

An Ostrowski like inequality for convex functions and applications

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ABSTRACT

In this paper we point out an Ostrowski type inequality for convex functions which complement in a sense the recent results for functions of bounded variation and absolutely continuous functions. Applications in connection with the Hermite-Hadamard inequality are also considered.

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1. Introduction

In 1938, A. Ostrowski [9] proved the following integral inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty \quad (1.1)$$

provided f is differentiable and $\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In the last 5 years, many authors have concentrated their efforts in generalising (1.1) and have applied the obtained results in different fields, including Numerical Integration, Probability Theory and Statistics, Information Theory, etc. For a comprehensive approach in the field, see the recent book [5] where many other references may be found.

One direction of generalising (1.1) was pointed out by the author in [2] – [4]. Let us recall here a couple of the main results obtained in the above papers.

Theorem 1. Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i ($i = 0, \dots, k+1$) be $k+2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ ($i = 1, \dots, k$) and $\alpha_{k+1} = b$. If $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$, then we have the inequality:

$$\left| \int_a^b f(x) dx - \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_i) \right| \quad (1.2)$$

$$\leq \left[\frac{1}{2} \nu(h) + \max \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, i = 0, \dots, k-1 \right\} \right] V_a^b(f),$$

where $\nu(h) := \max \{h_i | i = 0, \dots, k-1\}$, $h_i := x_{i+1} - x_i$ ($i = 0, \dots, k-1$) and $V_a^b(f)$ is the total variation of f on $[a, b]$.

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If one would assume more for the function f , for example, absolute continuity, then the following result holds.

Theorem 2. Under the assumptions of Theorem 1 for I_k and α_i ($i = 0, \dots, k+1$) and if $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then

$$\left| \int_a^b f(x) dx - \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_i) \right| \quad (1.3)$$

$$\leq \begin{cases} \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \nu(h) + \max \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, i = 0, \dots, k-1 \right\} \right] \|f'\|_1, & \end{cases}$$

where $\|\cdot\|_p$ ($p \in [1, \infty)$) are the Lebesgue norms, i.e.,

$$\|h\|_\infty : = \operatorname{ess\,sup}_{t \in [a, b]} |h(t)|,$$

$$\|h\|_p : = \left(\int_a^b |h(t)|^p dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ and $\frac{1}{2}$ are best in the sense mentioned above.

In this paper, the case of convex functions $f : [a, b] \rightarrow \mathbb{R}$ is examined. Some particular cases in connection with the well known Hermite-Hadamard inequality for convex functions are also considered.

2. The Results

The following result holds.

Theorem 3. Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i ($i = 0, \dots, k + 1$) be $k + 2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ ($i = 1, \dots, k$) and $\alpha_{k+1} = b$. If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, then we have the inequality:

$$\begin{aligned} & \frac{1}{2} \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1})^2 f'_+(\alpha_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_-(\alpha_{i+1}) \right] \tag{2.1} \\ & \leq \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t) dt \\ & \leq \frac{1}{2} \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1})^2 f'_-(x_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_+(x_i) \right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. Using the integration by parts formula, we may prove the equality (see for example [3]):

$$\sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t) dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f'(t) dt \tag{2.2}$$

for any locally absolutely continuous function $f : (a, b) \rightarrow \mathbb{R}$.

Since f is convex, then it is locally Lipschitzian on (a, b) and thus the above equality holds. Also, we have

$$f'_+(x_i) \leq f'(t) \leq f'_-(\alpha_{i+1}) \quad \text{for a.e. } t \in [x_i, \alpha_{i+1}] \tag{2.3}$$

and

$$f'_+(\alpha_{i+1}) \leq f'(t) \leq f'_-(x_{i+1}) \quad \text{for a.e. } t \in [\alpha_{i+1}, x_{i+1}]. \tag{2.4}$$

Using (2.3) and (2.4), we may write that

$$\begin{aligned} f'_-(\alpha_{i+1}) \int_{x_i}^{\alpha_{i+1}} (t - \alpha_{i+1}) dt & \leq \int_{x_i}^{\alpha_{i+1}} f'(t) (t - \alpha_{i+1}) dt \tag{2.5} \\ & \leq f'_+(x_i) \int_{x_i}^{\alpha_{i+1}} (t - \alpha_{i+1}) dt \end{aligned}$$

and

$$\begin{aligned} f'_+ (\alpha_{i+1}) \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) dt &\leq \int_{\alpha_{i+1}}^{x_{i+1}} f'(t) (t - \alpha_{i+1}) dt \\ &\leq f'_- (x_{i+1}) \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) dt. \end{aligned} \quad (2.6)$$

Adding (2.5) and (2.6) and taking into account that

$$\int_{x_i}^{\alpha_{i+1}} (t - \alpha_{i+1}) dt = -\frac{1}{2} (\alpha_{i+1} - x_i)^2$$

and

$$\int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) dt = \frac{1}{2} (x_{i+1} - \alpha_{i+1})^2,$$

we get

$$\begin{aligned} &\frac{1}{2} \left[(x_{i+1} - \alpha_{i+1})^2 f'_+ (\alpha_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_- (\alpha_{i+1}) \right] \\ &\leq \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f'(t) dt \\ &\leq \frac{1}{2} \left[(x_{i+1} - \alpha_{i+1})^2 f'_- (x_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_+ (x_i) \right] \end{aligned} \quad (2.7)$$

for any $i = 0, \dots, k-1$.

