  the fibre  $E_P = ((A_1)_P, (B_1)_P)$  is isomorphic to  $H^1(L, \mathcal{H}om((A_1)_P, (B_1)_P))$ .

By semicontinuity, for every  $P \in T$  the general extension of  $(A_1)_P$  by  $(B_1)_P$  has middle term isomorphic to  $\mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L^{(r-1)(n-r-1)}$  and so it defines a rope embedded in the Grassmannian  $G_{r,n}$  for every  $P$ . In fact, we have a line  $L \subset G_{r,n}$  and an exact sequence as (4), which defines the scheme structure of the rope embedded in  $G_{r,n}$ . The family of such extensions is algebraic and projective. Furthermore, the degree and genus of the ropes we obtain are fixed because  $\deg(C) = \text{rank}_K(A_1)_P + 1 = \text{rank}_K(A_2)_P + 1$ , and  $g(C) = -\deg(\underline{\alpha}_1) = -\deg(\underline{\alpha}_2)$ . The Hilbert polynomial of the ropes is then independent of  $P$  and so the family is flat by [7], Ch. III, Theorem 9.9.  $\square$

As last result, we describe a parameter space for the set  $\Gamma_{\underline{\alpha}}$ . Of course, a similar statement holds for  $\Delta_{\underline{\beta}}$ .

**Proposition 5.7.** *The scheme structures of ropes in  $\Gamma_{\underline{\alpha}}$  are parameterized by a non-empty, irreducible, rational variety  $\mathcal{U}$  of dimension*

$$\dim \mathcal{U} = \deg \underline{\alpha}(\dim G - 1) + (n - 2)(d - 1) - \sum_{i,j=1}^{d-1} \binom{\alpha_i - \alpha_j + 1}{1}.$$

$\Gamma_{\underline{\alpha}}$  is parameterized by  $F_1(G) \times \mathcal{U}$ , where  $F_1(G)$  is the Fano variety of the lines in  $G$ .

*Proof.* Let  $\underline{\alpha} = (\alpha_1, \dots, \alpha_{d-1})$  and let  $\mathcal{A} = \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1)$ .

Every rope in  $\Gamma_{\underline{\alpha}}$  is uniquely determined by a pair  $(L, \mathcal{E})$  where  $L$  is a line in  $G$  and  $\mathcal{E} = \text{Im}(\varphi_A^*) \subset \mathcal{N}_{L|G}$ , with  $\varphi_A \in \text{Hom}(\mathcal{N}_{L|G}, \mathcal{A})$ . Then, the parameter space for the scheme structures of ropes in  $\Gamma_{\underline{\alpha}}$  supported on a fixed line is the quotient of the open subset  $\mathcal{U}$  of  $\text{Hom}(\mathcal{N}_{L|G}, \mathcal{A})$ , corresponding to surjective maps which do not drop rank in codimension 1, by the action of the automorphisms of  $\mathcal{A}$ .

Hence,  $\mathcal{U}$  is irreducible, rational of dimension

$$\begin{aligned} \dim \mathcal{U} &= h^0(\text{Hom}(\mathcal{N}_{L|G}, \mathcal{A})) - \dim \text{Aut } \mathcal{A} = \\ &= \deg(\underline{\alpha})(\dim G - 1) + (d - 1)(n - 2) - \sum_{i,j=1}^{d-1} \binom{\alpha_j - \alpha_i + 1}{1}. \end{aligned}$$

The last part of the statement is straightforward.  $\square$

*Remark 5.8.* We want to compute the dimension of  $F_1(G)$ . Each line  $L$  in  $G$  is determined by a surjective morphism  $\mathcal{O}_{\mathbb{P}^1}^n \rightarrow \mathcal{O}_{\mathbb{P}^1}^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{V}$  because of the universal property of the Grassmannian  $G$ , up to the automorphisms of  $\mathcal{V}$  and the ones of  $\mathbb{P}^1$ . Hence, the dimension of  $F_1(G)$  is

$$\dim F_1(G) = h^0(\mathcal{O}_{\mathbb{P}^1}^{n(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}^n(1)) - \dim \text{Aut } \mathcal{V} - 3 = \dim G + n - 3.$$

Of course, the parameter space for  $\Gamma_{\underline{\alpha}}$  reflects the properties of  $F_1(G)$ .

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