Algorithms for the computation of moduli spaces for semiquasihomogeneous singularities

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ABSTRACT

We present algorithms and their implementation in the computer algebra system SINGULAR 2.0 for the computation of equations for moduli spaces for semiquasihomogeneous singularities w.r.t. right equivalence. In addition, we describe the structure of the stabilizer group of Brieskorn-Pham singularities.

1. Introduction

One of the important achievements of singularity theory was the classification of some "generic" classes of hypersurface singularities via normal forms by V.I. Arnold, cf. Ch.15 in [2]. For more complicated classes of singularities the classification by normal forms seems to be impossible. In 1997 Greuel, Hertling, and Pfister came up with a geometric classification of semiquasihomogeneous hypersurface singularities with fixed part w.r.t. right- and contact equivalence by geometric methods, i.e., by providing a construction of moduli spaces for such singularities with some invariants fixed, cf. [9]. The moduli spaces w.r.t. right equivalence (and fixed principal part) turn out to be quotients of affine varieties by finite groups, which are again algebraic varieties. We describe algorithms and their implementation in the computer-algebra system SINGULAR 2.0 (cf. [10]) for computing equations of these moduli spaces given the principal part. In addition, we provide a structure theorem for the isometry groups of

Brieskorn-Pham polynomials and its application to the computation of moduli spaces (cf. Ch. 4 in [3]).

The objects to classify are singularities defined by semiquasihomogeneous power series with principal part f_0 w.r.t. right equivalence, where f_0 is a quasihomogeneous polynomial which has an isolated singularity at 0. Right equivalence leads to an algebraic group action of the isometry group of the polynomial f_0 (leaving the polynomial fixed) on the vector space spanned by the upper monomials of the Milnor algebra of f_0. The quotient $\mathcal{M}_{f_0}$ of the linearization of this action, which is an affine variety, is the desired coarse moduli space for semiquasihomogeneous singularities with principal part f_0 w.r.t. right equivalence.

We present the algorithms ARNOLDACTION, LINEARIZEACTION and MODEQN which are the essential parts of the SINGULAR 2.0 libraries `rinvar.lib` and `qhmoduli.lib`. Together with the `zeroset.lib` they provide constructive means to compute equations for $\mathcal{M}_{f_0}$ given the principal part f_0. The design and implementation of these three libraries is part of the author's diploma thesis, cf. [3].

Acknowledgments

2. Basic notation

In this section we provide necessary definitions from singularity theory and introduce unfoldings of power series. We do not introduce moduli spaces [1] since the existence of a moduli space is proved in [9] and technical details of the proof are not important for the computation of the equations.

In the sequel we denote the ring of convergent power series over $\mathbf{C}$ by $\mathbf{C}\{x_1, x_2, \ldots, x_n\}$. Let $0 \neq f \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ be s.t. $f(0) = 0$. The power series f has a **singularity** at 0 if $\frac{\partial f}{\partial x_1}(0) = \frac{\partial f}{\partial x_2}(0) = \ldots = \frac{\partial f}{\partial x_n}(0) = 0$. The **Jacobi ideal** of f is $j(f) := \left\langle \frac{\partial f}{\partial x_i} : 1 \leq i \leq n \right\rangle$ and the **Milnor algebra** of f is defined by $M_f := \mathbf{C}\{x_1, x_2, \ldots, x_n\}/j(f)$. The **Milnor number** μ_f of f is $\mu_f := \dim_{\mathbf{C}} M_f$. If there exists a neighborhood $U \subset \mathbf{C}^n$ of 0 s.t. 0 is the only singular point of f in U then 0 is called an **isolated singularity** of f. Note that the Milnor number gives an important characterization of isolated singularities, namely $\mu_f < \infty$ iff 0 is an isolated singularity of f.

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[1] We refer to Ch. 1 of [12] for a description or to, e.g., [9] for a definition.

In the sequel we fix a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbf{N}_+^n$. A monomial $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ is **quasihomogeneous of type** $(d; \mathbf{w})$ if $\deg_{\mathbf{w}}(\mathbf{x}^\alpha) := |\mathbf{x}^\alpha|_{\mathbf{w}} := \sum_{i=1}^n w_i \alpha_i$. A polynomial f is **quasihomogeneous of type** $(d; \mathbf{w})$ if all of its monomials are quasihomogeneous of type $(d; \mathbf{w})$. For a power series $0 \neq f \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ we define $ord_{\mathbf{w}}(f) = \min\{|\mathbf{x}^\alpha|_{\mathbf{w}} : \mathbf{x}^\alpha\}$ and $ord_{\mathbf{w}}(0) := -1$. The power series f is called **semiquasihomogeneous of type** $(d; \mathbf{w})$ **with principal part** f_0 if f_0 is a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0 and $f = f_0$ or $ord_{\mathbf{w}}(f - f_0) > d$.

By $Aut(\mathbf{C}\{x_1, x_2, \ldots, x_n\})$ we denote the group of local $\mathbf{C}$−algebra automorphisms. Two power series $f, g \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ are **right equivalent**, denoted by $f \sim_r g$, if there exists a $\varphi \in Aut(\mathbf{C}\{x_1, x_2, \ldots, x_n\})$ s.t. $f = \varphi(g)$. The power series $f \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ is k−**determined** w.r.t. $\sim_r$ if for $g \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ we have $f^{(k)} = g^{(k)} \leftrightarrow f \sim_r g$ where $f^{(k)}, g^{(k)}$ denote the k−jet (f, g truncated at degree k) respectively.

Theorem 1. *Let* $f \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ *be a power series with an isolated singularity at 0. Then* f *is* $(\mu_f + 1)$−*determined.*

Proof. We refer to Section 6.3 of [2] (p. 121ff). ∎

For the construction of a moduli space we need the notion of families of unfoldings of negative weight. Let $f \in \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ be semiquasihomogeneous with principal part f_0. The power series f can be considered as a function germ $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ which can be deformed as follows. An **unfolding** of f over a germ (T, t) is a map $F : (\mathbf{C}^n, 0) \times (T, t) \longrightarrow (\mathbf{C}, 0)$ s.t. the following diagram commutes,

$$
\begin{array}{ccc}
(\mathbf{C}^n, 0) & \hookrightarrow & (\mathbf{C}^n, 0) \times (T, t) \\
\downarrow f & & \downarrow \phi \\
(\mathbf{C}, 0) & \hookrightarrow & (\mathbf{C}, 0) \times (T, t) \\
\downarrow & & \downarrow \\
0 & \hookrightarrow & (T, t)
\end{array}
$$

where $\phi(x, t) = (F(x, t), t)$, and $F(x, t) = f(x) + g(x, t)$ for some $g \in \mathbf{C}^n\{\mathbf{x}, \mathbf{t}\}$ with $g(x, 0) = 0$. Two unfoldings F, F' of f are **right equivalent** if there exists an isomorphism $\Psi : (\mathbf{C}^n, 0) \times (T, 0) \to (\mathbf{C}^n, 0) \times (T, 0), \Psi(x, t) = (\psi(x, t), t)$ s.t. $F(\psi(x, t), t) = F'(x, t)$. A morphism $\varphi : (S, s) \to (T, t)$ induces an unfolding $\varphi^* F : (\mathbf{C}^n, 0) \times (S, s) \to (\mathbf{C}, 0)$ of f by $\varphi^* F(x, s') = F(x, \varphi(s'))$ for $s' \in (S, s)$ (**base change**). The unfolding F is **semiuniversal** if for each unfolding G of f over some base space (S, s) there exist a base space (S, s) and a morphism $\varphi : (S, s) \to (T, t)$ s.t. the unfolding G is right equivalent to the induced unfolding $\varphi^* F$ and the differential $d\varphi$ is unique. F is called an **unfolding of negative weight** over $(T, 0)$ if $F \sim_r f'(x) + g(x, t)$ for some semiquasihomogeneous power series f' with $g(x, 0) = 0$ and $\deg_x(g) > d$. If the germ (T, t) is replaced by a base space T then the map $F : (\mathbf{C}^n, 0) \times T \longrightarrow (\mathbf{C}, 0)$ is called a **family of unfoldings of negative weight over** S. Hence $\phi(x, t) = (F(x, t), t) = (F_t(x), t)$ and for each $t \in T$ the germ

$F : (\mathbf{C}^n, 0) \times (T, t) \longrightarrow (\mathbf{C}, 0)$ is an unfolding of negative weight of F_t over (T, t). If, in addition, for any $t \in T, F_t : (\mathbf{C}^n, 0) \times (T, t) \longrightarrow (\mathbf{C}, 0)$ is right equivalent to a semiquasihomogeneous power series with principal part f_0 then F is called a **family of unfoldings of negative weight with principal part f_0** or a **family of f_0^- −unfoldings** for short. For families of unfoldings right equivalence, base change and semiuniversality are defined as for unfoldings.

For B a monomial basis of the Milnor algebra of f_0 ($|B| < \infty$ since $\mu_{f_0} < \infty$) we define the set of **upper monomials** $B_- = \{m_1, m_2, \ldots, m_k\} = \{m \in B : \deg_{\mathbf{w}}(m) > d\}$ of M_{f_0} and the space $T_- = \mathbf{C}^k$. Given f_0, we define the family of f_0^- − unfoldings

$$F \quad : \quad \mathbf{C}^n \times T_- \longrightarrow \mathbf{C}^n,$$

$$F(x, t) \quad = \quad f_0(x) + \sum_{i=1}^{k} t_i m_i$$

and call it the **semiuniversal family of unfoldings of negative weight of semiquasihomogeneous power series with principal part f_0**. This name is justified by Proposition 3.4.5 and Theorem 3.2.3 below.

Example 1. The polynomial $f_0 = x^2 y + x^2 z + y^5 - z^5$ is semiquasihomogeneous of type $(5; 2, 1, 1)$. The Milnor number equals $\mu_{f_0} = 24$, hence f_0 has an isolated singularity at 0. Upper monomials are given by $B_- = \{y^3 z^3, x^2 y^3, x^2 y^2\}$ and the family of f_0^- −unfoldings is

$$F(x, y, z, t_1, t_2, t_3) = x^2 y + x^2 z + y^5 - z^5 + t_1 y^3 z^3 + t_2 x^2 y^2 + t_3 x^2 y^3.$$

3. Existence of a coarse moduli space

We briefly describe the construction of a coarse moduli space for semiquasihomogeneous power series with principal part f_0 given by Greuel, Hertling and Pfister in [9]. The building blocks are a Theorem of Greuel, Hertling and Pfister on the order of quasihomogeneous maps and a Theorem of Arnold on semiquasihomogeneous power series, which states that each semiquasihomogeneous power series with principal part f_0 is right equivalent to some F_t for $t \in T_-$. By [9] we only need to consider isomorphisms of order ≥ 0, hence we can construct, by comparing coefficients, the isometry group of f_0 and its induced action on T_-, the base space of the semiuniversal unfolding F, which turns out to be a finite group action. [9] proved that the quotient of this action is a coarse moduli space for semiquasihomogeneous power series with principal part f_0 w.r.t. right equivalence.

