

ON THE NONSQUARE CONSTANTS OF

$$L^{(\Phi)}[0, +\infty)$$

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Abstract

Let $L^{(\Phi)}[0, +\infty)$ be the Orlicz function space generated by N -function $\Phi(u)$ with Luxemburg norm. We show the exact nonsquare constant of it when the right derivative $\phi(t)$ of $\Phi(u)$ is convex or concave.

1 Introduction

Let X be a Banach space and $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of X . The nonsquare constants in the sense of James $J(X)$ and in the sense of Schaffer $g(X)$ are defined as:

$$J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in S(X)\}, \quad (1)$$

$$g(X) = \inf\{\max(\|x + y\|, \|x - y\|) : x, y \in S(X)\}. \quad (2)$$

Clearly, if $\dim X \geq 2$, then $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$. Ji and Wang [5] asserted

$$g(X) \cdot J(X) = 2 \quad (3)$$

for $\dim X \geq 2$.

It is proved[1] that $J(X) = 2$ if X fails to be reflexive. Nonsquareness is an important geometric property of Banach spaces which expose the intrinsic construction of a space according to the “shape” of the unit ball of the spaces. Therefore, it is interesting to investigate it in classical Banach spaces, for example, Orlicz spaces. It is showed[5] that $J(L^p) = \max(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}})(1 < p < \infty)$. However, examples for values of $J(X)$ for X to be reflexive except L^p remains unknown. In this paper, we deal with $J(X)$ when X is an Orlicz function space with Luxemburg norm.

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Let $\Phi(u) = \int_0^{|u|} \phi(t)dt$ be an N -function, i.e., $\phi(0) = 0$, ϕ is right continuous and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. The Orlicz function space $L^{(\Phi)}[0, \infty)$ is defined to be the set

$$L^{(\Phi)}[0, \infty) = \left\{ x(t) : \rho_{\Phi}(\lambda x) = \int_{[0, \infty)} \Phi(\lambda|x(t)|)dt < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm is expressed as

$$\|x\|_{(\Phi)} = \inf \left\{ c > 0 : \rho_{\Phi}\left(\frac{x}{c}\right) \leq 1 \right\}.$$

$\Phi(u)$ is said to satisfy the Δ_2 -condition for all $u \geq 0$, in symbol $\Phi \in \Delta_2$, if there exists $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $u \geq 0$. In what follows, we will frequently use Semenove indices of $\Phi(u)$:

$$\bar{\alpha}_{\Phi} = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \bar{\beta}_{\Phi} = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}. \quad (4)$$

2 Main Results

We first consider the lower bounds of $L^{(\Phi)}[0, \infty)$. The following result is refined from Ren[8].

Theorem 1. *Let $\Phi(u)$ be an N -function. Then the nonsquare constant of $L^{(\Phi)}[0, \infty)$, in the sense of James, satisfies*

$$\max \left(\frac{1}{\bar{\alpha}_{\Phi}}, 2\bar{\beta}_{\Phi} \right) \leq J(L^{(\Phi)}[0, \infty)). \quad (5)$$

Proof. To prove (5), we first show

$$\frac{1}{\bar{\alpha}_{\Phi}} \leq J(L^{(\Phi)}[0, \infty)). \quad (6)$$

Take a real number $u \in (0, \infty)$, choose measurable subsets G_1 and G_2 in $[0, \infty)$ such that $G_1 \cap G_2 = \emptyset$. and $\mu(G_1) = \mu(G_2) = \frac{1}{2u}$. Put

$$x(t) = \Phi^{-1}(2u)\chi_{G_1}(t) \text{ and } y(t) = \Phi^{-1}(2u)\chi_{G_2}(t),$$

where χ_{G_1} is the characteristic function of G_1 . Note that

$$\|\chi_{G_1}\|_{(\Phi)} = \|\chi_{G_2}\|_{(\Phi)} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(G_1)})} = \frac{1}{\Phi^{-1}(2u)}.$$

We have $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}.$$

Taking the supremum over $u \in (0, \infty)$, since the function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is right continuous at 0 and takes value on $[\frac{1}{2}, 1]$, we deduce that

$$J(L^{(\Phi)}[0, \infty)) \geq \sup_{u \in (0, \infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \sup_{u \in [0, \infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \frac{1}{\bar{\alpha}_\Phi}.$$

