The starlikeness and convexity of multivalent functions involving certain inequalities

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ABSTRACT

In the present paper, a theorem for the starlikeness and convexity of multivalent functions involving certain inequalities is given. Some interesting consequences of the main result are also mentioned.

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1. Introduction and definitions

Let $\mathcal{T}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ (p \in \mathcal{N} = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* and *multivalent* in the open unit disc $\mathcal{U} = \{z :\ z \in \mathcal{C} \text{ and } \mid z \mid < 1\}$. A function $f(z)$ belonging to $\mathcal{T}(p)$ is said to be *multivalently starlike of order* α in $\mathcal{U}$ if it satisfies the inequality:

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha,\ (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{1.2}$$

and, a function $f(z) \in \mathcal{T}(p)$ is said to be *multivalently convex of order* α in $\mathcal{U}$ if it satisfies the inequality:

$$\Re e\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha,\ (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}). \tag{1.3}$$

For the aforementioned definitions, one may refer to [1] and [2] (see also [11]). Further, a function $f(z) \in \mathcal{T}(p)$ is said to be in the subclass $\mathcal{T}_\lambda(p;\,\alpha)$ if it satisfies the inequality

$$\Re e\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\} > \alpha, \tag{1.4}$$

$$(z \in \mathcal{U};\ 0 \leq \lambda \leq 1;\ 0 \leq \alpha < p;\ p \in \mathcal{N}).$$

From the above definitions, the following subclasses of the classes $\mathcal{T}(p)$ and $\mathcal{T} \equiv \mathcal{T}(1)$ emerge from the families of functions $\mathcal{T}_\lambda(p;\,\alpha)$:

$$\mathcal{T}_\lambda(1;\,\alpha) = \mathcal{T}_\lambda(\alpha), \quad (0 \leq \lambda \leq 1;\ 0 \leq \alpha < 1, \tag{1.5}$$
$$\mathcal{T}_0(p;\,\alpha) = \mathcal{S}_p(\alpha), \quad (0 \leq \alpha < p;\ p \in \mathcal{N}), \tag{1.6}$$
$$\mathcal{T}_1(p;\,\alpha) = \mathcal{K}_p(\alpha), \quad (0 \leq \alpha < p;\ p \in \mathcal{N}), \tag{1.7}$$
$$\mathcal{S}_p(\alpha) \subseteq \mathcal{S}_p(0) = \mathcal{S}_p, \quad (0 \leq \alpha < p; p \in \mathcal{N}) \tag{1.8}$$
$$\mathcal{K}_p(\alpha) \subseteq \mathcal{K}_p(0) = \mathcal{K}_p, \quad (0 \leq \alpha < p;\ p \in \mathcal{N}), \tag{1.9}$$
$$\mathcal{T}_0(\alpha) = \mathcal{S}_1(\alpha) \subseteq \mathcal{S}(\alpha), \quad (0 \leq \alpha < 1), \tag{1.10}$$
$$\mathcal{T}_1(\alpha) = \mathcal{K}_1(\alpha) \subseteq \mathcal{K}(\alpha), \quad (0 \leq \alpha < 1), \tag{1.11}$$
$$\mathcal{T}_0(\alpha) = \mathcal{S}(\alpha) \subseteq \mathcal{S}(0) = \mathcal{S}, \quad (0 \leq \alpha < 1), \tag{1.12}$$
$$\mathcal{T}_1(\alpha) = \mathcal{K}(\alpha) \supseteq \mathcal{K}(0) = \mathcal{K}, \quad (0 \leq \alpha < 1). \tag{1.13}$$

The important subclasses in the Geometric function theory such as the multivalently starlike functions $\mathcal{S}_p(\alpha)$ of order $\alpha(0 \leq \alpha < p;\ p \in \mathcal{N})$ in $\mathcal{U}$, the multivalently convex functions $\mathcal{K}_0(\alpha)$ of order $\alpha(0 \leq \alpha < p;\ p \in \mathcal{N})$ in $\mathcal{U}$, the multivalently starlike functions $\mathcal{S}_p$ in $\mathcal{U}$, the multivalently convex functions $\mathcal{K}_p$ in $\mathcal{U}$, the starlike functions in $\mathcal{S}(\alpha)$ of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$, the convex functions $\mathcal{K}(\alpha)$ of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$, the starlike functions $\mathcal{S}$ in $\mathcal{U}$, and the convex functions K in $\mathcal{U}$, are seen to be easily identifiable with the aforementioned classes (*cf.*, *e.g.*, [1], [2], and [11]).