If we sum (2.7) over i from 0 to $k-1$ and use the identity (2.2), we deduce the desired result (2.1).

The sharpness will be proved in what follows for a particular case. ■

It is natural to consider the following particular case.

Corollary 1. *Let L_k and f be as in the above theorem. Then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{8} \sum_{i=0}^{k-1} \left[f'_+ \left(\frac{x_i + x_{i+1}}{2} \right) - f'_- \left(\frac{x_i + x_{i+1}}{2} \right) \right] (x_{i+1} - x_i)^2 \\ &\leq \frac{1}{2} \left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b) \right] \\ &\quad - \int_a^b f(t) dt \\ &\leq \frac{1}{8} \sum_{i=0}^{k-1} [f'_- (x_{i+1}) - f'_+ (x_i)] (x_{i+1} - x_i)^2. \end{aligned} \quad (2.8)$$

The constant $\frac{1}{8}$ in both inequalities is sharp.

The proof follows by the above theorem choosing $\alpha_i = \frac{x_{i-1} + x_i}{2}$, $i = 1, \dots, k$ and taking into account that (see also [2])

$$\begin{aligned} & \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_i) \\ &= \frac{1}{2} \left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b) \right]. \end{aligned} \quad (2.9)$$

The following corollary for equidistant partitioning also holds.

Corollary 2. *Let*

$$I_k : x_i := a + (b - a) \cdot \frac{i}{k} \quad (i = 0, \dots, k)$$

be an equidistant partitioning of $[a, b]$. If $f : [a, b] \rightarrow \mathbb{R}$ is convex on $[a, b]$, then we have the inequalities

$$\begin{aligned} 0 &\leq \frac{(b-a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'_+ \left[a + \left(i + \frac{1}{2} \right) \frac{b-a}{n} \right] \right. \\ &\quad \left. - f'_- \left[a + \left(i + \frac{1}{2} \right) \frac{b-a}{n} \right] \right\} \\ &\leq \frac{1}{k} \cdot \frac{f(a) + f(b)}{2} (b-a) \\ &\quad + \frac{b-a}{k} \sum_{i=1}^{k-1} f \left[\frac{(k-i)a + ib}{k} \right] - \int_a^b f(t) dt \\ &\leq \frac{(b-a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'_- \left[a + (i+1) \cdot \frac{b-a}{n} \right] - f'_+ \left[a + i \cdot \frac{b-a}{n} \right] \right\}. \end{aligned} \quad (2.10)$$

The following particular cases which hold when we assume differentiability conditions may be stated.

Corollary 3. *If $\alpha_i \in (a, b)$ for $i = 1, \dots, k$ are points of differentiability for f , then we have the inequality*

$$\begin{aligned} & \sum_{i=0}^{k-1} (x_{i+1} - x_i) \left(\frac{x_i + x_{i+1}}{2} - \alpha_{i+1} \right) f'(\alpha_{i+1}) \\ &\leq \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t) dt. \end{aligned} \quad (2.11)$$

If we denote by $\nu(I_n) := \max\{x_{i+1} - x_i | i = 0, \dots, k-1\}$, then the following corollary also holds.

Corollary 4. *If x_i ($i = 1, \dots, k-1$) are points of differentiability for f then*

$$\begin{aligned} & \frac{1}{2} \left[(x_1 - a) f(a) + \sum_{i=0}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b) \right] - \int_a^b f(t) dt \\ & \leq \frac{1}{8} [\nu(I_n)]^2 [f'_-(b) - f'_+(a)]. \end{aligned} \quad (2.12)$$

3. Some Particular Inequalities

- (1) If we choose $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = x \in (a, b)$, $\alpha_2 = b$, then from (2.1) we deduce (see also [6])

$$\begin{aligned} & \frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \quad (3.1) \\ & \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t) dt \\ & \leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_-(a) \right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [6]).

If $x = \frac{a+b}{2}$, then by (3.1) one deduces (see also [6])

$$\begin{aligned} 0 & \leq \frac{1}{8} (b-a)^2 \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] \quad (3.2) \\ & \leq \frac{f(a) + f(b)}{2} \cdot (b-a) - \int_a^b f(t) dt \\ & \leq \frac{1}{8} (b-a)^2 [f'_-(b) - f'_+(a)] \end{aligned}$$

and the constant $\frac{1}{8}$ in both inequalities is sharp (see for example [6]).

