

AN ABSTRACT VERSION OF THE CONCENTRATION COMPACTNESS PRINCIPLE

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Abstract

We prove an abstract version of concentration compactness principle in Hilbert space and show its applications to a range of elliptic problems on unbounded domains.

1 Introduction

In the present paper we propose a generalization of the concentration compactness principle by P.-L.Lions [9, 10, 11, 12]) from specific functional spaces to an abstract Hilbert space. One can define the concentration compactness approach as a (partial) recovery of compactness in non-compact sequences by selecting certain, vaguely speaking, moving bulks of mass.

If $\alpha_k \in \mathbb{R}^N$ is an unbounded sequence and u is a locally Lebesgue-integrable function on \mathbb{R}^N , the sequence $u_k := u(\cdot - \alpha_k)$ will converge to zero in the sense of distributions. This means that nonzero shift-invariant mappings are not weakly continuous. At the same time one can use weak convergence with different shifts to define a stronger topology in which some shift-invariant functionals become continuous. We say that a bounded sequence u_k converges to zero *weakly with concentration*, if $u_k(\cdot - \alpha_k) \rightharpoonup 0$ for any sequence of shifts α_k . As it is known, in the case of a Sobolev space over \mathbb{R}^N concentrated weak convergence yields convergence in L^p with subcritical p (cf. [8]).

Instead of shifts we consider a general set of linear operators that might be called *dislocations*. This set cannot be arbitrarily small or arbitrarily large: if it is compact then concentrated weak convergence is

the same as weak convergence, if it consists of all unitary operators, then the only sequences convergent weakly with concentration are constant sequences.

In Section 2 we prove, under certain assumptions about the set of dislocations, that any bounded sequence in a Hilbert space has a subsequence that can be written as an asymptotic sum of differently dislocated weak limits, with a remainder convergent to zero weakly with concentration. A tentative Banach space version has been proved in [18]. Applications of this decomposition require an analytic interpretation of concentrated weak convergence for specific choices of the space and the set of dislocations. Similar decompositions of sequences in Sobolev spaces have been in use since [13], but only under additional conditions on the sequence.

In Section 3 we deal with concretization of the abstract statement for Sobolev spaces with groups of shifts and dilations, and the product group thereof. The spaces involved there are $H^1(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$.

In Section 4 we study reduction of the above decompositions to Sobolev spaces over $\Omega \subset \mathbb{R}^N$. We introduce a class of open sets for which it is possible (*asymptotically contractive sets*) and give their geometric characterization. At the end of the section we give two examples of compactness results for Sobolev imbeddings.

Section 5 presents applications of the method to semilinear elliptic equations. We provide existence results for: a model problem on an asymptotically contractive set; a problem on exterior domain with symmetry; a problem with a critical exponent on unbounded asymptotically contractive domains.

2 Abstract concentration compactness in Hilbert space

Let H be a separable Hilbert space and let D be a bounded set of bounded linear operators on H , closed with respect to strong convergence and satisfying the following properties:

- (I) $g \in D \Rightarrow g^{-1} \in D, I \in D$;
- (II) if $g_k, h_k \in D$ and $g_k h_k^{-1}$ does not converge weakly to 0, then there exists a renamed strongly convergent subsequence of

$g_k h_k^{-1}$ such that $s\text{-}\lim g_k h_k^{-1} \neq 0$;

(III) $g_k \in D, u_k \rightharpoonup 0 \Rightarrow g_k^* g_k u_k \rightharpoonup 0$.

We shall call D a set of dislocations. Dislocations do not have to form a group, however, for a group of unitary operators one can replace conditions (I-III) by a single condition as below.

Proposition 2.1. *Let D be a group of unitary operators in H . Then D is a set of dislocations if the following condition is satisfied:*

(IIa) *if $g_k \in D$ does not converge weakly to zero, then there exists a renamed strongly convergent subsequence of g_k such that $s\text{-}\lim g_k \neq 0$.*

Proof. Condition (I) follows from the definition of a group. Condition (III) is tautological since D consists of isometries and $g_k^* g_k = id$. Condition (II) rewritten for a group becomes (IIa).

Definition 2.2. *Let $u, u_k \in H$. We will say that u_k converges to u weakly with concentration (with respect to a given set of dislocations D), which we will denote as*

$$u_k \xrightarrow{cw} u,$$

if for all $\varphi \in H$,

$$\limsup_{k \rightarrow \infty} \sup_{g \in D} (g(u_k - u), \varphi) = 0. \quad (1)$$

The goal of this section is to prove the theorem on decomposition of sequences into dislocated weak limits:

Theorem 2.3. *Let $u_k \in H$ be a bounded sequence and let D be a set of dislocations. Then there exist $w^{(n)} \in H, g_k^{(n)} \in D, k, n \in \mathbb{N}$ such that for a renumbered subsequence*

$$w^{(n)} = w\text{-}\lim g_k^{(n)-1} u_k, \quad g_k^{(1)} = id, \quad (2)$$

$$g_k^{(n)-1} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m, \quad (3)$$

$$\sum_{n \in \mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2, \tag{4}$$

and

$$u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \xrightarrow{cw} 0, \tag{5}$$

where the series is absolutely convergent for every k .