The purpose of considering inequality (1.4) is to obtain general results which combine certain types of inequalities concerning functions belonging to the classes $\mathcal{S}_p(\alpha)$, $\mathcal{K}_p(\alpha)$, $\mathcal{S}_p$, $\mathcal{K}_p$, $\mathcal{S}(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{S}$, and $\mathcal{K}$. Some interesting corollaries are also deduced from our main results. Other interesting results involving certain inequalities and/or multivalent functions were also studied, for example, by Owa *et al.* ([8], [9], [10]), and Irmak *et al.* ([3], [4], [5]).

2. Main Result

Before stating and proving our main result, we require the following assertion (popularly known as Jacks's Lemma):

Lemma (*cf.*, [6], [7]). *Let the function $x(z)$ be non-constant and regular in the unit disc $\mathcal{U}$, such that $w(0) = 0$. If $\mid w(z) \mid$ attains its maximum value on the circle $\mid z \mid = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0), \quad (c \geq 1). \tag{2.1}$$

Theorem. *Let a function $f(z)$ belong to the class $\mathcal{T}(p)$. Define a function $F(z)$ by*

$$F(z) = (1-\lambda) f(z) + \lambda z f'(z), \quad (0 \leq \lambda \leq 1), \tag{2.2}$$

and if $F(z)$ satisfies anyone of the following inequalities:

$$\left| \frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF;(z)}{F(z)} - p} - 1 \right| < \frac{1}{2p-\alpha}, \; (z \in \mathcal{U}; 0 \leq \alpha < p; \, p \in \mathcal{N}), \tag{2.3}$$

$$\left| 1 + z \left(\frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) \right| < \frac{p-\alpha}{2p-\alpha}, \; (z \in \mathcal{U}; 0 \leq \alpha < p; \, p \in \mathcal{N}), \tag{2.4}$$

$$\left| \frac{F(z)}{zF'(z)} \left(1 + \frac{zF''(z)}{F'(z)} \right) - 1 \right| < \frac{p-\alpha}{(2p-\alpha)^2}, \; (z \in \mathcal{U}; 0 \leq \alpha < p; \, p \in \mathcal{N}), \tag{2.5}$$

$$\left| \frac{zF'(z)}{F(z)} \left[1 + z \left(\frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) \right] \right| < p - \alpha, \; (z \in \mathcal{U}; 0 \leq \alpha < p; \, p \in \mathcal{N}), \tag{2.6}$$

$$\Re e \left\{ \frac{zF'(z)}{F(z)} \left(\frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 \right) \right\} < 1, \; (z \in \mathcal{U}; 0 \leq \alpha < p; \, p \in \mathcal{N}), \tag{2.7}$$

then $f(z) \in \mathcal{T}_\lambda(p; \alpha)$.

Proof. Let $f(z) \in \mathcal{T}(p)$. then from (1.1) and (2.1), we find that

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} \\ &= \frac{p[1+\lambda(p-1)] + \sum_{k=p}^{\infty} k[1+\lambda(k-1)] a_k z^{k-p}}{1 + \lambda(p-1) + \sum_{k=p}^{\infty} [1+\lambda(k-1)] a_k z^{k-p}}, \\ &\qquad (z \in \mathcal{U}; \; 0 \leq \lambda \leq 1; p \in \mathcal{N}). \end{aligned} \tag{2.8}$$

Now, define a function $w(z)$ by

$$\frac{zF'(z)}{F(z)} - p = (p-\alpha) w(z), \; (z \in \mathcal{U}; \; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.9}$$

then the function $w(z)$ is analytic in $\mathcal{U}$, and $w(0) = 0$. It follows from the above definition (2.9) that

$$1 + \frac{zF''(z)}{F'(z)} - p = (p-\alpha)w(z)\left(1 + \frac{zw'}{w(z)}\frac{1}{p+(p-\alpha)w(z)}\right), \tag{2.10}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < p; p \in \mathcal{N}).$$

Hence, from (2.8) and (2.9), we have

$$F_1(z) = \frac{1 + \frac{zF''(z)}{zF'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 = \frac{zw'(z)}{w(z)}\frac{1}{p+(p-\alpha)w(z)}, \tag{2.11}$$

$$F_2(z) = 1 + z\left(\frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)}\right) = \frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}, \tag{2.12}$$

$$F_3(z) = \frac{F(z)}{zF'(z)}\left(1 + \frac{zF''(z)}{F'(z)}\right) - 1 = \frac{(p-\alpha)zw'(z)}{[p+(p-\alpha)w(z)]^2}, \tag{2.13}$$

$$F_4(z) = \frac{zF'(z)}{F(z)}\left[1 + z\left(\frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)}\right)\right] = (p-\alpha)zw'(z), \tag{2.14}$$

$$F_5(z) = \frac{zF'(z)}{F(z)}\left(\frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1\right) = \frac{zw'(z)}{w(z)}, \tag{2.15}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < p; p \in \mathcal{N}).$$