We only provide those proofs, which are important for the algorithms of Section 4 or which are not given in [9].

3.1. The order of maps

Let $R = \mathbf{C}\{x_1, x_2, \ldots, x_n\}$ be the ring of convergent power series, let $Aut(R)$ be the group of local $\mathbf{C}-$algebra automorphisms of R and let $\mathbf{w} = (w_1, w_2, ..., w_n) \in \mathbf{N}^n$ be weights. By R_d we denote the ideal generated by all power series of weighted order greater or equal to d.

Definition 1. *(a) Let $id \neq \varphi : R \to R$ be a ring homomorphism. The **weighted order** of φ, denoted by $ord_{\mathbf{w}}(\varphi)$, is the maximal integer d s.t. $ord_{\mathbf{w}}(\varphi(x_i) - x_i) \geq w_i + d$ for $1 \leq i \leq n$.*
(b) For $d \geq 0$ we define $Aut_{\geq d}(R) := \{\varphi \in Aut(R) : ord_{\mathbf{w}}(\varphi) \geq d\} \cup \{id\}$ and $Aut_{>d}(R) := Aut_{\geq d+1}(R)$.

If f is semiquasihomogeneous of type $(d; \mathbf{w})$ and φ is an automorphism of $\mathbf{C}\{x_1, x_2, \ldots, x_n\}$ of order 0 then $\varphi(f)$ is semiquasihomogeneous of type $(d; \mathbf{w})$. The converse, due to Greuel, Hertling and Pfister (cf. Theorem 2.1 in [9]), is by no means trivial and plays an important role for the construction of the moduli space.

Theorem 2. *[9] Let f and g be semiquasihomogeneous power series of type $(d; \mathbf{w})$ and let $\varphi \in Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$. Then*

$$f = \varphi(g) \Longrightarrow ord_{\mathbf{w}}(\varphi) \geq 0.$$

Proof. We refer to [9], Theorem 2.1. ∎

Hence it suffices to consider only automorphisms $\varphi \in Aut(R)$ which satisfy $ord_{\mathbf{w}}(\varphi) \geq 0$. For the subgroups $Aut_{\geq k}$ we have the following result.

Proposition 1. *For $q > p \geq 0$ $Aut_{\geq q}(R) \subset Aut_{\geq p}(R)$ is a normal subgroup.*

Proof. We refer to Proposition 12.4.2 in [2] (p. 203). ∎

Note that if f is semiquasihomogeneous with principal part f_0 then right equivalence need not preserve the principal part.

3.2. Arnold's Theorem on semiquasihomogeneous power series

In this section we present a result of V.I. Arnold, cf. [1] (or Ch. 12.6, in [2], p.209), which says that any semiquasihomogeneous power series with principal part f_0 is right equivalent to $f_0 + \sum_{\alpha \in B_-} c_\alpha \mathbf{x}^\alpha$ for some complex numbers c_α. Unfortunately, they are not unique in general.

The proof of Arnold's Theorem provides a construction of a morphism of weighted order > 0 which maps a given semiquasihomogeneous power series (with terms of arbitrarily high degree) with principal part f_0 to some $F(\mathbf{x}, \mathbf{c}), \mathbf{c} \in T_-$, where F is the semiuniversal family of $f_0^- -$ unfoldings. This construction is of fundamental

importance for the existence of an induced action of the stabilizer $G^{\mathbf{w}}_{f_0}$ of f_0 on T_-. We follow the presentation in Ch. 12.5 and 12.6 of [2] (p. 206ff).

In the sequel let $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0. The following Lemma will be needed to prove Arnold's Theorem.

Lemma 1. *(a) Let f be a power series of weighted order d and φ be an automorphism of the form $\varphi(\mathbf{x}) = \mathbf{x} - \mathbf{v}$ of order $\delta > 0$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Then*

$$\varphi(f) = f - \sum \frac{\partial f}{\partial x_i} v_i + r$$

for some r s.t. $\mathrm{ord}_{\mathbf{w}}(r) > d + \delta$.
(b) Any semiquasihomogeneous power series $f_0 + f_1$ with fixed part f_0 and $\mathrm{ord}_{\mathbf{w}}(f_1) > d$ and $d' > d$ can be transformed by a polynomial diffeomorphism of order > 0 to $f_0 + f_1'$ s.t. the terms of degree less than d' of f_1 and f_1' coincide, and the terms of f_1' of degree d' reduce to $\sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} = d'} c_\alpha \mathbf{x}^\alpha$ for some $c_\alpha \in \mathbf{C}$.

Proof. (a) We refer to the Lemma in Ch. 12.5 in [2] (p.207).
(b) Let g_1 denote the sum of all terms of degree d' in f_1, let g_1' denote the terms of f_1 of degree $< d'$ and set $g_1'' = f_1 - g_1 - g_1'$. Since $\{\mathbf{x}^\alpha : \alpha \in B_-, |\alpha|_{\mathbf{w}} = d'\}$ is a basis of monomials of M_{f_0} of degree d' we can write g_1 as

$$g_1 = \sum_{i=1}^n \frac{\partial f_0}{\partial x_i} v_i + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} = d'} c_\alpha \mathbf{x}^\alpha$$

for some $c_\alpha \in \mathbf{C}$. Since g_1 is quasihomogeneous, $v_1, v_2, \ldots, v_n$ can be chosen to be quasihomogeneous of degree $d' - d + w_1, d' - d + w_2, \ldots, d' - d + w_n$ respectively. Consider the morphism φ defined by $\varphi(x_i) = x_i - v_i$. We have $\mathrm{ord}_{\mathbf{w}}(\varphi(x_i) - x_i) = \mathrm{ord}_{\mathbf{w}}(v_i) = w_i + d' - d$, so the order of φ equals $d' - d > 0$. By applying (a) to $f = f_0 + f_1$ we obtain

$$\varphi(f) = f_0 \quad - \quad \sum_{i=1}^n \frac{\partial f_0}{\partial x_i} v_i + r_0 + f_1 - \sum_{i=1}^n \frac{\partial f_1}{\partial x_i} v_i' + r_1$$

$$= f_0 \quad + \quad \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} = d'} c_\alpha \mathbf{x}^\alpha + g_1'$$

$$+ \quad \left(r_0 + g_1'' - \sum_{i=1}^n \frac{\partial f_1}{\partial x_i} v_i + r_1 \right)$$

where the (weighted) order of the terms in the bracket is greater than d'. ■

Theorem 3. *[1] Let f be a semiquasihomogeneous power series with principal part f_0. Then*

$$f \sim_r f_0 + \sum_{\alpha \in B_-} c_\alpha \cdot \mathbf{x}^\alpha.$$

for some $c_\alpha \in \mathbf{C}$.

Proof. Let $d' > d$ be the minimal degree of a term in f not appearing in f_0 and set $d'' := \max\{|\alpha|_{\mathbf{w}} : \alpha \in B_-\}$. We use Lemma 3.2.1(a) to obtain a diffeomorphism φ_1 s.t. $ord_{\mathbf{w}}(\varphi_1) \geq d' - d$ and $\varphi_1(f) = f_0 + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} < d'+1} c_\alpha \mathbf{x}^\alpha + f_1$ where $ord_w(f_1) > d'$. We successively apply Lemma 3.2.1(b) and obtain in the $k-$th step

$$\varphi_k \circ \varphi_{k-1} \circ \ldots \circ \varphi_1(f) = f_0 + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} < d'+k} c_\alpha \mathbf{x}^\alpha + f_k$$

where $ord_{\mathbf{w}}(f_k) \geq d' + k$. Note that φ_k does not change the coefficients c_α where $|\alpha|_{\mathbf{w}} < d' + k$. If $d' + k > d''$ then no new terms of the form $c_\alpha \mathbf{x}^\alpha$, $\alpha \in B_-$, are introduced. Since f has an isolated singularity at the origin we have $\mu(f) < \infty$ by Theorem 2.1. Note that the Milnor numbers of f and f_0 coincide, hence there exists $N > 0$ s.t. $ord(f_N) > \mu(f_0)$ (not the weighted order) and therefore

$$f_0 + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}} < d'+k} c_\alpha \mathbf{x}^\alpha \sim_r \varphi_N \circ \varphi_{N-1} \circ \ldots \circ \varphi_1(f) \sim_r f$$

via some map $\varphi \in Aut(\mathbf{C}\{\mathbf{x}\})$.   ■

    In the next section, Arnold's Theorem will be used to construct a group action on T_-, where the order of the morphism, constructed in the proof above, is of fundamental importance.

Remark 1. (a) Arnold's Theorem provides a correspondence between semiquasihomogeneous power series with principal part f_0 and points of T_-. Unfortunately, this correspondence is not unique which can be seen, e.g., by applying the $\mathbf{C}^*-$action (which is of order 0) of f_0 to F. For maps of order > 0 the correspondence is unique, as shown in Proposition 2.3 in [9].
(b) By Proposition 2.3 of [9], the map φ constructed in the proof of Arnold's Theorem has order > 0.
(c) In the proof of Arnold's Theorem it suffices to compute the morphisms φ_k for $1 \leq k \leq d'' - d' + 1$ because we are only interested in the coefficients c_α and

$$\varphi_{d''-d'+1} \circ \ldots \circ \varphi_1(f) = f_0 + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}}} c_\alpha \mathbf{x}^\alpha + f_{d''-d'+1}$$

and $ord_{\mathbf{w}}(f_{d''-d'+1}) > d''$. The application of φ_k for $k > d'' - d' + 1$ increases the weighted order of $f_{k'-1}$ and does not change $f_0 + \sum_{\alpha \in B_-, |\alpha|_{\mathbf{w}}} c_\alpha \mathbf{x}^\alpha$.

