Long-time dynamics of an integro-differential equation describing the evolution of a spherical flame

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ABSTRACT

This article is devoted to the study of a flame ball model, derived by G. Joulin, which satisfies a singular integro-differential equation. We prove that, when radiative heat losses are too important, the flame always quenches; when heat losses are smaller, it stabilizes or quenches, depending on an energy input parameter. We also examine the asymptotics of the radius for these different regimes.

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0. Introduction

We are interested here in the study of the following equation, derived in [4], furthering the study [9]:

$$R\partial_{1/2}R = R\mathrm{Log}\ R + Eq(t) - \lambda R^3, \quad R(0) = 0, \quad \partial_{1/2}R = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\dot{R}(s)}{\sqrt{t-s}}. \qquad (01)$$

It describes the evolution of a spherical flame, initiated by a point source energy input $Eq(t)$, and at which are applied heat losses, represented here by the parameter λ, in the burnt gases. The intensity of this energy input is measured by the positive constant E, and its time evolution is described by the function q. This one is a smooth, non-negative function, with connected support and unit total mass, and its initial values satisfy the following assumption:

$$q(t) \sim q_0 t^\beta \text{ as } t \to 0 \text{ with } 0 \le \beta < 1/2. \tag{$\mathcal{Q}$}$$

Also, q tends to 0 as $t \to +\infty$.

Equation (01) is an asymptotic model for the classical thermo-diffusive model for flame propagation, with simple chemistry $A \to B$:

$$\begin{cases} T_t - \Delta T = BR_0 \exp\left(\dfrac{-1}{2\varepsilon T_*}\right) \delta_{r=\rho_\varepsilon(t)} - \varepsilon F, \\ Y_t - \dfrac{\Delta Y}{Le} = -BR_0 \exp\left(\dfrac{-1}{2\varepsilon T_*}\right) \delta_{r=\rho_\varepsilon(t)}. \end{cases}$$

Here, $T(t,x)$ is the temperature, $Y(t,x)$ is the mass fraction of the reactant, and $\rho_\varepsilon(t,x)$ is the flame radius; R_0 is the radius of the stationnary flame, T_* is the temperature at the flame sheet (defined by the location of the δ function). The number $Le > 0$ is the Lewis number, i. e. the ratio between thermal and molecular diffusion, B is a multiplicative constant, and the function F is a heat-loss term, chosen to have the form:

$$F(t,x) = \begin{cases} F(T) \text{ if } r \le \rho_\varepsilon(t), \\ 0 \text{ otherwise.} \end{cases}$$

It is proved, by means of a 3-scale asymptotic expansion, that $\rho_\varepsilon(t/\varepsilon^2)$ satisfies (01) asymptotically. In the course of the computation, the term λ comes out as: $\lambda = F(T_*)/6T_*^2$.

The study for (01) can also be done in the more general case where

$$R\partial_{1/2}R = f(R) + Eq(t). \tag{02}$$

The function f is polynomial, negative at $x = 0$, and has several zeroes. Qualitative results on this equation are similar to the ones presented above: existence of critical energies, and quenching, stabilization, or propagation phenomena; the ideas of the proof are identical. See Part 5 at the end of the paper.

Equation (01) has been studied in [2] with no heat losses. The following threshold phenomenon has been proved: there exists a critical energy $E_{cr}(q) > 0$ such that, if $E < E_{cr}(q)$, the flame quenches; if $E > E_{cr}(q)$, it propagates; if $E = E_{cr}(q)$, the flame stabilizes to 1. Our goal is to find a similar result when $\lambda > 0$ and to study the asymptotics.

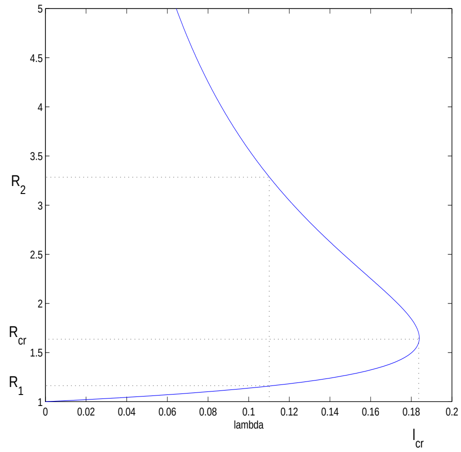

Figure 1: Solutions of Log $R = \lambda R^2$

The long-time behaviour of the radius depends on the value of the parameter λ: we will see later that the equation

$$\text{Log } R = \lambda R^2 \tag{03}$$

plays a fundamental part in the asymptotic study. If $\lambda = \lambda_{cr} = 1/2e$, (03) has a unique solution $R_{cr} = \sqrt{e}$; if $\lambda > \lambda_{cr}$, it has not any solution; if $\lambda < \lambda_{cr}$, it has two solutions: $R_1 < R_{cr} < R_2$ - see Fig. 1. The main results of the paper are the following:

Theorem 0.1. *Assume $\lambda < \lambda_{cr}$.*
(i) If q is positive on $\mathbb{R}_+^*$, (01) has a unique global solution $R(t)$. Moreover, there exists $E_{cr}(q) > 0$ such that:
- if $E < E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = 0$,
- if $E > E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = R_2$,
- if $E = E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = R_1$.
(ii) If there exists $t_0 > 0$ such that q is positive only on $]0, t_0[$, then (01) admits a unique solution $R(t)$. Moreover, there exists $E_{cr}(q) > 0$ such that:
- if $E \geq E_{cr}(q)$, the previous results hold.
- if $E < E_{cr}(q)$, there exists a finite $t_{max} > t_0$ such that the life span of $R(t)$ is $[0, t_{max}]$; moreover, $\lim_{t\to t_{max}} R(t) = 0$.

In particular, unlike the case $\lambda = 0$, the flame cannot propagate any longer. Nevertheless, if heat losses are small enough, we conserve the same structure of solution with a threshold phenomenon. When $\lambda \geq \lambda_{cr}$, the results are quite different:

Theorem 0.2. *Assume $\lambda > \lambda_{cr}$. Let us denote by $[0, t_0]$ the support of the function q.*
(i) If $t_0 = +\infty$, (01) has a unique global solution, satisfying: $\lim_{t\to+\infty} R(t) = 0$.
(ii) If $t_0 < +\infty$, there exists a finite $t_{max} > t_0$ such that the life span of $R(t)$ is $[0, t_{max}]$; moreover, $\lim_{t\to t_{max}} R(t) = 0$.

Thus, the flame always quenches when $\lambda > \lambda_{cr}$, whatever the energy input is. The case $\lambda = \lambda_{cr}$ presents more difficulties: its study forces us to consider energy inputs which do not satisfy any longer the assumption $\int_{\mathbb{R}_+} q = 1$.

Theorem 0.3. *Assume $\lambda = \lambda_{cr}$.*
(i) If q is a smooth, non-negative function, with connected support and unit total mass, and if there exists $\alpha > 1/2$ such that $q(t) = O_{+\infty}(t^{-\alpha})$, the results of Theorem 0.2 hold again.
(ii) If there exists a positive constant C such that $q(t) \sim C/\sqrt{t}$ as $t \to +\infty$, there exists $E_{cr}(q) > 0$ such that:
- if $E < E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = 0$.
- if $E \geq E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = R_{cr}$, and there exists $t_0 > 0$ such that $R(t) \geq R_{cr}$ for $t \geq t_0$.