We claim that $\mid w(z) \mid< 1$ in $\mathcal{U}$. For otherwise (by Jack's Lemma) there exists a point $z_0 \in \mathcal{U}$ such that

$$z_0 w'(z_0) = cw(z_0), \tag{2.16}$$

where

$$\mid w(z_0) \mid= 1,\ (c \geq 1).$$

Therefore, the equations (2.11)-(2.15) in conjunction with (2.16) yield:

$$\mid F_1(z_0) \mid= \left|\frac{z_0 w'(z_0)}{w(z_0)}\frac{1}{p+(p-\alpha)w(z)}\right| = \frac{c \mid w(z_0) \mid}{\mid p+(p-\alpha)w(z_0) \mid} \geq \frac{1}{2p-\alpha}, \tag{2.17}$$

$$\mid F_2(z_0) \mid= \left|\frac{(p-\alpha)z_0 w'(z_0)}{p+(p-\alpha)w(z)}\right| = \frac{c(p-\alpha) \mid w(z_0) \mid}{\mid p+(p-\alpha)w(z_0) \mid} \geq \frac{p-\alpha}{2p-\alpha}, \tag{2.18}$$

$$\mid F_3(z_0) \mid= \left|\frac{(p-\alpha)z_0 w'(z_0)}{p+(p-\alpha)w(z)}\right| = \frac{c(p-\alpha) \mid w(z_0) \mid}{\mid p+(p-\alpha)w(z_0) \mid^2} \geq \frac{p-\alpha}{(2p-\alpha)^2}, \tag{2.19}$$

$$\mid F_4(z_0) \mid= c(p-\alpha) \mid w(z_0) \mid\geq p-\alpha, \tag{2.20}$$

$$\Re e\{F_5(z_0)\} = \Re e\left\{\frac{zw'(z_0)}{w(z_0)}\right\} = c \geq 1, \tag{2.21}$$

$$(0 \leq \alpha < p;\ p \in \mathcal{N}),$$

which contradict our assumptions (2.3)-(2.7), respectively. Therefore $| w(z) |< 1$ holds true for all $z \in \mathcal{U}$. From the definition (2.9) yields

$$\left| \frac{zF'(z)}{F(z)} - p \right| = (p-\alpha) \mid w(z) \mid < p - \alpha, \; (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}), \tag{2.22}$$

which implies that

$$\Re e \left\{ \frac{zF'(z)}{F(z)} \right\} = \Re e \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right\} > \alpha, \tag{2.23}$$
$$(z \in \mathcal{U};\, 0 \leq \lambda \leq 1;\, 0 \leq \alpha < p;\, p \in \mathcal{N}),$$

and hence $f(z) \in \mathcal{T}_\lambda(p;\, \alpha)$.

We mention now some interesting corollaries for the classes $\mathcal{T}_\lambda(\alpha)$, $\mathcal{S}_p(\alpha)$, $\mathcal{K}_p(\alpha)$, $\mathcal{S}_p$, $\mathcal{K}_p$, $\mathcal{S}(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{S}$, and $\mathcal{K}$ which are easily deducible from the main result.

Corollary 1. *Let a function $f(z)$ defined by* (1.1) *belong to the class $\mathcal{T}(p)$. If $f(z)$ satisfies anyone of the following inequalities:*

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} - 1 \right| < \frac{1}{2p-\alpha}, (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}) \tag{2.24}$$

$$\left| 1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right| < \frac{p-\alpha}{2p-\alpha}, \; (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}), \tag{2.25}$$

$$\left| \frac{f(z)}{zf'(z)} \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < \frac{p-\alpha}{(2p-\alpha)^2}, \; (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}) \tag{2.26}$$

$$\left| \frac{zf'(z)}{f(z)} \left[1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right] \right| < p - \alpha, \; (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}), \tag{2.27}$$

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \left(\frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} - 1 \right) \right\} < 1, \; (z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}), \tag{2.28}$$

then $f(z) \in \mathcal{S}_p(\alpha)$.

Corollary 2. *Let a function $f(z)$ defined by* (1.1) *belong to the class $\mathcal{T}(p)$. If $f(z)$ satisfies anyone of the following inequalities:*

$$\left| \frac{1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} - p}{1 + \frac{zf''(z)}{f'(z)} - p} - 1 \right| < \frac{1}{2p-\alpha}, \tag{2.29}$$

$$(z \in \mathcal{U};\, 0 \leq \alpha < p;\, p \in \mathcal{N}),$$

$$\left|z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{f''(z)}{f'(z)}\right)\right|<\frac{p-\alpha}{2p-\alpha}, \tag{2.30}$$
$$(z\in\mathcal{U};\ 0\leq\alpha<p;\ p\in\mathcal{N}),$$