Finally we show

$$2\bar{\beta}_\Phi \leq J(L^{(\Phi)}[0, \infty)). \tag{7}$$

For every real number $v > 0$, choose measurable subsets E_1, E_2 in $[0, \infty)$ such that $E_1 \cap E_2 = \emptyset$ and $\mu(E_1) = \mu(E_2) = \frac{1}{2v}$. Put

$$x(t) = \Phi^{-1}(v)[\chi_{E_1}(t) + \chi_{E_2}(t)] \text{ and } y(t) = \Phi^{-1}(v)[\chi_{E_1}(t) - \chi_{E_2}(t)],$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{2\Phi^{-1}(v)}{\Phi^{-1}(2v)}.$$

Taking the supremum over $v \in (0, \infty)$ we also have

$$J(L^{(\Phi)}[0, \infty)) \geq 2\bar{\beta}_\Phi.$$

Hence (5) follows from (6) and (7). ■

Assume Φ satisfies Δ_2 -condition for all u . Ji and Wang([5], Theorem 3) offered a couple of formulas:

(i) If $\phi(t)$ is a concave function, then

$$g(L^{(\Phi)}[0, \infty)) = \inf \left\{ k_x > 0 : \rho_\Phi\left(\frac{2x}{k_x}\right) = 2, \rho_\Phi(x) = 1 \right\}; \tag{8}$$

(ii) If $\phi(t)$ is convex, then

$$J(L^{(\Phi)}[0, \infty)) = \sup \left\{ k_x > 0 : \rho_{\Phi}\left(\frac{2x}{k_x}\right) = 2, \rho_{\Phi}(x) = 1 \right\}. \quad (9)$$

We now extend the above representatives and deduce the upper bounds.

Theorem 2. *Suppose $\phi(t)$ be the right derivative of $\Phi(u)$. We have*

(i) *If $\phi(u)$ is concave , then*

$$J(L^{(\Phi)}[0, \infty)) \leq \frac{1}{\bar{\alpha}_{\Phi}}; \quad (10)$$

(ii) *If $\phi(u)$ is convex , then*

$$J(L^{(\Phi)}[0, \infty)) \leq 2\bar{\beta}_{\Phi}. \quad (11)$$

Proof. If $\Phi \notin \Delta_2$, which is equivalent to $\bar{\beta}_{\Phi} = 1$, then $L^{(\Phi)}[0, \infty)$ is nonreflexive and hence $J(L^{(\Phi)}[0, \infty)) = 2$ according to the results in Chen[1] or Hudzik[4]. Since $\phi(t)$ is concave implies $\Phi \in \Delta_2$ (see Krasnoselskiĭ and Rutickii[6],p.26), we only need to check (11) when $\phi(t)$ is convex, but this is trivial since $J(l^{(\Phi)}) = 2 = 2\beta_{\Phi}^0 = 2\bar{\beta}_{\Phi}$. Therefore it suffices for us to prove (10) and (11) for $\Phi \in \Delta_2$.

We first prove (10) for $\Phi(u) \in \Delta_2$, which is equal to

$$g(L^{(\Phi)}[0, \infty)) \geq 2\bar{\alpha}_{\Phi} \quad (12)$$

when $\phi(t)$ is concave in view of (3) and (8).

Let $H_{\Phi}(u) = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}$, then $\Phi^{-1}(2u) = H_{\Phi}(u) \cdot \Phi^{-1}(u)$. Put $x = \Phi^{-1}(u)$, then $u = \Phi(x)$ and

$$2\Phi(x) = \Phi[H_{\Phi}(\Phi(x)) \cdot x]. \quad (13)$$

Therefore, when $u = \Phi(x(t)) \geq 0$ we have

$$\begin{aligned} \rho_{\Phi}\left(\frac{2x(t)}{2\bar{\alpha}_{\Phi}}\right) &= \rho_{\Phi}\left(\frac{x(t)}{\bar{\alpha}_{\Phi}}\right) \geq \rho_{\Phi}\left(\frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} \cdot x(t)\right) \\ &= \rho_{\Phi}[H_{\Phi}(u) \cdot x(t)] = 2\rho_{\Phi}(x(t)) = 2 \end{aligned}$$

for $\rho_\Phi(x(t)) = 1$. It follows that (12) and hence (10) holds.