Our last results deal with the speed of convergence of the radius:

Theorem 0.4. *(i) If the flame quenches in a finite time t_{max}, there exists $\alpha > 0$ such that:*

$$R(t) = \frac{\sqrt{\pi}}{2}\sqrt{t_{max} - t} + O(t_{max} - t)^{1/2+\alpha} \text{ as } t \to t_{max}.$$

(ii) If the function q is positive and if the flame quenches in an infinite time, then

$$R(t) \sim f(t), \text{ with } f \, Log \, f + Eq = 0.$$

Theorem 0.5. *Assume $q(t) = O_{+\infty}(t^{-\alpha})$ with $\alpha > 1/2$. If $\lambda < \lambda_{cr}$ and $\lim_{t\to+\infty} R(t) = R_i$ $(i = 1, 2)$, then*

$$R(t) = R_i + \frac{R_i^2}{(1 - 2\lambda R_i^2)\sqrt{\pi}} \frac{1}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right).$$

The first results obtained about the stabilization and the quenching of the flame correspond to the ones described in [2]. The asymptotic behaviour is proved using the semilinear parabolic equation:

$$u_t - u_{xx} = 2\delta_{x=0}\left(\text{Log } u - \lambda u^2 + \frac{Eq(t)}{u}\right), \quad x \in \mathbb{R}.$$

But the consideration of heat losses forces us to be more careful: on the one hand, when $0 < \lambda < \lambda_{cr}$, we have a similar threshold phenomenon, but the stabilization of the flame for high energies needs another proof: the reminder term $-\lambda R^2$ is all the more important as $R(t)$ is large, and the demonstration written in [2] is no longer valid. On the other hand, in order to study the case $\lambda = \lambda_{cr}$, we have to introduce a larger type of energy inputs.

Theorems 0.1 to 0.5 are illustrated in the pictures 2 and 3; the numerical scheme used to obtain these numerical pictures is presented in [3].

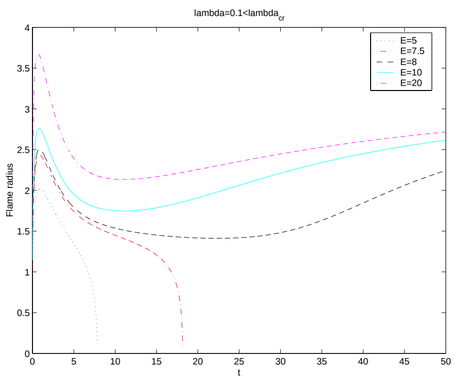

Figure 2: Evolution of the radius when $\lambda < \lambda_{cr}$ and E variable

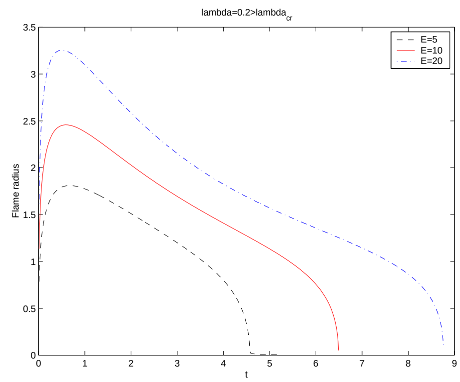

Figure 3: Evolution of the radius when $\lambda > \lambda_{cr}$ and E variable

The study of the quenching phenomenon is a very classical problem - see e.g. [12]. In particular, the heat equation has often been studied, for example in [7]. However, in this paper, the results concern equations of the form $u_t - u_{xx} = f(u)$, with f polynomially singular at $u = 0$; we consider here a weaker singularity in Log . In [5], is studied a parabolic equation of the following type:

$$u_t - u_{xx} = \delta_{x=0} f(u),$$

but the function f is monotone, as opposed to our problem. Moreover, the asymptotic results are formally proved. We have therefore been led to introducing new arguments to prove our results rigorously.

Nevertheless, the integro-differential equation stays an important tool for the asymptotics in case of stabilization; indeed, the radius R_1 is unstable, and the techniques used for the quenching cannot allow us to conclude here. Therefore, for the study of our problem, it is essential to use both diffusive and integral formulations. This systematic combination of both formulations has not, to our knowledge, been done.

This paper is organized in five sections. In the two first sections, we study the asymptotic behaviour of the radius, whose analysis follows the ideas of [2]. In the subsequent sections, we are interested in the asymptotics of the regimes obtained previously: Section 3 is devoted to the quenching case, and Section 4 to the stabilization case. In the last section, we extend these results to the more general parabolic equations coming from the integro-differential equations (02).

1. Long-time behaviour of the flame for any λ non critical

The proof of the local existence for (01), and of the conditions of global existence is omitted, being identical to the one written in [2]. Let us just only recall the main results which will be useful later.

Equation (01) can be rewritten by:

$$R = I_{1/2}\left(\text{Log } R + \frac{Eq}{R} - \lambda R^2\right), \tag{11}$$

where $I_{1/2}f = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^t \frac{f(s)}{\sqrt{t-s}}ds$ is the Abel integrator of order $1/2$. In the sequel, B denotes the Euler Beta function, defined by:

$$B(\alpha,\beta) = \int_0^1 \sigma^{\alpha-1}(1-\sigma)^{\beta-1}d\sigma.$$

Proposition 1.1. *Let us assume q positive on $[0,t_0]$, $q(0) = q_0$. Then, there exists $t_1 \in]0,t_0]$ such that (11) admits a solution in $\mathcal{C}^{3/2}([0,t_1])$ satisfying:*

$$R(t) \sim_0 R_0 t^{1/4}, \text{ with } R_0^2 = \frac{Eq_0}{\sqrt{\pi}}B(3/4,1/2).$$

In the case $q(0) = 0$, we also obtain a local solution of (11) in a vicinity of 0 with a similar equivalent. In order to prove the existence of a unique maximal solution, the flame radius is expressed as the trace at $x = 0$ of a function $u(t,x)$, solution of the following parabolic equation:

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}\left(\text{Log } u + \dfrac{Eq}{u} - \lambda u^2\right), & x \in \mathbb{R}, \\ u(0,.) = 0. \end{cases} \tag{12}$$

This formulation has been used in [6] to study Volterra integral equations, and here it is essential to characterize the long-time behaviour of the flame. We will then deal with more general Cauchy problems:

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}\left(\text{Log } u + \dfrac{Eq}{u} - \lambda u^2\right), \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{13}$$

where u_0 is an even, Lipschitz, square-integrable, non-negative function. This is equivalent to solving:

$$\begin{cases} u_t - u_{xx} = 0, \quad x > 0, \\ u_x(t,0) = -\left(\text{Log } u(t,0) + \dfrac{Eq(t)}{u(t,0)} - \lambda u^2(t,0)\right), \\ u(0,.) = u_0. \end{cases} \tag{14}$$

Theorem 1.2. *Let q satisfying the assumption $(\mathcal{Q})$. We suppose there exists $t_0 > 0$ such that $q(t) > 0$ on $]0, t_0[$, and $q(t) = 0$ if $t \geq t_0$.*
(i) If $t_0 = +\infty$, (13) has a unique global positive solution, except at $t = 0$. Moreover, u is $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathbb{R}^*$ and $t \mapsto u(t,0)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+^*$.
(ii) If $t_0 < +\infty$, (13) has a unique maximal solution u, defined on an interval $[0, t_{max}[$, positive, except at $t = 0$. Moreover, u is $\mathcal{C}^\infty$ on $]0, t_{max}[$. If $t_{max} < +\infty$, there exists $t_n \to t_{max}$ such that $\lim_{n\to+\infty} u(t_n, 0) = 0$.

In particular, an immediate consequence of this theorem is the existence of a solution of (11). The uniqueness of u is based on the following comparison principle:

Theorem 1.3. *Let q_1 and q_2 be two smooth functions satisfying the assumptions of Theorem 1.2. Let u_1 and u_2 the solutions of (13) with respectively $Eq = E_1 q_1, E_2 q_2$, $u_0 = u_{01}, u_{02}$ and $\lambda = \lambda_1, \lambda_2$. Let $[0,T]$ an interval on which both u_1 and u_2 are defined. Then, if $E_1 q_1 \leq E_2 q_1$, $u_{01} \leq u_{02}$ and $\lambda_1 \geq \lambda_2$, $u_1 \leq u_2$ on $[0,T] \times \mathbb{R}$.*

In the sequel, u_E will denote the solution of (12) and $R_E(t) := u_E(t,0)$ will be the corresponding radius of the flame.