If one would assume that $x \in (a, b)$ is a point of differentiability, then

$$(b-a) \left(\frac{a+b}{2} - x \right) f'(x) \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t) dt. \quad (3.3)$$

- (2) If we choose $a = x_0 < x < x_2 = b$ and the numbers $\alpha_0 = a$, $\alpha \in (a, x]$, $\beta \in [x, b)$ and $\alpha_3 = b$, then by Theorem 3, we deduce

$$\begin{aligned} & \frac{1}{2} \left[(x - \alpha)^2 f'_+(\alpha) - (\alpha - a)^2 f'_-(\alpha) + (b - \beta)^2 f'_+(\beta) - (\beta - x)^2 f'_-(\beta) \right] \\ & \leq (\alpha - a) f(a) + (\beta - \alpha) f(x) + (b - \beta) f(b) - \int_a^b f(t) dt \quad (3.4) \\ & \leq \frac{1}{2} \left[(x - \alpha)^2 f'_-(x) - (\alpha - a)^2 f'_+(a) + (b - \beta)^2 f'_-(b) - (\beta - x)^2 f'_+(x) \right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

- (a) Note that if we let $\alpha \rightarrow a+$ and $\beta \rightarrow b-$, then from (3.4), by taking into account firstly that $(x - \alpha)^2 f'_+(a) \leq (x - \alpha)^2 f'_+(\alpha)$ and $-(\beta - x)^2 f'_-(b) \leq -(\beta - x)^2 f'_-(\beta)$, we may deduce the inequality obtained in [7]:

$$\begin{aligned} & \frac{1}{2} \left[(b - x)^2 f'_+(x) - (x - a)^2 f'_-(x) \right] \quad (3.5) \\ & \leq \int_a^b f(t) dt - (b - a) f(x) \\ & \leq \frac{1}{2} \left[(\beta - x)^2 f'_-(b) + (x - a)^2 f'_+(a) \right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [7]).

If in (3.5) we choose $x = \frac{a+b}{2}$, then (see also [7])

$$\begin{aligned} 0 & \leq \frac{1}{8} (b - a)^2 \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right) \right] \quad (3.6) \\ & \leq \int_a^b f(t) dt - (b - a) f\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{8} (b - a)^2 [f'_-(b) - f'_+(a)] \end{aligned}$$

and the constant $\frac{1}{8}$ is sharp in both inequalities.

We may state now the following result for convex functions improving Hermite-Hadamard integral inequalities.

Proposition 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then

$$\begin{aligned} 0 &\leq \frac{1}{8}(b-a) \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt - f \left(\frac{a+b}{2} \right) \\ &\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{8}(b-a) [f'_-(b) - f'_+(a)]. \end{aligned} \quad (3.7)$$

The constant $\frac{1}{8}$ is sharp in both parts.

If one would assume that $x \in (a, b)$ is a differentiability point for f , then we have the inequality [7]

$$(b-a) \left(\frac{a+b}{2} - x \right) f'(x) \leq \int_a^b f(t) dt - (b-a) f(x). \quad (3.8)$$

(b) If we choose $\alpha = \frac{a+x}{2}$ and $\beta = \frac{x+b}{2}$, then by (3.4) we have the three point inequality:

$$\begin{aligned} 0 &\leq \frac{1}{8} \left\{ (x-a)^2 \left[f'_+ \left(\frac{a+x}{2} \right) - f'_- \left(\frac{a+x}{2} \right) \right] \right. \\ &\quad \left. + (b-x)^2 \left[f'_+ \left(\frac{x+b}{2} \right) - f'_- \left(\frac{x+b}{2} \right) \right] \right\} \\ &\leq \frac{1}{2} [(x-a)f(a) + f(x)(b-a) + (b-x)f(b)] - \int_a^b f(t) dt \\ &\leq \frac{1}{8} \left\{ (x-a)^2 [f'_+(x) - f'_-(a)] + (b-x)^2 [f'_-(b) - f'_+(x)] \right\} \end{aligned} \quad (3.9)$$

for any $x \in (a, b)$. The constant $\frac{1}{8}$ is sharp in both parts.