One can prove (11) analogously by (9). ■

We obtain the main result from the above theorems:

Theorem 3. *Let $\Phi(u)$ be an N -function, $\phi(t)$ be the right derivative of $\Phi(u)$. Then*

(i) *If $\phi(t)$ is concave, then*

$$J(L^{(\Phi)}[0, \infty)) = \frac{1}{\bar{\alpha}_\Phi}; \tag{14}$$

(ii) *If $\phi(t)$ is convex, then*

$$J(L^{(\Phi)}[0, \infty)) = 2\bar{\beta}_\Phi. \tag{15}$$

Remark 4. If the index function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is decreasing or increasing on interval $[0, \infty)$, then the indices $\bar{\alpha}_\Phi$ and $\bar{\beta}_\Phi$ take the values at either end of the interval. The author[10] found that if $F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}$ is increasing(decreasing) on $(0, \Phi^{-1}(u_0)]$ then $G_\Phi(u)$ is also increasing(decreasing) on $(0, \frac{u_0}{2}]$, respectively. Rao and Ren[7] gave interrelations between Semenov and Simonenko indices:

$$2^{-\frac{1}{A_\Phi}} \leq \alpha_\Phi \leq \beta_\Phi \leq 2^{-\frac{1}{B_\Phi}}, \quad 2^{-\frac{1}{A_\Phi^0}} \leq \alpha_\Phi^0 \leq \beta_\Phi^0 \leq 2^{-\frac{1}{B_\Phi^0}},$$

where

$$\begin{aligned} A_\Phi &= \liminf_{t \rightarrow \infty} \frac{t\phi(t)}{\Phi(t)}, & B_\Phi &= \limsup_{t \rightarrow \infty} \frac{t\phi(t)}{\Phi(t)}; \\ A_\Phi^0 &= \liminf_{t \rightarrow 0} \frac{t\phi(t)}{\Phi(t)}, & B_\Phi^0 &= \limsup_{t \rightarrow 0} \frac{t\phi(t)}{\Phi(t)}; \end{aligned}$$

and

$$\begin{aligned} \alpha_\Phi &= \liminf_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, & \beta_\Phi &= \limsup_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \\ \alpha_\Phi^0 &= \liminf_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, & \beta_\Phi^0 &= \limsup_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}. \end{aligned}$$

When the index function $F_{\Phi}(t)$ is monotonic, the limits $C_{\Phi} = \lim_{t \rightarrow \infty} F_{\Phi}(t)$ and $C_{\Phi}^0 = \lim_{t \rightarrow 0} F_{\Phi}(t)$ must exist and we have

$$\alpha_{\Phi} = \beta_{\Phi} = \lim_{u \rightarrow \infty} G_{\Phi}(u) = 2^{-\frac{1}{C_{\Phi}}}, \quad \alpha_{\Phi}^0 = \beta_{\Phi}^0 = \lim_{u \rightarrow 0} G_{\Phi}(u) = 2^{-\frac{1}{C_{\Phi}^0}}. \quad (16)$$

This makes it easier to calculate the indices in Theorem 3.

Example 5. Observe the N -function(see Gallardo[2])

$$\Phi_{p,r}(u) = |u|^p \ln^r(1 + |u|), \quad 1 \leq p < \infty, 0 < r < \infty.$$

It is easy to check the right derivative of $\Phi_{p,r}(u)$, $\phi(t)$ is convex when $1 \leq p < \infty, 2 \leq r < \infty$. The index function

$$F_{\Phi_{p,r}}(t) = \frac{t\Phi'_{p,r}(t)}{\Phi_{p,r}(t)} = p + \frac{rt}{(1+t)\ln(1+t)}$$

is decreasing from $p+r$ to p on $[0, \infty)$ since

$$\frac{d}{dt}\Phi_{p,r}(t) = \frac{r[\ln(1+t) - t]}{(1+t)^2 \ln^2(1+t)} < 0.$$

So $C_{\Phi_{p,r}}^0(t) = \lim_{t \rightarrow 0} F_{\Phi_{p,r}}(t) = p+r$. According to (16) in the above remark and Theorem 3 we have

$$J(L^{(\Phi_{p,r})}[0, \infty)) = 2\bar{\beta}_{\Phi_{p,r}} = 2\beta_{\Phi_{p,r}}^0 = 2 \cdot 2^{-\frac{1}{p+r}} = 2^{1-\frac{1}{p+r}}. \quad (17)$$

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