We can now turn to the asymptotic behaviour of the radius. An important tool in the forthcoming study will be the usual theorem on sub and supersolutions [13]. But, before everything, let us quickly explain the importance of the parameter λ: the stationnary solutions of

$$u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \lambda u^2)$$

are the constants R satisfying:

$$\text{Log } R = \lambda R^2,$$

hence the values of λ_{cr} and the distinction we have to make between the cases $\lambda < \lambda_{cr}$, $\lambda = \lambda_{cr}$ and $\lambda > \lambda_{cr}$. On the other hand, these heat losses introduced in this model, prevent the flame from expanding:

Proposition 1.4. *If $\lambda > 0$, the solution u_E of (13) is bounded.*

Proof. Let $\overline{u}(t,x) = C \in \mathbb{R}_+^*$. For any $E < E_0 = \dfrac{1}{\|q\|_\infty}(\lambda C^3 - C\text{Log } C)$ (with $C < R_1$ or $C > R_2$ if $\lambda \leq R_{cr}$), $\overline{u}$ is a supersolution of

$$-u'' = 2\delta_{x=0}\left(\text{Log } u + \frac{Eq}{u} - \lambda u^2\right)$$

and $u_E \leq C$ if $E < E_0$ and $u_0 \leq C$. ■

We begin the study of the asymptotic behaviour of the flame ball by the case $\lambda > \lambda_{cr}$, for which heat losses are too important and the flame always quenches; this is a consequence of the following proposition:

Proposition 1.5. *Assume $\lambda > \lambda_{cr}$.*
(i) If the function q is positive on $\mathbb{R}_+^*$, the solution of (12) is global and $\lim\limits_{t\to+\infty} R_E(t) = 0$.
(ii) If q is compactly supported, R_E quenches in finite time.

Proof. (i) Let $t_\varepsilon \geq 0$ be such that $q(t) \leq \varepsilon/E_0$ for all $t \geq t_\varepsilon$, where E_0 checks:

$$\text{Log } C + \frac{\varepsilon}{C^2} - \lambda C^2 \leq 0,$$

the constant C being defined in the proof of Proposition 1.4. Then, by Theorem 1.3, we have $u_E \leq u_\varepsilon$ for $t \geq t_\varepsilon$, where u_ε is the unique solution of

$$\begin{cases} (\partial_t - \partial_{xx})u_\varepsilon = 2\delta_{x=0}\left(\text{Log } u_\varepsilon + \dfrac{\varepsilon}{u_\varepsilon} - \lambda u_\varepsilon^2\right), \\ u_\varepsilon(t_\varepsilon, .) = C. \end{cases} \tag{15}$$

Yet, since C is a supersolution of the stationnary equation associated to (15), u_ε converges towards the greatest zero, smaller than C, of the function:

$$u \mapsto u\text{Log } u + \varepsilon - \lambda u^3$$

as t tends to $+\infty$. For ε small enough, this function has a unique zero $u_{\varepsilon,\infty}$ and $u_{\varepsilon,\infty} \to 0$ as $\varepsilon \to 0$. Hence, $\lim\limits_{t\to+\infty} u_E(t,x) = 0$.

(ii) Identical to [2], Proposition 4.2. ■

We can now turn to the case λ subcritical for which we conserve the threshold phenomenon obtained in [2]; the results are summarized in the following theorems:

Theorem 1.6. *Assume $\lambda < \lambda_{cr}$ and $q > 0$ on $\mathbb{R}_+^*$. Then, (11) has a unique global solution $R_E(t)$ and there exists $E_{cr}(q) > 0$ such that:*

- *if* $E < E_{cr}(q)$, $\lim_{t\to+\infty} R_E(t) = 0$,
- *if* $E > E_{cr}(q)$, $\lim_{t\to+\infty} R_E(t) = R_2$,
- *if* $E = E_{cr}(q)$, $\lim_{t\to+\infty} R_E(t) = R_1$.

Theorem 1.7. *Assume* $\lambda < \lambda_{cr}$ *and* q *is compactly supported, with support* $]0, t_0[$, *then (11) has a unique solution* $R_E(t)$. *Moreover, there exists* $E_{cr}(q) > 0$ *such that:*
- *if* $E \geq E_{cr}(q)$, *the previous results hold again.*
- *if* $E < E_{cr}(q)$, R_E *quenches in finite time.*

We are firstly going to study the asymptotic behaviour of the radius for small and high energies. In a second part, the existence and uniqueness of the threshold will be proved. Since the main arguments of the proof are identical to the case $\lambda = 0$, we only present its main lines and the necessary changes.

Let us introduce a family of subsolutions of:

$$-u'' = 2\delta_{x=0}(\text{Log } u - \lambda u^2). \tag{16}$$

Its only even solutions are the functions

$$\phi_b(x) = -(\text{Log } b - \lambda b^2)|x| + b,$$

and the functions

$$\underline{\phi}_b = \max(\phi_b, 0), \quad R_1 < b < R_2, \tag{17}$$

are compactly supported subsolutions of (16).

Similarly, we get a family of supersolutions of

$$-u'' = 2\delta_{x=0}\left(\text{Log } u + \frac{\varepsilon}{u} - \lambda u^2\right)$$

for any small positive ε: the functions

$$\overline{\varphi}_b(x) = \min(\varphi_b, R_1), \quad 0 < b < R_{1,\varepsilon}, \tag{18}$$

with:

$$\varphi_b(x) = -\left(\text{Log } b + \frac{\varepsilon}{b} - \lambda b^2\right)|x| + b,$$

and where $R_{1,\varepsilon}$ is the maximal zero smaller than R_1 of the function $u \mapsto \text{Log } u + \varepsilon/u - \lambda u^2$. Moreover, these supersolutions are valid again in the case $\lambda = \lambda_{cr}$, which will be useful in the next part.

Let us now begin with the behaviour of R_E for small energies, whose proof, based on the classical theorem on sub and supersolutions, is omitted.

Proposition 1.8. *(i) If q is compactly supported, there exists $E_0 > 0$ such that for all $E \leq E_0$, the solution quenches in finite time.*
(ii) Assume $q > 0$ on $\mathbb{R}_+^*$. Then, there exists $E_0 > 0$ such that $\forall E \leq E_0$,

$$\lim_{t \to +\infty} R_E(t) = 0.$$

When we are dealing with high energies, a stabilization of the radius is observed. This behaviour is different from the one described in [2]: because of the heat losses, the reminder term is all the more important as $R(t)$ is large, which prevents the flame from propagating:

Proposition 1.9. *Let q satisfy the assumptions of Theorem 1.2. There exists a positive energy E_1 such that, for all $E > E_1$, $\lim_{t \to +\infty} R_E(t) = R_2$.*

Proof. We prove that $\limsup_{t \to +\infty} u_E(t) \leq R_2$ as in [2]. The fact that $\liminf_{t \to +\infty} u_E(t) \geq R_2$ is based on a two-step argument. We first prove that $R(t)$ is large on a fixed - with respect to E - time interval. This allows us to use the argument of [2]: at the end of this time interval, the function u is above a subsolution of

$$u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \lambda u^2),$$

hence the conclusion using the usual theorem on sub and supersolutions. Let us turn to the first point.

We can suppose $q(0) > 0$ (if not, we begin from $t = \varepsilon$ small enough). Let $t_1 > 0$ be such that $q(t) \geq q_{min} > 0$ on $[0, t_1]$. Then, $R_E \geq \underline{R}$, solution of:

$$\underline{R} = I_{1/2}\left(\text{Log } \underline{R} + \frac{E q_{min}}{\underline{R}} - \lambda \underline{R}^2\right)$$

on $[0, t_1]$. Yet, by Proposition 1.1, $R(t) \sim R_0 t^{1/4}$ and this equivalent is only valid as long as $E/\underline{R} \gg \underline{R}^2$, namely $\underline{R} \ll E^{1/3}$ for instance. In particular, if $0 < \gamma \ll 1/6$, $\exists t_0 \in]0, t_1[$ such that $\underline{R}(t) \geq E^{2\gamma}$. Assume the existence of $t_2 \in]t_0, t_1[$ such that $\underline{R}(t_2) = E^\gamma$. Let us consider $\tilde{R}$, the unique solution of:

$$\tilde{R} = I_{1/2}(E^{1-\gamma}/(2\tilde{R})),$$

beginning at a time $t_E \in]t_0, t_2[$ and satisfying $\tilde{R}(t_2) = E^\gamma$; then,

$$\tilde{R}(t) = \sqrt{\frac{E^{1-\gamma}}{\sqrt{\pi}}}\sqrt{t - t_E},$$

and $\tilde{R}(t) = w(t, 0)$ with:

$$\begin{cases} w_t - w_{xx} = \delta_{x=0}\left(\dfrac{E^{1-\gamma}}{w}\right), \\ w(t_E, 0) = 0. \end{cases}$$

Also, for γ small enough, if $0 < R \leq E^{2\gamma}$, then

$$\operatorname{Log} R + \frac{Eq_{min}}{R} - \lambda R^2 \geq \frac{E^{1-\gamma}}{2R}.$$

Thus, $\underline{R} \geq \tilde{R}$ on $]t_E, t_1[$ which is impossible here. Indeed, $\partial_t \underline{R}(t_2) \leq 0$ whereas $\partial_t \tilde{R}(t_2) > 0$. Hence, $R(t) \geq E^{\gamma}$ on $[t_1/2, t_1]$ for $u_E(t, x) \geq \min(E^{2\gamma}, w(t, x))$. ■

Remark. If λ depends on the time and if there exists λ_1, λ_2 such that: $0 < \lambda_1 \leq \lambda(t) \leq \lambda_2 < \lambda_{cr}$, then, for sufficiently large energies, $R_2^1 \leq R(t) \leq R_2^2$, where R_2^i is the greatest critical radius associated to λ_i. In particular, the flame does not quench. We thank G. Joulin for pointing out to us this consequence.