Proof. We define, on a renumbered subsequence,

$$w^{(1)} =: \text{w-lim } u_k, g_k^{(1)} = id. \tag{6}$$

Let

$$v_k^{(1)} = u_k - g_k^{(1)} w^{(1)} \tag{7}$$

If $v_k^{(1)} \xrightarrow{cw} 0$, the theorem is verified with $w^{(n)} = 0, n \geq 2$. If not, there exist $g_k^{(2)}$ and $w^{(2)} \neq 0$ such that, on a renumbered subsequence,

$$g_k^{(2)-1} v_k^{(1)} \rightharpoonup w^{(2)}. \tag{8}$$

If $g_k^{(1)-1} g_k^{(2)} \neq 0$, then, applying this product to (8), and using (II), we obtain for a renamed subsequence,

$$g_k^{(1)-1} v_k^{(1)} - g_k^{(1)-1} g_k^{(2)} w^{(2)} \rightarrow 0. \tag{9}$$

Thus, on a renamed subsequence, noting that due to (I) $\lim g_k^{(1)-1} g_k^{(2)} w^{(2)} \neq 0$, we have:

$$\text{w-lim } g_k^{(1)-1} v_k^{(1)} \neq 0. \tag{10}$$

However, using (7) and (6), we get

$$\text{w-lim } g_k^{(1)-1} v_k^{(1)} = \text{w-lim } g_k^{(1)-1} u_k - w^{(1)} = 0. \tag{11}$$

This contradiction implies that

$$g_k^{(1)-1} g_k^{(2)} \rightharpoonup 0, \tag{12}$$

Recursively we define:

$$v_k^{(n)} = u_k - g_k^{(1)} w^{(1)} - \dots - g_k^{(n)} w^{(n)} \tag{13}$$

If for some n , $v_k^{(n)} \xrightarrow{cw} 0$, the statement is proved. Otherwise, there is $g_k^{(n+1)}$, and a $w^{(n+1)} \neq 0$ such that, on a subsequence

$$g_k^{(n+1)-1} v_k^{(n)} \rightharpoonup w^{(n+1)}. \tag{14}$$

An argument similar to that above shows that

$$g_k^{(p)-1} g_k^{(q)} \rightharpoonup 0 \text{ whenever } p \neq q. \tag{15}$$

From (13),(14),(15) follows (2). Let us now expand the left hand side of the inequality $\|v_k^{(n)}\|^2 \geq 0$ using (13):

$$\|u_k\|^2 + \sum_n \|g_k^{(n)} w^{(n)}\|^2 - 2 \sum_n (u_k, g_k^{(n)} w^{(n)}) + \sum_{m \neq n} (g_k^{(m)} w^{(m)}, g_k^{(n)} w^{(n)}) \geq 0. \tag{16}$$

The last term in (16) converges to zero due to (15) and (III), since $g_k^{(n)*} g_k^{(m)} = [g_k^{(n)*} g_k^{(n)}][g_k^{(n)-1} g_k^{(m)}]$. Let us estimate the third term.

$$\begin{aligned} \sum_n (u_k, g_k^{(n)} w^{(n)}) &= \sum_n (g_k^{(n)*} g_k^{(n)} g_k^{(n)-1} u_k, w^{(n)}) \\ &= \sum_n (g_k^{(n)*} g_k^{(n)} (g_k^{(n)-1} u_k - w^{(n)}), w^{(n)}) \\ &\quad + \sum_n (g_k^{(n)*} g_k^{(n)} w^{(n)}, w^{(n)}) \\ &= o(1) + \sum_n (g_k^{(n)*} g_k^{(n)} w^{(n)}, w^{(n)}). \end{aligned} \tag{17}$$

in the last step we have applied (III) and (14). Combining (16) and (17) we get:

$$\|u_k\|^2 \geq \sum_n \|g_k^{(n)} w^{(n)}\|^2 + o(1) \tag{18}$$

Due to (I) we can deduce from (18) that

$$\sum_n \|w^{(n)}\|^2 \leq C \limsup \|u_k\|^2, \quad C > 0. \tag{19}$$

It is easy to see, using (19) and (15), that the series $\sum g_k^{(n)} w^{(n)}$ (after passing to an appropriately rarefied subsequence) is absolutely convergent. Let $\epsilon > 0$. Note that by (19) there is a $M > 0$, independent of the selection of $w^{(n)}$'s such that the number of weak limits (14) with the norm exceeding $\epsilon/2$ is less than M . This implies that if on every iteration step we will be selecting a weak limit with the norm exceeding $\epsilon/2$ (if there is any), then for every $g_k \in D$, any weak subsequential limit of $g_k^{-1} v_k^{(M)}$ will have a norm less than ϵ .