Example 2. For $f_0 = x^2 z + x^2 y - z^5 + y^5$ we have to apply Arnold's Theorem to $f = f_0 + x^2 z^3$ because $x^2 z^3$ is no upper monomial, i.e., $f(\mathbf{x}) \neq F(\mathbf{x}, t_0)$ for all t_0. From $x^2 z^3 = \frac{1}{2}(xz^2 - xyz + xy^2)(\frac{\partial f_0}{\partial x}) - x^2 y^3$ we obtain the map $\varphi(x) =$

$x - \frac{1}{2}(xz^2 - xyz + xy^2), \varphi(y) = y, \varphi(z) = z.$ Application yields

$$
\begin{aligned}
\varphi(f) = \quad & f_0 - x^2 y^3 \\
& -\frac{3}{4} x^2 z^5 + \frac{3}{4} x^2 y z^4 - \frac{3}{4} x^2 y^2 z^3 + \frac{1}{4} x^2 y^3 z^2 - \frac{1}{4} x^2 y^4 z + \frac{1}{4} x^2 y^5 \\
& \frac{1}{4} x^2 z^7 - \frac{1}{2} x^2 y z^6 + \frac{3}{4} x^2 y^2 z^5 - \frac{1}{2} x^2 y^3 z^4 + \frac{1}{4} x^2 y^4 z^3
\end{aligned}
$$

Since the terms of the last two lines have degree $> 7 = \max\{\deg_{\mathbf{w}}(m) : m \in B_-\}$ we obtain $f \sim_r f_0 - x^2 y^3$ by Theorem 3.2.3 and Remark 3.2.1(c).

3.3. The induced group action

Any morphism $\sigma \in Aut(\mathbf{C}\{\mathbf{x}\})$ which leaves f_0 invariant induces a map $\Theta(\sigma) : T_- \to T_-$ by Arnold's Theorem. [9] proved that Θ induces an algebraic group action on T_- when applied to F and that morphisms of order > 0 preserve uniqueness (cf. the proposition below). For the construction of the moduli space we may therefore restrict ourselves to the factor group $Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ which is isomorphic to the group of quasihomogeneous morphisms of $\mathbf{C}\{\mathbf{x}\}$. Additionally, [9] showed that the group action induced by Θ on T_- is finite.

Proposition 2. *[9] For any semiquasihomogeneous power series f with principal part f_0 there exists an automorphism $\psi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ and a $t_0 \in T_-$ s.t. $\psi(f) = F_{t_0}$. If $\psi' \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ and $\psi'(F_{t_0}) = F_{t_0'}$ for some $t_0' \in T_-$ then $t_0 = t_0'$.*

Proof. We refer to Proposition 2.3 in [9]. ■

Hence only a morphism of order 0 may send a semiquasihomogeneous power series with principal part f_0 to a different point of T_- (e.g., the $\mathbf{C}^*-$ action of f_0). Note that morphisms of Arnold's Theorem have order > 0. Therefore we are interested in the following group.

Definition 2. *A morphism $\sigma \in Aut(\mathbf{C}\{\mathbf{x}\})$ is called* **quasihomogeneous** *(w.r.t. $\mathbf{w}$) iff σ maps quasihomogeneous elements to quasihomogeneous elements of the same degree. Set $G^{\mathbf{w}} := \{\sigma \in Aut(\mathbf{C}\{\mathbf{x}\}) : \sigma$ is quasihomogeneous w.r.t.$\mathbf{w}\}$.*

If σ is quasihomogeneous then $\sigma(x_i)$ is quasihomogeneous of degree w_i, i.e., the components σ_i are quasihomogeneous polynomials. By considering the natural inclusion $i : G^{\mathbf{w}} \hookrightarrow Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})$ and the projection $\pi : Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\}) \to Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ one obtains the following result.

Proposition 3. *The set $G^{\mathbf{w}}$ is an algebraic group, in particular,*

$$Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\}) \cong G^{\mathbf{w}}.$$

Proof. We refer to Proposition 2 in Ch. 12.4 (p. 203) in [2]. ∎

Definition 3. *The stabilizer group of f_0 is $G^{\mathbf{w}}_{f_0} := \{\sigma \in G^{\mathbf{w}} : \sigma(f_0) = f_0\}$.*

Note that $G^{\mathbf{w}}_{f_0}$ is an algebraic group since it is a closed subgroup of $G^{\mathbf{w}}$. Combining Theorem 3.1.2., Theorem 3.2.3 and the results above one can define a group action of $G^{\mathbf{w}}_{f_0}$ on T_- as follows.

Proposition 4. *[9] (a) For any $\sigma \in G^{\mathbf{w}}_{f_0}$ and $t \in T_-$ there exists a unique $s = \Theta(\sigma)(t)$ and an automorphism $\varphi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ s.t. $\varphi \circ \sigma(F_t) = F_s$.*
(b) The function $\Theta : G^{\mathbf{w}}_{f_0} \to Aut(T_-)$ is a group homomorphism.
(c) The components of $\Theta(\sigma)(t_i)$ are quasihomogeneous polynomials of degree $\deg_{\mathbf{w}}(m_i) - d$. In particular, the group $\Theta(G^{\mathbf{w}}_{f_0})$ acts algebraically on T_- and this action commutes with a $\mathbf{C}^*-$action on T_-.

Proof. We refer to Proposition 2.4 of [9] ∎

The induced group action of $G^{\mathbf{w}}_{f_0}$ on $T_- = \mathbf{C}^3$, where $f_0 = x^2 z + x^2 y - z^5 + y^5$, is given in Example 4.1.4. For the construction of the moduli space we are interested in the image of the stabilizer $G^{\mathbf{w}}_{f_0}$ under Θ.

Definition 4. *The group $E_{f_0} := \Theta(G^{\mathbf{w}}_{f_0})$ is the subgroup of $Aut(T_-)$ which is induced by the action of $G^{\mathbf{w}}_{f_0}$ on T_- provided by Arnold's Theorem.*

[9] proved that the quotient T_-/E_{f_0} is a coarse moduli space for semiquasihomogeneous power series with fixed principal part.

Remark 2. (a) Unfortunately (for computational purposes) the map φ of Proposition 3.3.2 cannot be omitted.
(b) It follows from (a) that, in general, E_{f_0} does not act linearly on T_-, even if $G^{\mathbf{w}}_{f_0} \subset GL_n(\mathbf{C})$. In order to compute the equations of T_-/E_{f_0} one has to linearize the action of E_{f_0} on T_- which is the main topic in Section 4.4.2.

The construction of the quotient T_-/E_{f_0} is considerably simplified by the fact that E_{f_0} is finite, hence reductive, so Hilbert's Finiteness Theorem implies that the quotient is an algebraic variety.

Theorem 4. *[9] Let $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0 and set $\overline{w}_i := \frac{w_i}{d}$.*
(a) The group $G^{\mathbf{w}}_{f_0}$ is finite if $\overline{w}_1 < \frac{1}{2}, ..., \overline{w}_{n-1} < \frac{1}{2}, \overline{w}_n \leq \frac{1}{2}$.
(b) The group E_{f_0} is finite.

Proof. We refer to Proposition 2.7 in [9], or for a more detailed treatment, to Theorem 2.3.56 in [3] (p. 36ff). ∎

Remark 3. Under the above hypotheses it is tempting, but wrong, to conclude that $G^{\mathbf{w}}_{f_0} \subset GL_n(\mathbf{C})$ or that $\oplus_{i=1}^{n} \mathbf{C}\frac{\partial f_0}{\partial x_i}$ is a $G^{\mathbf{w}}_{f_0}-$module.[2] Consider the quasihomogeneous

---

[2]For $G \subseteq G^{\mathbf{w}}_{f_0} \cap GL_n(\mathbf{C})$ Orlik and Wagreich proved that it is a $G-$module, cf. Lemma 5.3 or [15].

polynomial $f_0 = x^2 - xy^2 + y^4$ of type $(4; 2, 1)$ which has an isolated singularity at 0 $(\mu_{f_0}(0) = 3)$. Its stabilizer group $G_{f_0}^{\mathbf{w}}$ has order 8 and is generated by the two automorphisms σ, τ, where

$$\begin{aligned} \sigma(x) &= -x + y^2, \\ \sigma(y) &= y, \end{aligned}$$

and $\tau(x) = -x, \tau(y) = iy$. It is obvious that $G_{f_0}^{\mathbf{w}} \not\subseteq GL_2(\mathbf{C})$. Moreover, $\mathbf{C}\frac{\partial f_0}{\partial x} \oplus \mathbf{C}\frac{\partial f_0}{\partial y}$ is not a $G_{f_0}^{\mathbf{w}}$−module since $\sigma^{-1} \bullet \frac{\partial f_0}{\partial y} = 2y\frac{\partial f_0}{\partial x} + \frac{\partial f_0}{\partial y}$. Nonetheless, this is true for Brieskorn-Pham polynomials as we will show in Section 5.

3.4. Existence of the moduli space

We present the construction of coarse moduli spaces for semiquasihomogeneous singularities with principal part f_0 w.r.t. right equivalence given by Greuel, Hertling and Pfister (cf. Section 1 in [9]), whereas we do not introduce the notion of a moduli functor or moduli space. We just mention that the points of a moduli space are in $1:1$ correspondence with the isomorphism classes of semiquasihomogeneous singularities with principal part f_0.

Recall that the family F of f_0^-−unfoldings has been called the semiuniversal family of unfoldings of negative weight of semiquasihomogeneous power series with principal part f_0. This name is justified by the proposition below.

Proposition 5. *[9] Let f be a semiquasihomogeneous power series with principal part f_0.*
(a) For each $t \in T_-$ the restriction of F to the germ (T_-, t) is a semiuniversal unfolding of F_t of negative weight.
(b) T_- does not contain trivial subfamilies.
(c) If f_0 is neither simple nor simple elliptic then there exist $t, t' \in T_-$ arbitrary close to 0 s.t. $F_t \sim_r F_{t'}$ (that is, the unfolding F is not universal in any neighborhood of 0).

Proof. We refer to [9], Remark 1.1, Lemma 1.2 and Remark 1.4. ∎

If the singularity of f_0 is simple or simple elliptic, K. Saito proved that $T_- = \{0\}$, i.e., all power series with principal part f_0 are right equivalent to f_0 (cf. [17]).

The functor for our moduli problem is defined as follows where $\tau = alg$ if $\mathcal{S}$ denotes the category of algebraic spaces and $\tau = hol$ if $\mathcal{S}$ denotes the category of complex spaces.

$$\begin{aligned} Unf_{f_0,\tau}^- \quad &: \quad \mathcal{S} \longrightarrow (sets), \\ S \quad &\in \quad Obj(\mathcal{S}) \longmapsto \left\{ [\phi]_{\sim_r} : \phi \text{ a family of } f_0^- - \text{unfoldings over } S \right\}, \\ T \xrightarrow{\varphi} S \quad &\in \quad Mor(T, S) \longmapsto \left([\phi]_{\sim_r} \mapsto [\varphi^*\phi]_{\sim_r} \right). \end{aligned}$$

Proposition 6. *For the reduced point* $\{pt\} \in \mathcal{S}$ *we obtain*

$$Unf_{f_0,\tau}^-(\{pt\}) = \left\{ [f]_{\sim_r} : f \ \text{semiquasihomogeneous with principal part } f_0 \right\}.$$

Proof. By definition, a family ϕ of negative weight with principal part f_0 over $\{pt\}$ is a quasihomogeneous power series with principal part f_0. ∎

By Theorem 3.2.3 a quasihomogeneous power series f is right equivalent to F_t for some $t \in T_-$. Hence $Unf_{f_0,\tau}^-(\{pt\})$ is the desired set of equivalence classes classifying semiquasihomogeneous hypersurface singularities with principal part f_0 w.r.t. right equivalence. Greuel, Hertling and Pfister solved the geometric classification problem by providing a coarse moduli space for $Unf_{f_0,\tau}^-$.