The remaining of the study of the case $\lambda < \lambda_{cr}$ is the same as in [2]: let us consider the sets:

$$X_+ = \{E > 0, \lim_{t \to +\infty} R_E(t) = R_2\}$$

$$X_- = \{E > 0, \lim_{t \to +\infty} R_E(t) = 0 \text{ or } R_E \text{ quenches in a finite time}\}.$$

Proposition 1.10. *The sets X_+ and X_- are open subsets of $\mathbb{R}_+^*$.*

This fact follows from the two following results. The first one is a consequence of the non-increase of the zero number theorem - see [1] - which does not depend on the particular stucture of the nonlinearity:

Theorem 1.11. *Let $E > 0$. For any $t > 0$, the function $x \mapsto u_E(t, x) - R_1$ vanishes at most twice on $\mathbb{R}$.*

The proof is the same as in [2]. We deduce from this theorem a characterization of X_- when q is compactly supported.

Proposition 1.12. *Let us assume that q is compactly supported, of support $(0, t_0)$. Then,*

$$E \in X_- \Leftrightarrow \exists t_1 > t_0,\, R_E(t) < R_1.$$

Proposition 1.10 implies, by connectedness of $\mathbb{R}_+^*$, that $X_0 = \mathbb{R}_+^* \setminus (X_+ \cup X_-) \neq \emptyset$. In order to study this set X_0, let us introduce, for every Lipschitz function u_0, square-integrable, the ω-limit set of u_0 with respect to (13):

$$\omega(u_0) = \{\psi \in \mathcal{C}^\infty(\mathbb{R}_+^* \cup \mathbb{R}_-^*),\, \psi \neq 0 \text{ such that } \exists t_n \to +\infty / \lim_{n \to +\infty} u(t_n, .) = \psi\}. \quad (19)$$

This ω-limit set is characterized in the following proposition:

Proposition 1.13. *Assume $\omega(0) \neq \emptyset$. Then, there exists $\psi \in \omega(0)$ such that:*
(i) either $\psi \equiv R_1$,
(ii) or $\psi(x) > R_1$ for all $x \in \mathbb{R}$,
(iii) or $\psi(x) < R_1$ for all $x \in \mathbb{R}$.

If the first case is true, $S(t)\psi$ will converge to R_1, wich is our claim. Let us then prove that the two other cases are impossible. For any positive fixed L, we consider $S_L(t)$, the semi-group associated to the equation:

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \lambda u^2),\ x \in]-L, L[, \\ u(t, \pm L) = R_1,\ u(0, x) = u_0(x). \end{cases} \tag{110}$$

Proposition 1.14. *There exists $L_0 > 0$ such that for all $L > L_0$, $S_L(t)$ has two fixed points: the unstable solution R_1 and a stable solution $\psi_L^+ > R_1$. Moreover, $\lim_{L \to +\infty} \psi_L^+ = R_2$. Finally, let $u_0 \in \mathcal{C}^1([-L, L])$ an even function checking $u_0(\pm L) = R_1$ and $u_0(0) \neq R_1$. If $u_0 \geq R_1$, $S_L(t)u_0 \to \psi_L^+$ on every compact subset of $]-L, L[$.*

This proposition applies when the function q is compactly supported; when q is positive on $\mathbb{R}_+^*$, we have to look at the stability of the semi-group $S_{L,\varepsilon}(t)$, generated by the problem:

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}\left(\text{Log } u + \dfrac{Eq}{u} - \lambda u^2\right), \\ u(t, \pm L) = R_1, \end{cases}$$

where $\varepsilon > 0$, and we obtain a result similar to Proposition 1.14. With these tools, we can then prove that the set X_0 is reduced to a unique point R_{cr} and that the assertions (ii) and (iii) of Proposition 1.13 are impossible; thus,

$$\lim_{t \to +\infty} R_{E_{cr}}(t) = R_1.$$

To prove Theorems 0.1 and 0.2, it remains, in the case of a compactly supported function $q(t)$, to make more precise the behaviour of R_E near its quenching point: let $[0, t_0]$ be the support of q.

Lemma 1.15. *Let t_{max} be the quenching time. There exists $t_1 \in]t_0, t_{max}[$ such that $\dot{R}_E(t) < 0$ for $t \in]t_1, t_{max}[$.*

This lemma is proved in [2]; we just recall its main steps. Either $u(t, .)$ converges uniformly on every compact subset of $\mathbb{R}_+^*$, or there exist $\varepsilon > 0$, a sequence t_n converging to t_{max}, and $x_0 > 1$ such that $u(t_n, x_0) \geq \varepsilon$. In this case, parabolic regularity and the nonincrease of zero number theorem allow us to conclude. Otherwise, $u(t, .) \to u_\infty$ as $t \to t_{max}$. Therefore, $u \to 0$ on every compact subset of $\mathbb{R}_+^*$. We thus can extend u

by 0 for $t > t_{max}$, and we have a contradiction using Holmgren's uniqueness theorem. This lemma implies that, if the flame quenches in a finite time t_{max}, $\lim_{t \to t_{max}} R_E(t) = 0$. We have thus proved the dynamics of the model in case of non-critical heat losses.

2. Long-time behaviour of the flame for critical heat losses: proof of Theorem 0.3

We now study the asymptotic behaviour of the radius when $\lambda = \lambda_{cr}$. Let us notice that the behaviour of the function q plays a fundamental role here, as opposed to the previous part. Indeed, when $\lambda \neq \lambda_{cr}$, we only needed $q(t) \to 0$ as $t \to +\infty$ to prove the desired results; it will not be the case anylonger here. Let us first examine the case $q = O_{+\infty}(t^{-\alpha})$, where $\alpha > 1/2$; we will prove the second point of Theorem 0.3 later.

Lemma 2.1. *When $\lambda = \lambda_{cr}$, under the assumptions of Theorem 0.3 (i), if $q(t) = o(1/\sqrt{t})$, there exists $t_n \to +\infty$ such that $R(t_n) \to 0$.*

Proof. This directly comes from the integro-differential equation (11).

Indeed, suppose the result is false. Then, $R(t)$ is bounded away from 0 for t sufficiently large, and since $q \in L_1(\mathbb{R}_+)$, and Log $R - \lambda_{cr} R^2 \leq 0$ for any $R > 0$,

$$R = I_{1/2}\left(\text{Log } R + \frac{Eq}{R} - \lambda_{cr} R^2\right) \leq I_{1/2}\left(\frac{Eq}{R}\right) = o_{+\infty}(1),$$

which contradicts our assumption. ■

Therefore, the proof of Proposition 1.5 can be applied again when q is compactly supported, since Log $C - \lambda C^2 \leq 0$ for any positive constant C.