$$\left|\frac{f'(z)}{f'(z)+zf''(z)}\left(1+\frac{z[2f''(z)+zf'''(z)]}{f'(z)+zf''(z)}\right)-1\right|<\frac{p-\alpha}{(2p-\alpha)^2}, \tag{2.31}$$
$$(z\in\mathcal{U};\ 0\leq\alpha<p;\ p\in\mathcal{N}),$$

$$\left|z\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{f''(z)}{f'(z)}\right)\right|<p-\alpha, \tag{2.32}$$
$$(z\in\mathcal{U};\ 0\leq\alpha<p;\ p\in\mathcal{N}),$$

$$\Re e\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{1+\frac{z[2f''(z)+zf'''(z)]}{f'(z)+zf''(z)}-p}{1+\frac{zf''(z)}{f'(z)}-p}-1\right)\right\}<1, \tag{2.33}$$
$$(z\in\mathcal{U};\ 0\leq\alpha<p;\ p\in\mathcal{N}),$$

then $f(z)\in\mathcal{K}_p(\alpha)$.

Corollary 3. *Let a function* $f(z)$ *defined by belong to the class* $\mathcal{T}$*. If* $f(z)$ *satisfies anyone of the following inequalities:*

$$\left|\frac{\frac{zf''(z)}{zf'(z)}}{\frac{zf'(z)}{f(z)}-1}-1\right|<\frac{1}{2-\alpha},\ (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.34}$$

$$\left|1+z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right|<\frac{1-\alpha}{2-\alpha},\ (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.35}$$

$$\left|\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|<\frac{1-\alpha}{(2-\alpha)^2},\ (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.36}$$

$$\left|\frac{zf'(z)}{f(z)}\left[1+z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right]\right|<1-\alpha,\ (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.37}$$

$$\Re e\left\{\frac{zf'(z)}{f(z)}\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}-1}-1\right)\right\}<1,\ (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.38}$$

then $f(z)\in\mathcal{S}(\alpha)$.

Corollary 4. *Let a function* $f(z)$ *defined belong to the class* $\mathcal{T}$*. If* $f(z)$ *satisfies anyone of the following inequalities:*

$$\left|\frac{f'(z)}{zf''(z)}\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)-1\right|<\frac{1}{2-\alpha}, \tag{2.39}$$
$$(z\in\mathcal{U};\ 0\leq\alpha<1),$$

$$\left| z \left(\frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < \frac{1-\alpha}{2-\alpha}, \tag{2.40}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < 1),$$

$$\left| \frac{f'(z)}{f'(z) + zf''(z)} \left(1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} \right) - 1 \right| < \frac{1-\alpha}{(2-\alpha)^2}, \tag{2.41}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < 1),$$

$$\left| z \left(1 + \frac{zf''(z)}{f'(z)} \right) \left(\frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < 1 - \alpha, \tag{2.42}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < 1),$$

$$\Re e \left\{ \left(1 + \frac{zf''(z)}{f'(z)} \right) \left(\frac{f'(z)[2f''(z) + zf'''(z)]}{f''(z)[f'(z) + zf''(z)]} - 1 \right) \right\} < 1, \tag{2.43}$$

$$(z \in \mathcal{U};\ 0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{K}(\alpha)$.

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References

[1] P. L. Duren, *Univalent Functions*, Grundlehren der Matematischen Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[2] A. W. Goodman, *Univalent Functions*, Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[3] H. Irmak and Ö. F. Çetin, Some theorems involving inequalities on $p-$valent functions, Turkish J. Math. 23 (1999), 453-459.

[4] H. Irmak and Ö. F. Çetin, Some inequalities on $p-$valently starlike and $p-$valently convex functions, Bull. Natur. Sci. Engrg. Hacettepe Uni. Ser. B, 28 (1999), 71-76.

[5] H. Irmak and S. Owa, Certain inequalities for multivalent starlike and meromorphically multivalent starlike functions, Bull. Ins. Math. Academia Sinica, 31 (1) (2003), 11-21.

[6] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. 3 (1971), 469-474.

[7] S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65 (1978), 289-305.

[8] S. Owa, M. Nunokawa and H. Saitoh, Some inequalities involving multivalent functions, Ann. Polon. Math. 60 (2) (1994), 159-162.

[9] S. Owa, M. Nunokawa and S. Fukui, A criteiron for $p-$valently starlike functions, Internat. J. Math. and Math. Sci. 17 (1) (1994), 205-207.

[10] S. Owa, H. M. Srivastava, F.-Y. Ren and W.-Q. Yang, The starlikeness of a certain class of integral operators, Complex Variables 27 (1995), 185-1991.

[11] H. M. Srivastava, S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Yersey, London and Hong Kong, 1992.