Theorem 5. *[9]. Let* $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ *be a quasihomogeneous polynomial with an isolated singularity at* 0. *The quotient*

$$\mathcal{M}_{f_0,\tau} := T_- / E_{f_0}$$

is a coarse moduli space for the functor $Unf_{f_0,\tau}^-$ *where* $\tau = alg/hol$.

Proof. We refer to Theorem 1.3 and Remark 1.5 in [9]. ∎

Note that the quotient $\mathcal{M}_{f_0,alg}$ is an affine variety since E_{f_0} is finite and acts algebraically on T_-.

Example 3. The moduli space $\mathcal{M}_{f_0,alg}$ for quasihomogeneous power series w.r.t. right equivalence with $f_0 = x^4 + xy^5$ (W_{17}) is given by the variety in $\mathbf{C}^5$, defined by the equations

$$Y_4^2 - Y_3 Y_5, Y_3 Y_4 - Y_2 Y_5, Y_2 Y_4 - Y_1 Y_5, Y_3^2 - Y_1 Y_5, Y_2 Y_3 - Y_1 Y_4, Y_2^2 - Y_1 Y_3.$$

The computation of such equations is our main concern in the following section. We conclude this section by some remarks on the properties of the moduli space $\mathcal{M}_{f_0,alg}$.

Remark 4. (a) Since the action of $G_{f_0}^{\mathbf{w}}$ on T_- is given by quasihomogeneous polynomials, the quotient T_- / E_{f_0} admits a $\mathbf{C}^*$−action. It turns out that this action is preserved by the linearization, i.e., the linearized action of E_{f_0} commutes with the $\mathbf{C}^*$−action, too.

(b) Since E_{f_0} is finite, it is reductive, so $\mathcal{M}_{f_0,alg}$ is an affine variety by Hilbert's Finiteness Theorem. Finiteness of E_{f_0} implies that $\mathcal{M}_{f_0,alg}$ is even a geometric quotient. As an affine variety, the moduli space $\mathcal{M}_{f_0,alg}$ is reduced, irreducible, has the Cohen-Macaulay property, and admits a $\mathbf{C}^*$−action.

(c) A fine moduli space for $Unf_{f_0,\tau}^-$ does not exist even if $\mathcal{S}$ is the category of complex germs (cf. [9]). Suppose $\mathcal{M}$ is a fine moduli space. Then $\mathcal{M}$ is also a coarse moduli

space, so $\mathcal{M} \simeq T_- / E_{f_0}$. There exists a universal unfolding over the germ $(T_-/E_{f_0}, 0)$ which can be induced from the semiuniversal unfolding F over the germ $(T_-, 0)$ and vice versa. Since T_- does not contain trivial subfamilies, the semiuniversal family F over $(T_-, 0)$ would be universal, which contradicts Proposition 3.4.5(c).

4. Algorithmic construction of the moduli space

We present the main algorithm for computing equations of moduli spaces for semi-quasihomogeneous power series with fixed principal part w.r.t. right equivalence, subroutines for computing the induced action of $G_{f_0}^{\mathbf{w}}$ on T_- by Arnold's Theorem and for a linearization of this action, and an implementation in the computer algebra system SINGULAR 2.0 ([10]). All other subroutines are either well known algorithms in commutative algebra (cf., e.g., [19]) or are easy to implement. For a more detailed description we refer to Chapter 3 of [3].

Procedure names typed in `typewriter` denote built-in SINGULAR 2.0 commands, e.g., `std` denotes the Standard bases algorithm of SINGULAR 2.0.

4.1. The action of $G_{f_0}^{\mathbf{w}}$

For a quasihomogeneous polynomial $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ we obtain a group action of $G_{f_0}^{\mathbf{w}}$ on T_- by Proposition 3.3.4, namely

$$
\begin{aligned}
G_{f_0}^{\mathbf{w}} \times T_- &\longrightarrow T_- \\
(\sigma, t) &\longmapsto \Theta(\sigma)(t) = s
\end{aligned}
$$

where $s \in T_-$ is s.t. $\varphi \circ \sigma(F_t) = F_s$ for some $\varphi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$. We provide the algorithm ARNOLDACTION for the computation of the algebraic action of $G_{f_0}^{\mathbf{w}}$ on T_- which is an implementation of the proof of Arnold's Theorem (Theorem 3.2.3). The algorithm computes the morphism φ of order > 0 s.t.

$$
\varphi(\sigma(F_t)) = F_s + r(\mathbf{x})
$$

where $ord_{\mathbf{w}}(r) > d'' := \max\{|\alpha|_{\mathbf{w}} : \alpha \in B_-\}$. By Remark 3.2.1 we may omit $r(\mathbf{x})$, i.e., $\varphi(F_t) = F_s$. Since $ord_{\mathbf{w}}(\varphi) > 0$ Proposition 3.3.2 implies that $s \in T_-$ is uniquely determined by the morphism φ. It remains to compute the action $t_i \mapsto p_i(\mathbf{s}, \mathbf{t})$ by setting $p_i(\mathbf{s}, \mathbf{t}) :=$ coefficient of m_i in $\varphi(F_t)$ where $m_1, m_2, \ldots, m_k$ denote all upper monomials of f_0.

Algorithm ARNOLDACTION$(f_0, G_{f_0}^{\mathbf{w}}, B_-)$

    In: quasihomogeneous polynomial $f_0 \in \mathbf{K}[x_1, x_2, ..., x_n]$ with an isolated singularity at 0, equations for the defining ideal of $G_{f_0}^{\mathbf{w}}$ in $\mathbf{K}[s_1, s_2, \ldots, s_r]$, upper monomials $B_- = \{m_1, m_2, \ldots, m_k\}$.

    Out: $(p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), \ldots, p_k(\mathbf{s}, \mathbf{t}))$ polynomials defining the action of $G_{f_0}^{\mathbf{w}}$ on $T_- = \mathbf{K}^n$.

Note: basering has parameters $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_k$ and variables $x_1, x_2, \ldots, x_n$.

begin

$lb := \min \left(|m|_{\mathbf{w}} : m \in B_- \right)$;

$ub := \max \left(|m|_{\mathbf{w}} : m \in B_- \right)$;

$F := f_0 + \sum_{i=1}^{k} t_i m_i$;

for $d := lb$ **to** ub **do**

     $g :=$ sum of all monomials in F of weighted degree d;

     $g_1 := \texttt{reduce}(g, \texttt{std}(\texttt{jacob}(f_0)))$;

     $\mathbf{v} := \texttt{lift}(\texttt{jacob}(f_0), g - g_1)$; // $g - g_1 = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i} v_i$;

     $\varphi(x_i) := x_i - v_i(\mathbf{x})$;

     $F := \varphi(F)$;

     reduce F w.r.t. the ideal of $G^{\mathbf{w}}_{f_0}$ (consider $s_1, s_2, \ldots, s_r$ as variables)

end;

for $i := 1$ **to** k **do**

     $p_i :=$ coefficient of m_i in F;

end;

return$(\{p_1, p_2, \ldots, p_k\})$;

end.

Example 4. For $f_0 = x^2 z + x^2 y - z^5 + y^5$ the ideal of the stabilizer $G^{(2,1,1)}_{f_0}$ equals $\left\langle s_1^2 s_3^2 - s_3, s_2 s_3, s_1^2 s_2 + s_1^2 s_3 - 1, s_2^4 - s_3^4 + s_1^2, s_1^4 + s_2^3 - s_3^3, s_3^5 - s_1^2 s_3 \right\rangle$ where $G^{(2,1,1)}_{f_0}$ acts via the map $(x, y, z) \mapsto (s_1 x, s_3 y + s_2 z, s_2 y + s_3 z)$, both computed by $\textsc{StabEqn}$[3]. The action α of the group $G^{(2,1,1)}_{f_0}$ on $T_- = \mathbf{C}^3$ is computed by $\textsc{ArnoldAction}$ and is given by the following polynomials.

$$
\begin{aligned}
\alpha^* t_1 &= p_1(\mathbf{s}, t_1, t_2, t_3) = (s_3 - s_2) t_1, \\
\alpha^* t_2 &= p_2(\mathbf{s}, t_1, t_2, t_3) = \left(s_3^2 - s_2^2 \right) t_2 + 2 s_2^2 t_3^2, \\
\alpha^* t_3 &= p_3(\mathbf{s}, t_1, t_2, t_3) = (s_2 + s_3) t_3
\end{aligned}
$$

where $\mathbf{s} = (s_1, s_2, s_3)$ and the ideal of $G^{(2,1,1)}_{f_0}$ is contained in $\mathbf{C}[\mathbf{s}]$. In this case, the action of $G^{(2,1,1)}_{f_0}$ is not linear.

4.2. Linearization of a group action

As noted above, the algebraic action of $G^{\mathbf{w}}_{f_0}$ on T_- need not be linear, but this fact is irrelevant for the existence proof of the moduli space. Nonetheless, the equations of the moduli space are computed from generators of the invariant ring of the action of $G^{\mathbf{w}}_{f_0}$ and the corresponding algorithms only work for linear actions. Therefore we

---

[3]We refer to Section 4.3 or Ch. 3.4 of [3] for a description.

need an algorithm for linearizing the action of $G_{f_0}^{\mathbf{w}}$ on T_-. Linearizations of algebraic actions over algebraically closed fields always exist and the construction is well known (we follow Ch. 1.6. in [4]).

Let G be an algebraic group which acts algebraically on the affine variety X (both defined over an algebraically closed field $\mathbf{K}$ of characteristic 0) via

$$
\begin{aligned}
\alpha : G \times X &\longrightarrow X \\
(\sigma, x) &\longmapsto \sigma \bullet_\alpha x.
\end{aligned}
$$

A **linearization** of α consists of a closed morphism $\rho : G \to GL_n(\mathbf{K})$ and a closed embedding $\phi : X \to \mathbf{K}^n$ s.t.

$$
\phi(\sigma \bullet_\alpha t) = \rho(\sigma) \cdot \phi(t)
$$

where "$\cdot$" is the usual action of $GL_n(\mathbf{K})$. On the coordinate ring of X we obtain the morphism $\lambda_\sigma(f)(x) := f(\alpha(\sigma^{-1}, x)) = \alpha^*(f)(\sigma^{-1}, x)$ for $\sigma \in G, f \in \mathbf{K}[X]$ and $x \in X$, which is called **left-translation**.