When the function q is positive on $\mathbb{R}_+^*$, this proof is no longer valid. Let us examine the ω-limit set $\omega(0)$. We want to get $\omega(0) \neq \emptyset$. Yet, some results of the previous part ($\lambda < \lambda_{cr}$) are still true; we conserve Theorem 1.11 about the number of zeroes of $u_E - R_{cr}$, and the characterization of X_- when q is compactly supported (Proposition 1.12). Lemma 1.15 about the behaviour of the flame near its quenching point holds again. This implies the equivalent of Proposition 1.13:

Proposition 2.2. *Assume $\lambda = \lambda_{cr}$ and $\omega(0) \neq \emptyset$ (where $\omega(u_0)$ is defined in (19)). Then, there exists $\psi \in \omega(0)$ such that:*
(i) either $\psi \equiv R_{cr}$,
(ii) or $\psi(x) < R_{cr}$ for all $x \in \mathbb{R}$.

Because of Lemma 2.1, $R_{cr} \notin \omega(0)$, and the second assertion is false:

Lemma 2.3. *Assume $\omega(0) \neq \emptyset$. There cannot exist $\psi \in \omega(0)$ such that $\psi(x) < R_{cr}$ for all $x \in \mathbb{R}$.*

Proof. Let us consider $\psi \in \omega(0)$ such that $\psi(x) < R_{cr}$ for all $x \in \mathbb{R}$. Then, $S(t)\psi$ is solution of the problem:

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \lambda u^2), \\ u(0,.) = \psi, \end{cases}$$

and Proposition 2.2 implies that $S(t)\psi \to 0$ as $t \to +\infty$. Then, for $\varepsilon > 0$ small enough, there exist $b < R_{cr,\varepsilon}$ and t_0 sufficiently large so that:

$$Eq(t) \leq \varepsilon \text{ for all } t \geq t_0,$$

and

$$u_E(t_0, x) \leq \overline{\varphi}_b(x) \text{ for all } x \in \mathbb{R},$$

where $\overline{\varphi}_b$ was defined in the previous part as a supersolution of

$$-u'' = 2\delta_{x=0}(\text{Log } u - \lambda_{cr} u^2).$$

We can then conclude as in Proposition 1.5 that the flame quenches. ■

In the remaining part of this section, we are going to prove the second point of Theorem 0.3. Firstly, it will be seen that for sufficiently high energies, if $q(t) = 1/\sqrt{t}$ for instance, the flame stabilizes to R_{cr}. Then, we will obtain a threshold phenomenon, similar to the one described in the previous part, and we will end by the proof of the estimate on the critical energy.

Lemma 2.4. *Let $q(t) = 1/\sqrt{t}$. There exists $E_0 > 0$, such that, for all $E > E_0$, $\lim_{t \to +\infty} R(t) = R_{cr}$.*

Proof. Let us write the diffusive formulation (12) in the similarity variables (τ, η), given by:

$$\tau = \text{Log } (t+1), \quad \eta = \frac{x}{t+1},$$

when $q(t) = 1/\sqrt{t}$. There holds:

$$\begin{cases} u_\tau + \mathcal{L}u = 2\delta_{x=0}\left((e^{\tau/2} - 1)(\text{Log } u - \lambda_{cr} u^2) + \dfrac{E}{u}\right), \\ u(0,.) = 0, \end{cases} \tag{21}$$

where $\mathcal{L}$ denotes the differential operator:

$$\mathcal{L} = -\partial_{\eta\eta} - \frac{\eta}{2}\partial_\eta.$$

As in the previous part, we are going to use a subsolution compactly supported of:

$$u_\tau + \mathcal{L}u = 2\delta_{x=0}\left((e^{\tau/2} - 1)(\text{Log } u - \lambda_{cr} u^2) + \frac{E}{u}\right). \tag{22}$$

The function

$$\gamma(\eta) = R_{cr} - \frac{E}{R_{cr}} \int_0^\eta e^{-x^2/4} dx$$

is a stationary solution of (22), so that:

$$\underline{\gamma}(\eta) = \max\left(0, R_{cr} - \frac{E}{R_{cr}} \int_0^\eta e^{-x^2/4} dx\right)$$

is a compactly supported subsolution of (22) when $E > R_{cr}^2\sqrt{\pi} = e\sqrt{\pi}$. As in the proof of Proposition 1.9, we can estimate u from below by $\underline{\gamma}$ for E sufficiently large, and we conclude easily that $u \to R_{cr}$ as $\tau \to +\infty$. ■

Then, there exists $E_{cr}(q) > 0$ such that:
- if $E < E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = 0$.
- if $E \geq E_{cr}(q)$, $\lim_{t\to+\infty} R(t) = R_{cr}$.

Proposition 2.5. *When $q(t) = 1/\sqrt{t}$ and $\lambda = \lambda_{cr}$, we have $E_{cr} > e\sqrt{\pi}$.*

Proof. It directly follows from the fact that the stationary solution γ of (22) is positive for $E \leq R_{cr}^2\sqrt{\pi}$. ■

To end the proof of the asymptotic behaviour of the radius, we will need the following lemma, consequence of the non-increase zeroes number [1]

Lemma 2.6. *For any $t > 0$, the function $\eta \mapsto u_E(\tau, \eta) - \gamma(\eta)$ has at most two zeroes on $\mathbb{R}$.*

In particular, since $\underline{\gamma}$ is a subsolution of (22), we have:

Proposition 2.7. *If there exists $t_0 > 0$ such that $R_E(t_0) \geq R_{cr}$, then, for any $t \geq t_0$, $R_E(t) \geq R_{cr}$.*

We conclude the proof of Theorem 0.3 (ii) with the following proposition:

Proposition 2.8. *If $R_E(t) \to R_{cr}$ as $t \to +\infty$, there exists $t_0 > 0$ such that $R_E(t_0) > R_{cr}$.*

Proof. Let $E > 0$. Let us suppose that $R_E(t) \to R_{cr}$ as $t \to +\infty$ and that $R_E(t) < R_{cr}$ for any $t > 0$. Let us first notice that, when τ is sufficiently large, the functions:

$$\overline{\gamma}_C(\eta) = C - \frac{E}{C} \int_0^\eta e^{-x^2/4} dx$$

are supersolutions of (22), for any $C \in]C_-, C_+[$, with $C_- = \sqrt{E\sqrt{\pi}}$ and $C_+ = (R_{cr} + \sqrt{R_{cr}^2 + E\sqrt{\pi}})/2$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that $u_E(\tau, \eta) \leq \overline{\gamma}_C(\eta)$ for any $\eta \in \mathbb{R}$, and, for any $\tau \geq \tau_0$,

$$u_E(\tau, \eta) \leq \overline{\gamma}_C(\eta).$$

In particular $R_{cr} \notin \omega(0)$. Therefore, the flame quenches. ■

Remark. If we search formally an equivalent of $R - R_{cr}$ under the form $Ct^{-1/4}$, Propositions 2.7 and 2.8 imply that:

$$R(t) - R_{cr} \sim \sqrt{e\left(\frac{E}{R_{cr}} - \frac{R_{cr}}{\pi}\right)} \frac{1}{t^{1/4}}.$$

3. Asymptotic behaviour of the flame in the quenching case

This section is devoted to the proof of Theorem 0.4. Let us begin it by the quenching in finite time: we study the diffusive formulation in the similarity variables (τ, η), where:

$$\tau = -\text{Log } (t_{max} - t); \quad \eta = \frac{x}{\sqrt{t_{max} - t}}. \tag{31}$$

Let $(0, t_0)$ be the support of q. Then, for any $t \geq t_0$, the diffusive formulation can be rewritten under the form:

$$\begin{cases} \left(\partial_\tau + \mathcal{L} + \frac{1}{2}\right) u_E = 0, \ \eta > 0, \\ \partial_\eta u_E(\tau, 0) = -e^{-\tau/2}(\text{Log } u_E(\tau, 0) - \lambda u_E(\tau, 0)^2), \end{cases}$$

and since $u_E(\tau, 0) \to 0$ as $\tau \to +\infty$, we are therefore led to considering the problem:

$$\begin{cases} \left(\partial_\tau + \mathcal{L} + \frac{1}{2}\right) u_E = 0, \ \eta > 0, \\ \partial_\eta u_E(\tau, 0) = -e^{-\tau/2}\text{Log } u_E(\tau, 0), \end{cases} \tag{32}$$

where $\mathcal{L}$ is the differential operator:

$$\mathcal{L} = -\partial_{\eta\eta} + \frac{\eta}{2}\partial_\eta - \frac{1}{2}.$$