If in (3.9) we choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{32}(b-a)^2 \left[f'_+ \left(\frac{3a+b}{4} \right) - f'_- \left(\frac{3a+b}{4} \right) \right. \\ &\quad \left. + f'_+ \left(\frac{a+3b}{4} \right) - f'_- \left(\frac{a+3b}{4} \right) \right] \\ &\leq \frac{1}{2} \cdot \left[\frac{f(a) + f(b)}{2} + f \left(\frac{a+b}{2} \right) \right] (b-a) - \int_a^b f(t) dt \\ &\leq \frac{1}{32}(b-a)^2 \left[f'_-(b) - f'_+ \left(\frac{a+b}{2} \right) + f'_- \left(\frac{a+b}{2} \right) - f'_+(a) \right] \end{aligned} \quad (3.10)$$

If one would assume that f is differentiable in $\frac{a+b}{2}$, then we get the following reverse of Bullen's inequality

$$\begin{aligned} 0 &\leq \frac{1}{2} \cdot \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] (b-a) - \int_a^b f(t) dt \quad (3.11) \\ &\leq \frac{1}{32} (b-a)^2 [f'_-(b) - f'_+(a)]. \end{aligned}$$

The constant $\frac{1}{32}$ is sharp.

(c) Now, if we choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ and $x \in [\frac{5a+b}{6}, \frac{a+5b}{6}]$ in (3.4), then we have the inequalities

$$\begin{aligned} &\frac{1}{2} \left[\left(x - \frac{5a+b}{6}\right)^2 f'_+\left(\frac{5a+b}{6}\right) - \frac{(b-a)^2}{36} f'_-\left(\frac{5a+b}{6}\right) \right. \quad (3.12) \\ &\quad \left. + \frac{(b-a)^2}{36} f'_+\left(\frac{a+5b}{6}\right) - \left(\frac{a+5b}{6} - x\right)^2 f'_-\left(\frac{a+5b}{6}\right) \right] \\ &\leq \frac{b-a}{3} \left[\frac{f(a) + f(b)}{2} + 2f(x) \right] - \int_a^b f(t) dt \\ &\leq \frac{1}{2} \left[\left(x - \frac{5a+b}{6}\right)^2 f'_-(x) - \frac{(b-a)^2}{36} f'_+(a) \right. \\ &\quad \left. + \frac{(b-a)^2}{36} f'_-(b) - \left(\frac{a+5b}{6} - x\right)^2 f'_+(x) \right]. \end{aligned}$$

If in (3.12) we choose $x = \frac{a+b}{2}$, then we get the Simpson's inequality

$$\begin{aligned} &\frac{1}{18} (b-a)^2 \left[f'_+\left(\frac{5a+b}{6}\right) - \frac{1}{4} f'_-\left(\frac{5a+b}{6}\right) \right. \quad (3.13) \\ &\quad \left. + \frac{1}{4} f'_+\left(\frac{a+5b}{6}\right) - f'_-\left(\frac{a+5b}{6}\right) \right] \\ &\leq \frac{b-a}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t) dt \\ &\leq \frac{1}{18} (b-a)^2 \left[f'_-\left(\frac{a+b}{2}\right) - \frac{1}{4} f'_+(a) + \frac{1}{4} f'_-(b) - f'_+\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

If the function is differentiable on (a, b) , then we get

$$\begin{aligned} & -\frac{1}{24}(b-a)^2 \left[f' \left(\frac{a+5b}{6} \right) - f' \left(\frac{5a+b}{6} \right) \right] \\ & \leq \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f \left(\frac{a+b}{2} \right) \right] - \int_a^b f(t) dt \\ & \leq \frac{1}{72}(b-a)^2 [f'_-(b) - f'_+(a)]. \end{aligned} \quad (3.14)$$

References

- [1] G. Anastassiou, Ostrowski type inequalities, *Proc. Amer. Math. Soc.*, **123**(12) (1995), 3775-3781.
- [2] S.S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60** (1999), 495-508.
- [3] S.S. Dragomir, A generalisation of Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a, b]$, $1 < p < \infty$ and applications in numerical integration, *J. Math. Anal. Appl.*, **225** (2001), 605-626.
- [4] S.S. Dragomir, A generalisation of Ostrowski integral inequality for mappings whose derivatives belong to $L_1[a, b]$, and applications in numerical integration, *J. Computational Analysis and Appl.*, **3**(4) (2001), 343-360.
- [5] S.S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.
- [6] S.S. Dragomir, A generalised trapezoid type inequality for convex functions, (Preprint) *RGMA Res. Rep. Coll.*, **5**(1) (2002), Article 9. [ONLINE] <http://rgmia.vu.edu.au/v5n1.html>
- [7] S.S. Dragomir, An Ostrowski type inequality for convex functions, (Preprint) *RGMA Res. Rep. Coll.*, **5**(1) (2002), Article 5. [ONLINE] <http://rgmia.vu.edu.au/v5n1.html>
- [8] A.M. Fink, Bounds on the derivation of a function from its averages, *Czech. Math. J.*, **42** (1992), 289-310.
- [9] A. Ostrowski, Über die Absolutabweichung einer differentiihbaren Function von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 220-227.