Lemma 2. *Let G be an algebraic group acting algebraically on a variety X. Any finite dimensional vector subspace V of $\mathbf{K}[X]$ can be extended to a finite dimensional vector subspace W of $\mathbf{K}[X]$ s.t. $G \bullet_\alpha W \subseteq W$. Moreover a necessary and sufficient condition that W is invariant under left translation is*

$$
\alpha^* W \subseteq \mathbf{K}[G] \otimes W.
$$

Proof. Firstly, we assume $\dim_{\mathbf{K}}(V) = 1$ and construct a $G-$invariant vectorspace $W \supseteq V$. The general case follows by constructing $G-$invariant vectorspaces for each basis element of V and taking the sum of these vectorspaces.
Choose a decomposition

$$
\alpha^*(f) = \sum_{i=1}^{r} g_i \otimes h_i \in \mathbf{K}[G] \otimes \mathbf{K}[X]
$$

s.t. r is minimal. Applying left translation to f yields $\lambda_\sigma f(x) = \alpha^*(f)(\sigma^{-1}, x) = \sum_{i=1}^{r} g_i(\sigma^{-1})h_i(x)$ for all $\sigma \in G$ and $x \in X$. Hence

$$
\lambda_\sigma(f) = \sum_{i=1}^{r} g_i(\sigma^{-1})h_i
$$

so $h_1, h_2, \ldots, h_r$ span a $\mathbf{K}-$vectorspace $W \supseteq V$ s.t. $\lambda_\sigma(f) \in W$ for all $\sigma \in G$. To see that W is $G-$invariant take $\sigma, \tau \in G$ and choose $c_1, c_2, \ldots, c_r \in \mathbf{K}$ s.t. $\lambda_{\sigma^{-1}\tau}(f) = \sum_{i=1}^{r} c_i h_i$. We have $\lambda_\tau(f) = \lambda_{\sigma\sigma^{-1}\tau}(f) = \lambda_\sigma(\sum_{i=1}^{r} c_i h_i) = \sum_{i=1}^{r} c_i \lambda_\sigma(h_i)$, so $\lambda_\tau(f)$ is contained in the $\mathbf{K}-$vectorspace W_σ spanned by $\lambda_\sigma(h_1), \lambda_\sigma(h_2), ..., \lambda_\sigma(h_r)$. Since

r is minimal and $\dim_{\mathbf{K}}(W_\sigma) = r$ we have $W \cap W_\sigma = W$, therefore $\lambda_\sigma(h_i) \in W$ for $\sigma \in G$ and $1 \leq i \leq r$.

Now let $W \subset \mathbf{K}[X]$ be a finite-dimensional subspace and $\{f_i\} \cup \{h_j\}$ be a basis for $\mathbf{K}[X]$ s.t. $\{f_i\}$ spans W. For $f \in W$ and $\sigma \in G$ we have $\lambda_\sigma(f) = \sum r_i(\sigma^{-1})f_i + \sum s_j(\sigma^{-1})h_j$, where $\alpha^* f = \sum r_i \otimes f_i + \sum s_j \otimes h_j$, so $\sigma \cdot f \in W$ iff $s_j(\sigma^{-1}) = 0$ for all $\sigma \in G$. Varying $\sigma \in G$ and $f \in W$ we see that $\lambda_\sigma(W) \subseteq W$ iff $\alpha^* W \subseteq \mathbf{K}[G] \otimes W$. $\blacksquare$

Theorem 6. *(Existence of equivariant embeddings) Let G be an algebraic group acting algebraically on a variety X. There exists a finite dimensional vectorspace V, a closed embedding $\phi : X \hookrightarrow V$ and a morphism $\rho : G \to GL(V)$ s.t.*

$$\phi(\sigma \bullet_\alpha x) = \rho(\sigma) \cdot \phi(x)$$

for all $\sigma \in G$ and $x \in X$.

Proof. By Lemma 4.2.2 we may assume that $f_1, f_2, \ldots, f_m \in \mathbf{K}[X]$ are algebra generators for $\mathbf{K}[X]$ which span a $G-$invariant $\mathbf{K}-$vectorspace V s.t. $\alpha^* V \subset \mathbf{K}[G] \otimes V$. Hence there exist $m_{ij} \in \mathbf{K}[G]$ s.t. $\alpha^* f_i = \sum_{j=1}^{m} m_{ij} \otimes f_j$ which define a map $\rho : G \to GL_n(\mathbf{K}), \sigma \mapsto \left(m_{ij}(\sigma^{-1})\right)$. Note that the m_{ij}'s are unique and that the map ρ is a closed morphism of algebraic groups. For each $\sigma \in G$ and $x \in X$ we have

$$f_i(\sigma \bullet_\alpha x) = \sum_{j=1}^{m} m_{ij}(\sigma^{-1})f_j(x). \tag{1}$$

Define $\phi : X \to V = \mathbf{K}^m$ by $y_i := f_i(x)$ for $x \in X$, where $y_1, y_2, \ldots, y_m$ are coordinates on V. Obviously, the map ϕ is a morphism and, by definition, we have $\phi^*(y_i) = f_i$, i.e. ϕ^* is surjective, hence ϕ is a closed embedding of X. Now we can write (1) as

$$\phi(\sigma \bullet_\alpha x) = \rho(\sigma) \cdot \phi(x).$$

$\blacksquare$

In order to construct a linearization of $\alpha = (p_1, p_2, \ldots, p_n)$ we start with $B = \{t_1, t_2, \ldots, t_n\}, V = \langle B \rangle$ and apply Lemma 4.2.2 to each t_j. If some h_i of a minimal decomposition is not contained in V we add h_i to B. Each such h_i gives rise to a new component p_{k_i} of the linearized action of α. From a linearization $\phi : X \hookrightarrow \mathbf{K}^m$ of α and from the projection $\pi : \mathbf{K}^m \to \mathbf{K}^m/G \subset \mathbf{K}^r$ given by generators of the invariant ring of the linearized action of G we obtain equations for the quotient of X/G by computing the image of the composition $\pi \circ \phi$

$$X \hookrightarrow \mathbf{K}^m \longrightarrow \mathbf{K}^m/G \hookrightarrow \mathbf{K}^r.$$

Since ϕ is injective and the fibers of π are finite (provided that G is finite which is true for $G = E_{f_0}$), the map $\pi \circ \phi$ is finite and therefore it is closed. Note that this implies that the quotient X/G is geometric, cf., e.g., Ch. II.3.2 in [13] (p. 96).

The algorithm $\textsc{MinimalDecomposition}(p_i)$ returns a list $\{\mathbf{g}^{(i)}, \mathbf{h}^{(i)}\}$ s.t. $p_i = \sum_{j=1}^{n_i} g_j^{(i)} \otimes h_j^{(i)} \in \mathbf{K}[s_1, s_2, \ldots, s_m] \otimes \mathbf{K}[t_1, t_2, \ldots, t_k]$ is a minimal decomposition. For a description we refer to Chapter 3.6 of [3].

Algorithm $\textsc{LinearizeAction}(G, \{p_1, p_2, \ldots, p_k\})$
  In: Ideal of G in $\mathbf{K}[s_1, s_2, \ldots, s_r]$, polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), \ldots, p_k(\mathbf{s}, \mathbf{t})$ defining an action of G on $\mathbf{K}^k$.
  Out: $\{p'_1(\mathbf{s}, \mathbf{t}), p'_2(\mathbf{s}, \mathbf{t}), \ldots, p'_l(\mathbf{s}, \mathbf{t})\}$, defining a linear action of G on $\mathbf{K}^l$, $\{B_1(\mathbf{t}), B_2(\mathbf{t}), \ldots, B_l(\mathbf{t})\}$ an equivariant embedding $\mathbf{K}^k \hookrightarrow \mathbf{K}^l$.
  begin
  $B := \{t_1, t_2, \ldots, t_k\}$; $l := k$;
  $(p'_1, p'_2, \ldots, p'_k) := p_1, p_2, \ldots, p_k$;
  for $i := 1$ **to** k **do**
      $\{\mathbf{g}^{(i)}, \mathbf{h}^{(i)}\} := \textsc{MinimalDecomposition}(p_i)$;
      for $j := 1$ **to** n_i **do**
        if $h_j^{(i)} \notin \langle B \rangle_{\mathbf{K}}$ **then begin**
            $B := B \cup \{h_j^{(i)}\}$; $l = l + 1$; // extend basis
            $p'_l := h_j^{(i)}(p_1, p_2, \ldots, p_k)$; // new component of the action
            replace $h_j^{(i)}$ in $p'_1, p'_2, \ldots, p'_l$ by the new variable t_l;
        end if
      end
  end
  return$(\{p'_1, p'_2, \ldots, p'_l\}, B)$;
  end.

We continue with our running example in order to illustrate the algorithm.

Example 5. Let $f_0 = x^2 z + x^2 y - z^5 + y^5$. We have to extend the $\mathbf{C}-$vectorspace V, spanned by $\langle t_1, t_2, t_3 \rangle_{\mathbf{C}}$, to a $G_{f_0}^{\mathbf{w}}-$ invariant vectorspace W. The decomposition of the nonlinear component $p_2 = (s_3^2 - s_2^2)t_2 + 2s_2^2 t_3^2$ of $\alpha = (p_1, p_2, p_3)$ equals

$$\alpha^* t_2 = p_2(s_1, s_2, s_3, t_1, t_2, t_3) = \left(s_3^2 - s_2^2\right) \otimes t_2 + 2s_2^2 \otimes t_3^2.$$

Therefore we consider the vectorspace $W := \langle t_1, t_2, t_3, t_3^2 \rangle_{\mathbf{C}}$ and we set $t_4 = t_3^2$ and $\deg(t_4) = 2$. The new component of the action is given by

$$\alpha^* t_4 = \alpha^* t_3^2 = (s_2 + s_3)^2 t_3^2 = \left(s_2^2 + 2s_2 s_3 + s_3^2\right) t_4 = \left(s_2^2 + s_3^2\right) t_4$$

since $s_2 s_3$ is in the ideal of $G_{f_0}^{\mathbf{w}}$. Hence W is the desired $G_{f_0}^{\mathbf{w}}-$invariant vectorspace. The closed embedding $\phi : T_- \hookrightarrow \mathbf{C}^4$ is given by

$$(t_1, t_2, t_3) \longmapsto \left(t_1, t_2, t_3, t_3^2\right)$$

and the representation $\rho : G_{f_0}^{\mathbf{w}} \to GL_4(\mathbf{C})$ by

$$(s_1, s_2, s_3) \longmapsto \begin{pmatrix} s_3 - s_2 & 0 & 0 & 0 \\ 0 & s_3^2 - s_2^2 & 0 & 2s_2^2 \\ 0 & 0 & s_2 + s_3 & 0 \\ 0 & 0 & 0 & s_2^2 + s_3^2 \end{pmatrix}.$$