If we search solutions of (32) under the form: $e^{-\tau/2}(\tau\phi(\eta) + \psi(\eta))$, there holds:

$$\begin{cases} \mathcal{L}\phi = 0, \ \mathcal{L}\psi = -\phi, \\ \tau\phi'(0) + \psi'(0) = \frac{\tau}{2} - \text{Log } (\tau\phi(0) + \psi(0)). \end{cases}$$

This imposes $\phi(0) = 0$ and the solutions of this problem can be calculated explicitely, and we get:

$$\begin{cases} \phi(\eta) = \frac{\eta}{2}, \\ \psi(\eta) = \frac{\sqrt{\pi}}{2} - \eta\text{Log } \frac{\sqrt{\pi}}{2} + \eta \int_0^\eta \left(\frac{1}{\eta'} - \frac{1}{\eta'^2}\left(\int_0^{\eta'} e^{(\eta''^2 - \eta'^2)/4} d\eta'' \right.\right. \\ \qquad \left.\left. + \int_0^{+\infty} (e^{(-\eta''^2 - \eta'^2)/4} - e^{-\eta''^2/4}) d\eta''\right)\right) d\eta' \end{cases}$$

In the following, we are going to look for $u_E(\tau,\eta)$ under the form:

$$u_E(\tau,\eta) = e^{-\tau/2}(\tau\phi(\eta) + \psi(\eta) + v(\tau,\eta)),$$

where the function v checks:

$$\begin{cases} v_\tau + \mathcal{L}v = 0, \ \eta > 0, \\ v_\eta(\tau,0) = -\text{Log}\ \left(1 + \dfrac{2v(\tau,0)}{\sqrt{\pi}}\right). \end{cases} \tag{33}$$

Our goal is to prove that $v(\tau,0) = O(e^{-\omega\tau})$ for some $\omega > 0$. First, we will prove that $v(\tau,0)$ is bounded away from $-\dfrac{\sqrt{\pi}}{2}$; then, we will verify that $v(\tau,0) \to 0$ as $\tau \to +\infty$, and we will conclude by a stability argument.

Lemma 3.1. *The quantity $v(\tau,0)$ is bounded away from $-\dfrac{\sqrt{\pi}}{2}$.*

Proof. Since $u_E(\tau,\eta) \geq 0$, we already know that $v(\tau,0) \geq -\dfrac{\sqrt{\pi}}{2}$. We have to prove that $v(\tau,0) \neq -\dfrac{\sqrt{\pi}}{2}$: let us set $h = v_\tau$. Then,

$$\begin{cases} h_\tau + \mathcal{L}h = 4\delta_{\eta=0}\dfrac{h(\tau,0)}{2v(\tau,0)+\sqrt{\pi}}, \\ h(\tau,\eta) = \dfrac{|\eta|}{2} + O(1), \, \eta \to \pm\infty. \end{cases}$$

The nonincrease of the zero number theorem can be applied once again to the function h; yet, $h(\tau,\eta) \to \pm\infty$ as $\eta \to +\infty$ and this number of zeroes is finite and nonincreasing. Since the function h is odd, the sign of $h(\tau,0)$ is constant for large times, so that $v(\tau,0)$ converges to some value $\overline{v}$. Let us prove that $\overline{v} > -\dfrac{\sqrt{\pi}}{2}$. To do so, we define $\varphi(\tau) = v(\tau,0)$ and $w(\tau,\eta) = v(\tau,\eta) - \varphi(\tau)$. Then, the function w is solution of the problem:

$$\begin{cases} w_\tau + \mathcal{L}w = \dfrac{\varphi}{2} - \dot{\varphi}, \ \eta > 0, \\ w(\tau,0) = 0. \end{cases} \tag{34}$$

To study this problem, we are going to examine the decomposition of w in $L^2_\rho(\mathbb{R}_+)$, where $\rho(\eta) = e^{-\eta^2/4}$ and

$$L^2_\rho(I) = \{u \in L^2_{loc}(I), \, \rho u \in L^2(I)\}$$

for any unbounded interval I of $\mathbb{R}$. The eigenvalues of $\mathcal{L}$ in $L^2_\rho(\mathbb{R}_+)$ are $\lambda_k = (k-1)/2$, $k \in \mathbb{N}$, and the corresponding simple eigenfunctions are $e_k(\eta) = H_k(\eta/2)$, where H_k is the k^{th} Hermite polynomial. In $L^2_\rho(\mathbb{R}_+)$ with scalar product:

$$(u|v) = 2\int_0^{+\infty} u(\eta)v(\eta)e^{-\eta^2/4}d\eta,$$

the eigenvalues of $\mathcal{L}$ with Dirichlet conditions are $\mu_k = \lambda_{2k+1}$, and the eigenfunctions are $\tilde{e}_k = e_{2k+1}$. In particular,

$$\tilde{e}_0(\eta) = \frac{1}{2\pi^{1/4}}\eta.$$

If we express w in this basis:

$$w(\tau,\eta) = \sum_{k\geq 0} w_k(\tau)\tilde{e}_k(\eta), \tag{35}$$

the functions w_k are solutions of the differential equations:

$$\dot{w}_k + \mu_k w_k = \left(\frac{\varphi}{2} - \dot{\varphi}\right)(\tilde{e}_k|1), \tag{36}$$

so that:

$$w_k(\tau) = e^{-\mu_k\tau}w_k(0) + \int_0^\tau e^{-\mu_k(\tau-\sigma)}\left(\frac{\varphi(\sigma)}{2} - \dot{\varphi}(\sigma)\right)d\sigma\,(\tilde{e}_k|1). \tag{37}$$

Assume now that $\lim_{\tau\to+\infty}\varphi(\tau) = -\frac{\sqrt{\pi}}{2}$. Then,

$$w_0(\tau) = -\frac{\pi^{1/4}}{2}\tau + O(1) \text{ and } w_k(\tau) = O(1)(\tilde{e}_k|1) \text{ when } k\geq 1.$$

These estimates imply:

$$w(\tau,\eta) = -\frac{\tau}{4}\eta + O(1),$$

and then, there holds, in the $L^2_\rho(\mathbb{R}_+)$ sense:

$$u_E(\tau,\eta) = e^{-\tau/2}\left(\frac{-\sqrt{\pi}}{2}\tau\eta - \frac{\tau\eta}{4} + O(1)\right),$$

in contradiction with the non-negativity of u_E. Therefore, $\overline{v} > -\sqrt{\pi}/2$. ■

An immediate consequence of this lemma is that there exists some $k>0$ such that $u_E(\tau,0)\geq ke^{-\tau/2}$, hence the inequality:

$$|\text{Log}\, u_E(\tau,0)| \leq \tau/2 + C \tag{38}$$

for some $C>0$, which is the main tool of the following lemma:

Lemma 3.2. *There exists $C>0$ such that:*

$$|v(\tau,0)| \leq C(1+\tau).$$

Proof. Let us expand u_E in $L^2_\rho(\mathbb{R})$:

$$u_E(\tau,\eta) = \sum_{k\geq 0} u_k(\tau)e_k(\eta).$$

Problem (32) can be rewritten under the form:

$$\left(\partial_\tau + \mathcal{L} + \frac{1}{2}\right) u_E = 2\delta_{x=0}\mathrm{Log}\ u_E(\tau,0),$$

which implies the following equations on the functions w_k:

$$\dot{u}_k + \left(\lambda_k + \frac{1}{2}\right) u_k = 2e^{-\tau/2}\mathrm{Log}\ u_E(\tau,0)e_k(0). \tag{39}$$

Then,

$$u_k(\tau) = u_k(0)e^{-(\lambda_k+1/2)\tau} + 2\int_0^\tau e^{-(\lambda_k+1/2)(\tau-\sigma)}e^{-\sigma/2}\mathrm{Log}\ u_E(\sigma,0)d\sigma\, e_k(0).$$

If $k>0$, by the inequality (38), there holds $u_k(\tau)=O(\tau)$. If $k=0$,

$$u_0(\tau) = u_0(0) + 2\int_0^\tau e^{-\sigma/2}\mathrm{Log}\ u_E(\sigma,0)d\sigma\, e_0(0) = O(1),$$

which implies the expected result. ■

Proof of Theorem 0.4. Let us set $h=v_\tau$, $\overline{v}=\varphi(+\infty)$, with $\varphi(\tau)=v(\tau,0)$. We are firstly going to prove that $\overline{v}=0$.