The ideal of the variety $\phi(T_-)$ is generated by $t_4 - t_3^2$ and the generator is quasihomogeneous of type $(2; 1, 2, 1, 2)$. The group $G_{f_0}^{\mathbf{w}}$ acts linearly but not faithfully on $\phi(T_-)$. The ideal of the group $E_{f_0}' := \rho(G_{f_0}^{\mathbf{w}})$ computed by IMAGEGROUP[4] is

$$\mathcal{I}(E_{f_0}') = \left\langle s_5^5 - 1, s_4^2 - s_5, s_3 + s_4 s_5^3 - s_5, s_2 + s_3 - s_5, s_1 - s_5^3 \right\rangle$$

and may be simplified[5] to $\mathcal{I}(E_{f_0}') = \left\langle s_4^{10} - 1 \right\rangle$ so the linear action on $\phi(T_-)$ (and $\mathbf{C}^4$)is given by

$$(s_4, t_1, t_2, t_3, t_4) \longmapsto \left(s_4^6 t_1, s_4^7 t_2 - s_4^7 t_4 + s_4^2 t_4, s_4 t_3, s_4^2 t_4 \right).$$

Note that $|G_{f_0}^{\mathbf{w}}| = 20$ and $|E_{f_0}'| = 10$.

The algorithm LINEARIZEACTION preserves $\mathbf{C}^*-$ actions.

Proposition 7. *Let α be an action of G on $\mathbf{C}^k$ which commutes with a $\mathbf{C}^*-$action and let $\mathbf{w}$ be s.t. the defining polynomials of α are quasihomogeneous w.r.t. $\mathbf{w}$, where $t_1, t_2, \ldots, t_k$ are coordinates on $\mathbf{C}^k$ and the variables $s_1, s_2, \ldots, s_r$ of $\mathbf{C}[G]$ have weight 0. The linearized action, constructed by LINEARIZEACTION, commutes with a $\mathbf{C}^*-$action, i.e., the polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_l(\mathbf{s}, \mathbf{t})$ are quasihomogeneous w.r.t. to the extended weight $\mathbf{w}'$ where $\deg_{\mathbf{w}'}(t_i) = \deg_{\mathbf{w}}(t_j)$ if $i > k$ and t_i comes from the decomposition of $\alpha^* t_i$. In particular, the ideal of $\phi(T_-)$ is quasihomogeneous w.r.t. the weights of $t_1, t_2, \ldots, t_l$.*

Proof. We refer to Proposition 3.6.69 of [3] (p. 56). ∎

4.3. Computation of equations for the moduli space

Given a semiquasihomogeneous polynomial f_0 of type $(d; \mathbf{w})$ we compute equations for a coarse moduli space of semiquasihomogeneous power series with principal part f_0 as follows. Firstly, we compute equations for the stabilizer $G_{f_0}^{\mathbf{w}}$ by means of STABEQN and upper monomials for M_{f_0} with UPPERMONOMIALS (see below). Then ARNOLDACTION yields an algebraic action α of $G_{f_0}^{\mathbf{w}}$ on T_- which is linearized by (ϕ, ρ), obtained from LINEARIZEACTION (note that an action yields a representation). Equations for E_{f_0}', the image of $G_{f_0}^{\mathbf{w}}$ under ρ, are computed and QUOTIENTEQUATIONS (based on Derksen's Algorithm, cf. [5], and IMAGEVARIETY) provides

---

[4] The resulting equations may contain more variables than the equations of the input.

[5] Since the simplification is done by `elimpart` (cf. [8]) and by hand, it is not contained in the current version of `qhmoduli.lib`.

equations for an embedding of the moduli space. Below we give a brief description of the algorithms UPPERMONOMIALS, STABEQN, IMAGEVARIETY, IMAGEGROUP and QUOTIENTEQUATIONS. For a thorough description we refer to Chapter 3 in [3].

1. UPPERMONOMIALS: Computes the set $\{m_1, m_2, \ldots, m_k\}$ of upper monomials of a monomial basis of the Milnor algebra $\mathbf{C}\{x_1, x_2, \ldots, x_n\}/j(f_0)$ of f_0. This requires a standard basis computation in a local ring.

2. STABEQN: Computes equations of the stabilizer group $G_{f_0}^{\mathbf{w}} \subset G^{\mathbf{w}} \subset Aut(\mathbf{C}\{\mathbf{x}\})$ of f_0 by comparing coefficients of f_0 and $\sigma(f_0)$ where the elements $\sigma \in G^{\mathbf{w}}$ satisfy $\sigma(x_i) = \sum_{\langle \mathbf{w}, \alpha \rangle = w_i} s_{i,\alpha} \mathbf{x}^{\alpha}, \det(\frac{\partial \sigma(x_i)}{\partial x_j}) \neq 0$ and $s_{i,\alpha}$ are new indeterminates. The equations are simplified by using the command `elimpart` from the SINGULAR 2.0 library `presolve.lib`, cf. [8]. Since f_0 has an isolated singularity at 0 the zeroset of the resulting ideal is a group. This step needs the command `radical`.

3. IMAGEVARIETY and IMAGEGROUP: Computes the ideal I of the image of T_- under the morphism

$$\Phi : T_- \hookrightarrow \mathbf{C}^l \to \mathbf{C}^l/E'_{f_0} \hookrightarrow \mathbf{C}^m$$

given by $\Phi^*(y_i) = h_i$. Since $h_1, h_2, \ldots, h_m$ are quasihomogeneous ($\mathbf{C}^l/E'_{f_0}$ admits a $\mathbf{C}^*-$action) we apply the Hilbert-driven standard bases algorithm (`stdhilb`), which proves to be much more efficient in our application (cf. [18]). The coarse moduli space for $Unf_{f_0}^-$ is given by the variety $\mathcal{V}(I)$, defined by the ideal I. In particular, as sets we have

$$\mathcal{V}(I) \simeq Unf_{f_0}^- (\{pt\}).$$

IMAGEGROUP computes equations of E'_{f_0} from the equations of $G_{f_0}^{\mathbf{w}}$ and the linearized polynomials $p'_1, p'_2, \ldots, p'_l$ of the action of $G_{f_0}^{\mathbf{w}}$. May introduce additional variables.

4. QUOTIENTEQUATIONS: Given equations of an algebraic group G, of a variety X, and polynomials defining a linear action of G on X, QUOTIENTEQUATIONS computes equations of an embedding of the quotient X/G. Generators of the invariant ring $\mathbf{C}[t_1, t_2, \ldots, t_l]^G$ are computed by Derksen's algorithm ([5]), which requires equations for G and polynomials defining the action. In general, one has to apply the Reynolds operator, in which case G must be finite. In the implementation one can choose to use equations for $G_{f_0}^{\mathbf{w}}$ or E'_{f_0} to compute the invariant ring w.r.t. the linearized action (running times may be different).

In the algorithm the group $G_{f_0}^{\mathbf{w}}$ is described by its defining ideal and the action of $G_{f_0}^{\mathbf{w}}$ on T_- is represented by polynomials $p_1(\mathbf{s}, \mathbf{t}), \ldots, p_k(\mathbf{s}, \mathbf{t})$ where $\mathbf{s} = (s_1, s_2, \ldots, s_r)$, $\mathbf{t} = (t_1, t_2, \ldots, t_k)$. The linearized action is represented by the polynomials

$p_1(\mathbf{s}, \mathbf{t}'), ..., p_l(\mathbf{s}, \mathbf{t}')$ where $l \geq k$ and $\mathbf{t}' = (t_1, t_2, \ldots, t_l)$, which also encode the representation ρ. From a theoretical point of view, the computation of E'_{f_0} is not necessary, but this may simplify the computation of the Reynolds operator since $|E'_{f_0}| \leq |G^{\mathbf{w}}_{f_0}|$. The change of the basering, which occurs at several occasions in the SINGULAR 2.0 implementation, is omitted.

Algorithm MODEQN(f_0)

> **In:** quasihomogeneous polynomial f_0 with an isolated singularity at 0
> **Out:** equations $f_1, f_2, \ldots, f_q$ of the moduli space of $Unf^-_{f_0}$.
> **begin**
>> $\mathbf{m} :=$ UPPERMONOMIALS(f_0); //
>> $k :=$ **size**$(\mathbf{m})$;
>> $G^{\mathbf{w}}_{f_0} :=$ STABEQN(f_0); // variables $s_1, s_2, \ldots, s_r$
>> // variables are $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_k, x_1, x_2, \ldots, x_n$
>> $F_t := f_0 + \sum_{i=1}^{k} t_i \mathbf{m}[i]$;
>> $\{p_1(\mathbf{s}, \mathbf{t}), ..., p_k(\mathbf{s}, \mathbf{t})\} :=$ ARNOLDACTION$(G^{\mathbf{w}}_{f_0}, F_t)$;
>> $(\{p'_1(\mathbf{s}, \mathbf{t}'), ..., p'_l(\mathbf{s}, \mathbf{t}')\}, \phi) :=$ LINEARIZEACTION$(G^{\mathbf{w}}_{f_0}, \{p_1, p_2, \ldots, p_k\})$;
>> // variables are $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_k, t_{k+1}, ..., t_l$
>> $E'_{f_0} :=$ IMAGEGROUP$(G^{\mathbf{w}}_{f_0}, p'_1, p'_2, \ldots, p'_l)$;
>> $\{g_1, g_2, \ldots, g_l\} := \{\phi_1(t_1, \ldots, t_k), \phi_2(t_1, \ldots, t_k), ..., \phi_l(t_1, \ldots, t_k)\}$;
>> // variables are $s_1, s_2, \ldots, s_{r'}, t_1, t_2, \ldots, t_k, t_{k+1}, ..., t_l$;
>> $\{g'_1, g'_2, \ldots, g'_{l'}\} :=$ IMAGEVARIETY$(\{0\}, \{g_1, g_2, \ldots, g_l\})$;
>> // IMAGEVARIETY provides the ideal of the embedding of T_-
>> $\{f_1, ..., f_q\} :=$ QUOTIENTEQUATIONS $(E'_{f_0}, \{p'_1, p'_2, \ldots, p'_l\}, \{g'_1, ..., g'_{l'}\})$;
>> **return**$(\{f_1, f_2, \ldots, f_q\})$;
> **end;**

4.4. SINGULAR **2.0 Implementation**

All algorithms from the previous section are implemented in the SINGULAR 2.0 libraries `qhmoduli.lib,rinvar.lib` and `zeroset.lib`. A short description is given below, a full description for all algorithms and their options is contained in [3].