Case 1: $\overline{v}\neq 0$ and $\overline{v}\neq +\infty$.
Applying the same technique as in the proof of Lemma 3.1, based on the decomposition of w in $L^2_\rho(\mathbb{R}_+)$, we get:

$$w(\tau) = \frac{\overline{v}\tau\eta}{2\sqrt{\pi}} + O(1).$$

Hence,

$$w_\eta(\tau,0) = \frac{\overline{v}\tau}{2\sqrt{\pi}} + O(1) \tag{310}$$

for large times. On the other hand, the boundary condition in (33) implies:

$$\lim_{\tau\to+\infty} w_\eta(\tau,0) = -\mathrm{Log}\ \left(1+2\frac{\overline{v}}{\sqrt{\pi}}\right). \tag{311}$$

The signs of the quantities in (310) and (311) are different, hence a contradiction.

Case 2: $\overline{v} = +\infty$
Let us expand v in $L^2_\rho(\mathbb{R})$:

$$v(\tau,\eta) = \sum_{k\geq 0} v_k(\tau) e_k(\eta).$$

Because of the differential problem checked by v,

$$\begin{aligned} v_k(\tau) &= e^{-\lambda_k(\tau-\tau_0)} v_k(\tau_0) + 2\int_{\tau_0}^{\tau} e^{-\lambda_k(\tau-\sigma)} \mathrm{Log}\left(1 + \frac{2v(\tau,0)}{\sqrt{\pi}}\right) d\sigma\, e_k(0) \\ &\geq e^{-\lambda_k(\tau-\tau_0)/2} v_k(\tau_0) \end{aligned}$$

for some $\tau_0 > 0$ sufficiently large. Yet,

$$v_0(\tau_0) = (e_0|v(\tau_0)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2/4} v(\tau_0,\eta) d\eta.$$

Since $\overline{v} = +\infty$, the function $v_0(\tau_0)$ is positive for τ_0 large enough and v grows at least like $e^{\tau/2}$, contradicting Lemma 3.2.

Therefore $\overline{v} = 0$, and problem (33) can be rewritten as:

$$\begin{cases} v_\tau + \mathcal{L}v = 0, \ \eta > 0, \\ v_\eta(\tau,0) = -\dfrac{2v(\tau,0)}{\sqrt{\pi}} + O(v(\tau,0)^2). \end{cases}$$

This allows us to use a stability argument. Indeed, there does not exist any function ϕ, solution of

$$\mathcal{L}\phi = 0, \quad \phi'(0) = -\frac{2\phi(0)}{\sqrt{\pi}}.$$

Then, the spectrum of the linearized operator around the null solution does not contain 0, and we end the proof by applying a stable manifold argument in $L^2_\rho(\mathbb{R})$ - see [8] -, since the nonlinear terms are only localized at one point. ■

We have thus proved the asymptotic behaviour of the radius of the flame when it quenches in finite time. Let us now go on with studying the quenching in infinite time. In the remaining of this section, the function f will denote the solution of the equation:

$$f \mathrm{Log}\, f + Eq = 0. \tag{312}$$

The proof of Theorem 0.4 (ii) will be given when $\lambda = 0$: it is the same for any $\lambda > 0$.

Lemma 3.3. *Let us assume that q is positive and smooth on $\mathbb{R}_+^*$, and that the flame quenches. Then, there exists $t_0 > 0$ and $C_1, C_2 > 0$ such that $C_1 f(t) \leq u(t,0) \leq C_2 f(t)$ for all $x \in \mathbb{R}$, $t \geq t_0$.*

Proof. Let us pick $\tau_0 > 0$. The function f can be seen as the trace at $x = 0$ of the function v, where:

$$\begin{cases} v_t - v_{xx} = 2\delta_{x=0}\partial_{1/2} f, \\ v(\tau_0, .) = f(\tau_0). \end{cases}$$

There exists $C_2(\tau_0) > 1$ such that $u(\tau_0, .) \leq C_2 f(\tau_0)$ when τ_0 is sufficiently large. Moreover, let $w = u - v$. Then, there holds:

$$\begin{cases} w_t - w_{xx} = 2\delta_{x=0}\left(\text{Log }(v+w) + \dfrac{Eq}{v+w} - \partial_{1/2} f\right), \\ w(\tau_0, .) \leq 0. \end{cases} \tag{313}$$

Because of the smoothness of q and the fact that $f \to 0$, the quantity:

$$\text{Log } C_2 - \left(\frac{1}{C_2} - 1\right) \text{Log } f - \partial_{1/2} f$$

is negative for C_2 large enough, and 0 is a supersolution of (313). Therefore, $u(\tau, .) \leq C_2 f(\tau)$ for any $\tau \geq \tau_0$.

Moreover, the function

$$\gamma(\tau, \eta) = \max\left(C_1 f(\tau) - K\int_0^\eta e^{-x^2/4} dx, 0\right)$$

is a subsolution of (33) for any $K > 0$ and $C_1 > 0$; since there exists $K > 0$, $C_1 > 0$ such that $\gamma(\tau_0, \eta) \leq u(\tau_0, \eta)$, we have:

$$\gamma(\tau, \eta) \leq u(\tau, \eta) \text{ for any } \tau \geq \tau_0,\ \eta \in \mathbb{R},$$

which ends the proof of this lemma. ■

Proof of Theorem 0.4 (*ii*): Let $w(t) = \dfrac{u(t,x)}{f(t)} - 1$. Then,

$$w_t - w_{xx} + \frac{f_t}{f}(1+w) = \frac{2}{f(t)}\delta_{x=0}\left(\text{Log }(1+w) + \text{Log } f(t)\frac{w}{1+w}\right).$$

Let us notice that, because of the lemma above, there exists $C > 0$ such that $w \leq \overline{w}$, where:

$$\overline{w}_t - \overline{w}_{xx} + C\frac{f_t}{f} = 2\delta_{x=0}\overline{w},$$

which yields:

$$\partial_{1/2}\overline{w}(t,0) = C\overline{w}(t,0) + \frac{1}{\sqrt{\pi}}\int_{t_0}^t \frac{f_t(s)}{f(s)\sqrt{t-s}} ds. \tag{314}$$

For t_0 sufficiently large, the smoothness of the function q implies that

$$\overline{f}(t) = \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{f_t(s)}{f(s)\sqrt{t-s}} ds \to 0 \text{ as } t \to +\infty.$$

Applying the Laplace transform to (314), we get:

$$\overline{w}(t,0) = H * \overline{f}, \text{ where } H = O(\min(t^{-1/2}, t^{-3/2})),$$

so that $\overline{w}(t,0) \to 0$ as $t \to +\infty$.

In the same manner, we can see that $w \geq \underline{w}$, with $\underline{w}(t,0) \to 0$ as $t \to +\infty$, and $u(t,0) \sim f(t)$ as claimed. ∎

4. Asymptotic behaviour of the flame in case of stabilization

Let us begin this last section by the study of the case $\lambda < \lambda_{cr}$, for which the flame stabilizes to R_1 or R_2, depending on whether E is equal or above $E_{cr}(q)$. Since R_1 is an unstable fixed point, we cannot use as previously a stability argument, and the proof of Theorem 0.5 will be based upon the integro-differential equation (01). Let us just notice that this proof is also valid for the stabilization towards the stable fixed point R_2. In the following, we assume $\lambda < \lambda_{cr}$ and $R(t)$ converges to R_i $(i = 1,2)$ as $t \to +\infty$.