1. `qhmoduli.lib`: Contains the main algorithms for computing equations for moduli spaces for semiquasihomogeneous power series with fixed principal part (e.g., MODEQN, ARNOLDACTION and STABEQN).

2. `rinvar.lib`: Contains algorithms for computing invariant rings of reductive groups which are given by equations and algebraic actions (i.e., defined by polynomials). The computation of the invariant ring is based on Derksen's algorithm (NULLCONE), cf. [5]. Note that the Reynolds operator is implemented for finite groups only.

3. `zeroset.lib`: Contains algorithms for computing the zeroset of a zero-dimensional ideal (ZEROSET) and factorization of univariate polynomials (FACTOR) over finite extensions of **Q**.

4.4.1. The example $f_0 = x^2z + x^2y - z^5 + y^5$

....

We show how to compute the moduli space of semiquasihomogeneous hypersurface singularities w.r.t. the fixed part $f_0 = x^2z + x^2y - z^5 + y^5$ by means of the SINGULAR 2.0 library `qhmoduli.lib`.

```
> LIB ''qhmoduli.lib'';
> ring R = 0,(x,y,z),ls;
> poly f = -z5+y5+x2z+x2y;
> list stab = StabEqn(f); > def R1 = stab[1]; setring R1;
> stabid;
stabid[1]=s(2)*s(3)
stabid[2]=s(1)^2*s(2)+s(1)^2*s(3)-1
stabid[3]=s(1)^2*s(3)^2-s(3)
stabid[4]=s(2)^4-s(3)^4+s(1)^2
stabid[5]=s(1)^4+s(2)^3-s(3)^3
stabid[6]=s(3)^5-s(1)^2*s(3)
> vdim(stabid); 20
> setring R;
> StabOrder(f);
20
> ideal B = UpperMonomials(f);
> B;
B[1]=y3z3
B[2]=x2y3
B[3]=x2y2
> def R2 = ArnoldAction(f,stab,B);
> setring R2;
> actionid;
actionid[1]=-s(2)*t(1)+s(3)*t(1)
actionid[2]=-s(2)^2*t(2)+2*s(2)^2*t(3)^2 +s(3)^2*t(2)
actionid[3]=s(2)*t(3)+s(3)*t(3)
> nvars(stab[1]);
3
> def R3 = LinearizeAction(stabid, actionid,nvars(stab[1]));
> setring R3;
> actionid;
actionid[1]=-s(2)*t(1)+s(3)*t(1)
actionid[2]=-s(2)^2*t(2)+2*s(2)^2*t(4)+s(3)^2*t(2)
```

```
actionid[3]=s(2)*t(3)+s(3)*t(3)
actionid[4]=s(2)^2*t(4)+s(3)^2*t(4)
// 'ImageGroup' is not necessary
> def S = ImageGroup(groupid, actionid)⁶;
> setring S;
> groupid;
groupid[1]=s(2)+s(3)-s(5)
groupid[2]=s(4)^2-s(5)
groupid[3]=s(1)*s(5)+s(3)*s(4)-s(4)*s(5)
groupid[4]=s(1)*s(4)+s(3)-s(5)
groupid[5]=s(3)^2-2*s(3)*s(5)
groupid[6]=s(1)*s(3)-s(1)*s(5)+s(4)*s(5)
groupid[7]=s(1)^2+s(4)^2-2*s(5)
groupid[8]=-s(1)+s(5)^3
groupid[9]=s(3)*s(5)^2+s(4)-s(5)^3
groupid[10]=s(1)*s(5)^2-1
> ideal G = groupid,t(1),t(2),t(3),t(4);
> vdim(std(G));
10
// continue
setring R3;
> def R4 = QuotientEquations(groupid,actionid,embedid);
> setring R4;
> size(std(imageid)); // elimpart is not used !
72
> dim(std(imageid)); 3
// short version
> def T = ModEqn(f); setring T;
> size(modid); // 'elimpart' is used !
56
> dim(std(modid));
3
```

4.4.2. Performance

....

We provide 6 examples with increasing complexity and runtime. The first table contains information about the singularity, the group action and the invariant ring together with the embedding of the moduli space while the second table contains a profile of the computation. We consider the following examples

---

[6]Recall that the algorithm may introduce additional variables to those contained in `groupid`.

1.	E_{19}	$xy^7 + x^3$	4.	$x^2y + x^2z + y^5 - z^5$
2.	V_{19}^{*1}	$x^2y + y^4z + z^4$	5.	$x^5y - xy^5$
3.	Z_{23}	$y^{11} + x^3y$	6.	$x^8 + 14x^4y^4 + y^8$

where examples $1, 2, 3$ are taken from [20]. Note that stabilizers of Example 3,4,5 contain matrices with more than one entry per row. A "?" denotes that the computation was aborted due to time or memory constraints (more than 12h or 700MB). For more examples we refer to Chapter 4.2 in [3] (p. 70ff).

Nr.	μ	t_-	$|G_{f_0}^{\mathbf{w}}|$	$|E'_{f_0}|$	l	$\#I$	$\#E$	ub
1	19	2	21	21	2	8	21	21
2	19	3	32	16	3	22	191	16
3	23	3	33	33	3	35	673	33
4	24	3	20	10	4	21	56	10
5	25	3	144	?	3	?	?	?
6	49	10	192	?	> 10	?	?	?

Here μ denotes the Milnor number, t_- the dimension of T_-, l is the dimension of the ambient space of the embedding of T_- via ϕ, $\#I, \#E$ are the numbers of generators for the invariant ring $\mathbf{C}[t_1, t_2, \ldots, t_l]^{E'_{f_0}}$ respectively ideal of the quotient (deg-lex. Gröbner basis) and ub denotes highest degree occurring in the basis of the invariant ring. Note that $\dim T_- = l$ implies $\phi = id$.

In the table below we provide running times (measured in seconds) for these examples computed in Singular-2-0-0 running under Linux on a 1GHz Athlon with 768 MB RAM. Here we denote by $G_{f_0}^{\mathbf{w}}$ the time for STABEQN, by Θ the time for ARNOLDACTION, by ρ the time for LINEARIZEACTION, by E'_{f_0} the time for IMAGEGROUP, by NC the time for NULLCONE, by R1, R2 the timer for computing the Reynolds operator, resp. applying the operator to all non-invariant generators, and by Eq the time for IMAGEIDEAL. In the last column we provide the running time of MODEQN.

Nr.	$G_{f_0}^{\mathbf{w}}$	Θ	ρ	E'_{f_0}	NC	R1	R2	Eq.	tot.
1	0.02	0.02	0.00	0.01	0.02	0	0	0.07	0.13
2	0.03	0.05	0.00	0.01	0.06	0	0	15.46	15.60
3	0.02	0.04	0.01	0.15	0.00	0	0	829.35	829.56
4	0.05	0.15	0.04	0.01	0.16	0.90	0.00	4.80	5.10
5	0.03	0.37	0.01	?	?	?	?	?	?
6	0.04	6910.00	?	?	?	?	?	?	?

5. Brieskorn-Pham singularities

A polynomial $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ is called a **Brieskorn-Pham** polynomial if $f_0 = \sum_{i=1}^n x_i^{a_i}$ for $a_1, a_2, \ldots, a_n \in \mathbf{N}$. Brieskorn-Pham polynomials appeared, e.g., in the study of exotic spheres pursued by Milnor, Brieskorn and others in the 1960's. For

semiquasihomogeneous power series with principal part a Brieskorn-Pham polynomial the computation of the moduli space may be much simpler because of the following two facts :

- Generators for $G^{\mathbf{w}}_{f_0}$ can be written down without computation.

- The image of Θ is contained in $GL_k(\mathbf{C})$, i.e., the induced group action of $G^{\mathbf{w}}_{f_0}$ on T_- is linear.

Hence one may compute all elements of $G^{\mathbf{w}}_{f_0}$ or E_{f_0}, which are needed to compute the Reynolds operator, by matrix multiplication instead of computing the zeroset of a $0-$dim. ideal (the ideal of $G^{\mathbf{w}}_{f_0}$ or E_{f_0}). In general, the Reynolds operator must be used since Derksen's algorithm computes generators for the ideal generated by $\mathbf{C}[t_1, t_2, \ldots, t_m]^{G^{\mathbf{w}}_{f_0}}_+$, which are not necessarily invariant. This might be a significant improvement in the computation, cf., e.g., Example 5.7.

In order to prove the structure theorem on $G^{\mathbf{w}}_{f_0}$ we need the following Lemma, which was published (without proof) in 1991 by Orlik and Wagreich.

Lemma 3. *[15] Let $f \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be quasihomogeneous and $G = G^{\mathbf{w}}_{f_0} \cap GL_n(\mathbf{C})$. Then*

$$\bigoplus_{i=1}^n \mathbf{C} \frac{\partial f}{\partial x_i}$$

is a $G-$module.

Proof. For $\sigma \in G^{\mathbf{w}}_{f_0} \cap GL_n(\mathbf{C})$ we have $f_0 = f_0 \circ \sigma$ (composition of functions) and therefore $\frac{\partial f_0}{\partial x_i}(x) = \frac{\partial f_0 \circ \sigma}{\partial x_i}(x) = \sum_{j=1}^n \frac{\partial f_0}{\partial x_j}(\sigma \cdot x) \frac{\partial \sigma_j}{\partial x_i}(x)$. Since $\sigma \in GL_n(\mathbf{C})$ the terms $\frac{\partial \sigma_j}{\partial x_i}(x)$ are constants which we denote by $\lambda_j \in \mathbf{C}$. We obtain $\frac{\partial f_0}{\partial x_i}(x) = \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j} \right)(\sigma \cdot x)$. Now

$$
\begin{aligned}
\sigma \cdot \frac{\partial f_0}{\partial x_i}(x) &= \sigma \cdot \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j} \right)(\sigma \cdot x) = \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j} \right)(\sigma^{-1}\sigma \cdot x) \\
&= \sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}(x).
\end{aligned}
$$

$\blacksquare$

By using the previous lemma we obtain a theorem about the structure of the stabilizer groups of Brieskorn-Pham polynomials.

Theorem 7. *Let $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be a Brieskorn-Pham polynomial.*
(a) For $f_0 = \sum_{i=1}^n x_i^d$ the group $G^{\mathbf{w}}_{f_0}$ is generated by the diagonal matrix containing precisely one $d-$th root of 1 and all $n \times n$ permutation matrices, provided that $d > 2$.