Lemma 4.1. *If there exists some $k \in \mathbb{R}$ such that $R(t) - R_i = kt^{-1/2} + o(t^{-1/2})$, then:*

$$k = \frac{R_i^2\sqrt{\pi}}{(1-2\lambda R_i^2)\sqrt{\pi}}.$$

Proof. We look formally for an equivalent of $R(t) - R_i$ under the form $kt^{-1/2}$. It has been assumed that there exists $\alpha > 1/2$ such that $q(t) = O(t^{-\alpha})$; and so, for large times,

$$I_{1/2}\left(\frac{Eq}{R}\right) \leq C\int_0^t \frac{ds}{s^\alpha\sqrt{t-s}} = C\frac{B(\alpha-1,1/2)}{t^{\alpha-1/2}} = o\left(\frac{1}{\sqrt{t}}\right).$$

On the other hand,

$$\begin{aligned} I_{1/2}(\text{Log } R - \lambda R^2) &\sim \int_0^t \frac{\text{Log }(R_i + ks^{-1/2}) - \lambda(R_i + ks^{-1/2})^2}{\sqrt{\pi(t-s)}} ds, \\ &\sim \int_0^t \frac{(k/R_i - 2\lambda R_i k)s^{-1/2}}{\sqrt{\pi(t-s)}} ds, \\ &\sim \frac{k(1-2\lambda R_i^2)}{R_i}\sqrt{\pi}, \end{aligned}$$

hence the desired result. ∎

Lemma 4.2. *There exists some $k \in \mathbb{R}$ such that*

$$R(t) - R_i = kt^{-1/2} + o(t^{-1/2}).$$

Proof. Let us pick $t_0 > 0$. Then, for any $t \geq t_0$,

$$R(t) = \mathcal{T}(t-t_0)u(t_0)(0) + \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\text{Log } R(s) + Eq(s)/R(s) - \lambda R(s)^2}{\sqrt{t-s}} ds, \tag{41}$$

where $\mathcal{T}$ is the heat semi-group. This comes from the diffusive formulation (12). Let us set $\rho(t) = R(t) - R_i$. In the sequel, we are going to estimate ρ, using the integro-differential equation (41). Since $R(t) \to R_i$ as $t \to +\infty$ and

$$\text{Log } R - \lambda R^2 = \chi(R - R_i) + f(R - R_i) \text{ as } R \to R_i,$$

with $\chi = \dfrac{1}{R_i} - 2\lambda R_i$ and $f(R - R_i) = O(R - R_i)^2$, (41) implies:

$$\rho(t) = \mathcal{T}(t-t_0)u(t_0)(0) + \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{1}{\sqrt{t-a}} \left(\chi\rho + f(\rho) + \frac{Eq}{R} \right) ds.$$

From now on, we will write for any function f:

$$I_{1/2} f = \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{f(s)}{\sqrt{t-s}} ds,$$

even if it is not the real Abel integrator. Let $\gamma \in \mathcal{C}_0^\infty(\mathbb{R}_+^*)$ such that $\gamma(t_0) = \rho(t_0)$. Setting $\rho := \rho - \gamma$ and $g(t, \rho) = f(\rho - \gamma) - f(-\gamma)$, there holds:

$$\rho = \chi I_{1/2}\rho + I_{1/2}(g(\rho)) + I_{1/2}\left(\frac{Eq}{R} + f(-\gamma) - \chi\gamma \right) + \gamma - R_i + \mathcal{T}(t-t_0)u(t_0)(0).$$

Let us define:

$$\phi = \mathcal{T}(t-t_0)u(t_0)(0) + I_{1/2}\left(\frac{Eq}{R} + f(-\gamma) - \chi\gamma \right) + \gamma - R_i.$$

Then,

$$\rho = \chi I_{1/2}\rho + I_{1/2}(g(\rho)) + \phi. \tag{42}$$

Since $\rho(t_0) = 0$ and $g(\rho(t_0)) = 0$, taking the half derivative and the derivative of (42), then adding the two equations, we get:

$$\partial_{1/2}\rho = \chi\rho + g(\rho) + \partial_{1/2}\phi, \tag{43}$$

$$\dot{\rho} = \chi\partial_{1/2}\rho + \partial_{1/2}g(\rho) + \phi', \tag{44}$$

$$\dot{\rho} = \chi^2\rho + \partial_{1/2}g(\rho) + \chi g(\rho) + \phi' + \chi\partial_{1/2}\phi. \tag{45}$$

Yet, there exists $k_0 \geq 0$ such that:

$$\mathcal{T}(t-t_0)u(t_0)(0) = k_0 t^{-1/2} + o(t^{-1/2}) \text{ and } \frac{d}{dt}\mathcal{T}(t-t_0)u(t_0)(0) = -\frac{k_0}{2}t^{-3/2} + o(t^{-3/2}).$$

Moreover, since γ is compactly supported and because of the assumption on the asymptotic behaviour of q, there exists $k_1, k_2 \in \mathbb{R}$ such that:

$$\partial_{1/2}\phi = k_1 t^{-1/2} + o(t^{-1/2}) \text{ and } \phi' = k_2 t^{-1/2} + o(t^{-1/2}).$$

Let us look at the quantities $g(\rho)$ and $\partial_{1/2}g(\rho)$. On the one hand, we have $g(\rho) = O(\rho^2)$ for large times, so that there exists $\varepsilon_1 > 0$ such that $|g(\rho)| \leq \varepsilon_1|\rho|$ for t_0 large enough. On the other hand,

$$\partial_{1/2}g(\rho) = \frac{1}{\sqrt{\pi}}\int_{t_o}^{t} \frac{\dot{\rho}g'(t,\rho)}{\sqrt{t-s}}ds.$$

Parabolic regularity - see [10] - applied to (42) implies that $\dot{\rho}(t) \to 0$ as $t \to +\infty$. Moreover, $g'(\rho) = O(\rho)$ for t large enough, and therefore, there exists $\varepsilon_2, \varepsilon_3 > 0$ such that:

$$\begin{aligned} |\partial_{1/2}g(\rho)| &\leq \varepsilon_2|\partial_{1/2}\rho| \\ &\leq \varepsilon_3|\rho| + k_3 t^{-1/2} + o(t^{-1/2}) \end{aligned}$$

by (43). Finally, there exists $\varepsilon > 0$ and $k \in \mathbb{R}$ such that:

$$|\dot{\rho} - (\chi^2 - \varepsilon)\rho| \leq (k + \varepsilon)t^{-1/2},$$

hence the expected result. ■

5. Generalizations

In this part, we are dealing with a generalization of the study we have done just above. We are going to treat parabolic equations of the following type:

$$\begin{cases} u_t - u_{xx} = \delta_{x=0}\left(f(u) + \dfrac{Eq(t)}{u}\right), & x \in \mathbb{R}, \\ u(0,.) = 0, \end{cases} \tag{51}$$

where f is polynomial, and negative at $x = 0$. We recover on such an equation a similar threshold phenomenon to the one described just above with the example of the spherical flame. In particular, there exist critical energies, and the dynamics of equation (51) will depend on the different positive zeroes of f. Let us denote them by $x_1 < \cdots < x_p$.

We can prove easily that $x_1, x_3, \ldots, x_{2k+1}, \ldots$ are unstable solutions, and that the critical energies are associated to these zeroes: if $E = E_{cr}^k$, then, $\lim_{t\to+\infty} u(t,0) = x_{2k+1}$. If $E < E_{cr}^1$, the solution will quench - in finite time if q is compactly supported - in infinite time otherwise. If $E_{cr}^{k-1} < E < E_{cr}^k$, then $u(t,.)$ will converge to the stable solution x_{2k}. Indeed, a mere adaptation of Proposition 1.9 yields the existence of a small closed interval of energies such that $u(t,0)$ converges towards the zero x_{2k}; the remaining of the proof is analogous.

From now on, let us restrict our attention to the case $E > E_{cr}^{[p/2]}$. The asymptotic behaviour depends now on the number of zeroes of f: if this one is even ($p = 2q$), as in the case treated in this paper, the largest zero is a stable solution; and if $E > E_{cr}^q$, then $\lim_{t\to+\infty} u(t,0) = x_{2q}$. More interesting is the case where the number of positive zeroes of f is odd: in [2] is studied equation (51) with $f(u) = \operatorname{Log} u$; it is not a polynomial function as the one we are dealing with in this part, but it gives an idea of what can happen, namely the propagation of the solution: $u(t,0) \to +\infty$ as t tends to $+\infty$. Nevertheless, (51) has been studied in [11] and [14]; they proved that blow-up occurs when the growth of f is at least quadratic. When f has linear growth, we get the same behaviour as in [2], namely we have global existence and $u(t,0) \to +\infty$ as $t \to +\infty$.

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