(b) Let $f_0 = \sum_{i=1}^{m} f_0^{(d_i)}$ where $f_0^{(d_i)} = \sum_{j=r_{i-1}+1}^{r_i} x_j^{d_i}$ s.t. $d_i, r_i \in \mathbf{N}, 0 < d_i < d_j$ for $i < j$ and $r_0 = 0, r_i < r_{i+1}$. The stabilizer of f_0 is

$$G_{f_0}^{\mathbf{w}} = \bigoplus_{i=1}^{m} G_{f_0^{(d_i)}}^{\mathbf{w}}.$$

Proof. (a) Since all weights are equal (to 1) the only quasihomogeneous automorphisms are matrices, i.e., $G_{f_0}^{\mathbf{w}} \subset GL_n(\mathbf{C})$. Let $M = \bigoplus_{i=1}^{n} \mathbf{C} \frac{\partial f_0}{\partial x_i} = \mathbf{C} x_1^{d-1} \oplus \mathbf{C} x_2^{d-1} \oplus \ldots \oplus \mathbf{C} x_n^{d-1}$ and assume w.l.o.g. that $A = (a_{ij}) \in G_{f_0}^{\mathbf{w}}$ is s.t. a_{11} and a_{12} are not equal to 0. By Lemma 5.3 M is a $G_{f_0}^{\mathbf{w}}$−module, but $A^{-1} \cdot x_1^{d-1}$ contains terms like $a_{11}^{d-2} a_{12} x_1^{d-2} x_2$, and is therefore not contained in M, a contradiction. Hence any matrix in $G_{f_0}^{\mathbf{w}}$ contains precisely one nonzero entry per row, which must be a d−th root of unity. Since f_0 is invariant under permutations the claim follows.

(b) By assumption we have $\deg_{\mathbf{w}}(x_1) = \deg_{\mathbf{w}}(x_2) = \ldots = \deg_{\mathbf{w}}(x_{r_1}) > \deg_{\mathbf{w}}(x_{r_1+j})$ and for $r_{i-1} < k \leq r_i$ each automorphism $\varphi \in G_{f_0}^{\mathbf{w}}$ is of the form

$$\varphi(x_k) = \sum_{j=r_{i-1}+1}^{r_i} \lambda_{kj} x_j + c_k R_k(x_{r_i+1}, x_{r_i+2}, \ldots, x_n)$$

for some $\lambda_{kj}, c_k \in \mathbf{C}$ and $R_k \in \mathbf{C}[x_{r_i+1}, x_{r_i+2}, \ldots, x_n]$ with $\deg_{\mathbf{w}}(R_k) = w_k$ s.t. $\det(\frac{\partial \varphi}{\partial x_i}(0)) \neq 0$. We show that $c_k = 0$ and that the linear part of φ satisfies part (a) for each block $x_{r_{i-1}+1}, x_{r_{i-1}+2}, \ldots, x_{r_i}$ of variables which proves the claim. Let i be minimal s.t. $\varphi(x_k)$ is not as required for some $r_{i-1} < k \leq r_i$ and define $\varphi'(x_k) = x_k$ for $1 \leq k \leq r_{i-1}$, $\varphi'(x_k) = \sum \lambda_{kj} x_j$ for $r_{i-1} < k \leq r_i$ and $\varphi'(x_k) = \varphi(x_k)$ otherwise. By assumption, $\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{r_{i-1}})$ only affect $x_1, x_2, \ldots, x_{r_{i-1}}$ which cannot be changed by $\varphi(x_{r_{i-1}+j})$ for $j > 0$ and $x_{r_{i-1}+1}, x_{r_{i-1}+2}, \ldots, x_{r_i}$ are affected by $\varphi(x_{r_{i-1}+1}), \varphi(x_{r_{i-1}+2}), \ldots, \varphi(x_{r_i})$. Note that $\varphi(f_0^{(d_i)} + f_0^{(d_{i+1})} + \ldots + f_0^{(d_m)}) = f_0^{(d_i)} + f_0^{(d_{i+1})} + \ldots + f_0^{(d_m)}$. Hence $\varphi'(f_0^{(d_i)}) = f_0^{(d_i)} + R'(x_{r_i+1}, x_{r+2}, \ldots, x_n)$ and by part (a) the coefficients λ_{kl} in $\varphi(x_k)$ have the desired form. Now suppose some $c_k \neq 0$ for some $r_i \leq k \leq r_{i+1} - 1$. Then, for a monomial $\mathbf{x}^\alpha$ in R_k, the term $(\lambda_{kj} x_j)^{d_i-1} \mathbf{x}^\alpha$ cannot be canceled from $\varphi(f_0^{(d_i)} + f_0^{(d_{i+1})} + \ldots + f_0^{(d_m)})$, a contradiction to $\varphi \in G_{f_0}^{\mathbf{w}}$. Hence we obtain

$$\sum_{i=1}^{j} f_0^{(d_i)} = \varphi\left(\sum_{i=1}^{j} f_0^{(d_i)}\right) = \varphi\left(\sum_{i=1}^{j-1} f_0^{(d_i)}\right) + \varphi(f_0^{(d_j)}) = \sum_{i=1}^{j-1} f_0^{(d_i)} + \varphi(f_0^{(d_j)}),$$

hence $f_0^{(d_j)} = \varphi(f_0^{(d_j)})$ for $1 \leq j \leq r$. ∎

Now we make use of the knowledge of the structure of $G_{f_0}^{\mathbf{w}}$ to show that the induced group-action of $G_{f_0}^{\mathbf{w}}$ on T_- is linear.

Theorem 8. *If f_0 is a Brieskorn-Pham polynomial then $E_{f_0} \subset GL_k(\mathbf{C})$, i.e., the induced action of $G_{f_0}^{\mathbf{w}}$ on T_- is linear.*

Proof. By Theorem 5.7 $G_{f_0}^{\mathbf{w}} \subset GL_n(\mathbf{C})$ is generated by diagonal and permutation matrices so we may assume that $f_0 = \sum_{i=1}^n x_i^{d_i}$. For diagonal matrices $\sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$ it is clear that $\Theta(\sigma) \in E_{f_0}$ since

$$\sigma^{-1} \bullet \mathbf{x}^\alpha = (\sigma_1 x_1)^{\alpha_1} \cdot (\sigma_2 x_2)^{\alpha_2} \cdot \ldots \cdot (\sigma_n x_n)^{\alpha_n}$$

induces a diagonal matrix in $E_{f_0} \subset Aut(T_-)$. Now let B_- be the set of upper monomials of the Milnor algebra $M_{f_0} = \mathbf{C}[x_1, x_2, \ldots, x_n]/\langle x_i^{d_i-1} \rangle$. Since any permutation is a product of transpositions it suffices to consider the transposition $\pi = (ij)$. Note that any monomial in B_- can be written as $m = x_i^{\alpha_i} x_j^{\alpha_j} \cdot m'$, where m' does not contain x_i and x_j and $\alpha_i < d_i - 1, \alpha_j < d_i - 1$. The application of the transposition $\pi \in G_{f_0}^{\mathbf{w}}$ to m yields

$$\pi \cdot \left(x_i^{\alpha_i} x_j^{\alpha_j} \cdot m' \right) = x_i^{\alpha_j} x_j^{\alpha_i} \cdot m' \in M_-$$

since $\alpha_i, \alpha_j < d_i - 1 = d_j - 1$ and induces a permutation matrix in $E_{f_0} \subset Aut(T_-)$. In both cases the map ψ from Theorem 3.2.3 is the identity. ∎

Still the computation of moduli spaces for semiquasihomogeneous power series with principal part a Brieskorn-Pham polynomial may fail due to the intrinsic complexity of the invariant ring of E_{f_0} and the quotient T_-/E_{f_0}. We illustrate the situation with 2 examples.

Example 6. Let $f_0 = x^4 + y^8$ be a Brieskorn-Pham polynomial of type $(8; 2, 1)$. The semiuniversal family of negative weight is given by $F(x, y, t_1, t_2) = x^4 + y^8 + t_1 x^2 y^6 + t_2 x^2 y^5$. We have $\mu_{f_0}(0) = 21$, $|G_{f_0}^{(2,1)}| = |E_{f_0}| = 32$ and $T_- = \mathbf{C}^2$. The invariant ring equals $\mathbf{C}[t_1, t_2, t_3, t_4]^{G_{f_0}^{\mathbf{w}}} = \mathbf{C}[t_1^4, t_1^2 t_2^4, t_2^8]$ and the embedding of the moduli space $\mathcal{M} = T_-/E_{f_0}$ in $\mathbf{C}^3$ is given by the variety $\mathcal{V}(y_1 y_3 - y_2^2)$.

Example 7. Let $f_0 = x^3 + y^3 + z^7$ be a Brieskorn-Pham polynomial of type $(21; 7, 7, 3)$. We have $\mu_{f_0}(0) = 24$, $|G_{f_0}^{(7,7,3)}| = |E_{f_0}| = 126$ and $T_- = \mathbf{C}^5$. The invariant ring $\mathbf{C}[t_1, t_2, t_3, t_4, t_5]^{E_{f_0}}$ is generated by 162 homogeneous invariants $h_1, h_2, \ldots, h_{162}$ with degrees ranging from 3 to 42. Among the 162 generators delivered by Derksen's algorithm, 107 are invariant. Even the computation of the zeroset of $G_{f_0}^{(7,7,3)}$ takes ca. 2h25' (without applying the Reynolds operator) but we can write down the generators of $G_{f_0}^{\mathbf{w}}$ and compute the induced action by hand (or SINGULAR 2.0). Then all matrices of E_{f_0} are computed (12.68sec) and the Reynolds operator is applied to each non-invariant generator (1589.30sec). In order to compute the equations one has to eliminate the variables t_1, t_2, t_3, t_4, t_5 in the ideal $\langle y_1 - h_1, y_2 - h_2, ..., y_{162} - h_{162} \rangle$.

6. Conclusion

We have described algorithms and their implementation in SINGULAR 2.0 for computing equations of moduli spaces of semiquasihomogeneous power series with principal part f_0 w.r.t. right equivalence. The algorithms are contained in the SINGULAR 2.0 libraries `qhmoduli.lib` and `rinvar.lib` and some auxiliary algorithms are contained in `zeroset.lib`. The implementation is not restricted to specific classes of principal parts, but limitations arise from the intrinsic complexity of Gröbner bases and Arnold's Theorem.

If one considers the coarser relation contact equivalence instead of right equivalence, then a coarse moduli space still exists, but one has to fix more invariants and the construction is much more involved. In particular, the corresponding group need not be reductive! We refer to [9] for the existence proof. For a construction in the case of space curve singularities we refer to [6].

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