Let $\varphi \in H$, $\|\varphi\| = 1$ and $g_k \in D$. Then

$$\limsup |(g_k^{-1} v_k^{(M)}, \varphi)| \leq 2|(w^{(M+1)}, \varphi)| \leq 2\|w^{(M+1)}\| \leq 2\epsilon \quad (20)$$

And so

$$\limsup |(g_k^{-1}(u_k - \sum_n g_k^{(n)} w^{(n)}), \varphi)| = 0, \quad (21)$$

which verifies (5). ■

3 Concentration compactness in Sobolev spaces

Assume first that $H = H_0^1(\mathbb{R}^N)$ and that D is a group of shift operators $g_\alpha u = u(\cdot + \alpha)$, $\alpha \in \mathbb{R}^N$. Shifts are unitary operators on H and it is easy to see that they verify condition (IIa). First,

$$g_{\alpha_k} \rightharpoonup 0 \Leftrightarrow |\alpha_k| \rightarrow \infty. \quad (22)$$

A bounded sequence α_k has a convergent subsequence, so that $u_k \rightharpoonup 0 \Rightarrow (g_{\alpha_k} u_k, \varphi) = (u_k, g_{-\alpha_k} \varphi)$, and the latter expression has the same limit as $\lim (u_k, g_{-\lim \alpha_k} \varphi) = 0$. Theorem 2.3 now can be quoted together with (22) while the analytical meaning of concentrated weak convergence is clarified by the following statement. We will use the following notation: $2^* = \frac{2N}{N-2}$ for $N > 2$ and $2^* = \infty$ when $N = 2$.

Lemma 3.1. *Let u_k be a bounded sequence in $H^1(\mathbb{R}^N)$ and $p \in (2, 2^*)$. Then $u_k \xrightarrow{cw} 0 \Leftrightarrow \|u_k\|_{L^p(\mathbb{R}^N)} \rightarrow 0$.*

Proof. The *if* implication is an elementary corollary of Lemma 6 in [8] and can be also found in [10]. For the *only if* statement note that due

to the shift invariance of L^p -norm, all L^p -weak limits of sequences $g_k u_k$ are equal to zero. Since the sequence u_k is bounded H^1 -norm, L^p -weak convergence implies weak convergence in H^1 and thus, concentrated weak convergence. ■

An immediate consequence of this lemma is

Lemma 3.2. *Let $H = H^1(\mathbb{R}^N)$, let D be the group of shifts and let $u_k, w^{(n)}$, and $g_k^{(n)}$ be as in Theorem 2.3. If $F : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and for every $\epsilon > 0$ there is a $C_\epsilon < \infty$ and a p_ϵ such that $2 < p_\epsilon < 2^*$ and*

$$|F(s)| \leq \epsilon(|s|^2 + |s|^{2^*}) + C_\epsilon |s|^{p_\epsilon} \tag{23}$$

then

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k) = \sum_n \int_{\mathbb{R}^N} F(w^{(n)}). \tag{24}$$

Proof. It is easy to see from Lemma 3.1 and (23) that $u \rightarrow \int_{\mathbb{R}^N} F(u)$ is a continuous map with respect to concentrated weak convergence. This reduces the proof to the case of finite sums, i.e. one can assume without loss of generality that $u_k = \sum_{n=1}^M w^{(n)}(\cdot + \alpha_k^{(n)})$. Note that by (22) and (3), $|\alpha_k^{(n)} - \alpha_k^{(m)}| \rightarrow \infty$ for $m \neq n$, so that for every $\epsilon > 0$ one can find $R > 0$ and $k_0 \in \mathbb{N}$ such that for all $k > k_0$ the balls $B_R(-\alpha_k^{(n)})$ will be disjoint and for $p = 2, 2^*$ one will have

$$\int_{\mathbb{R}^N \setminus B_R(-\alpha_k^{(n)})} |w^{(n)}|^p \leq \epsilon, \tag{25}$$

as well as

$$\int_{\mathbb{R}^N \setminus B_R(-\alpha_k^{(n)})} |w^{(m)}|^p \rightarrow 0 \text{ for } m \neq n \text{ as } k \rightarrow \infty. \tag{26}$$

This together with (23) allows, with an error of order ϵ , to reduce integration to the balls $B_R(-\alpha_k^{(n)})$, where the contribution of the terms $w^{(m)}(\cdot + \alpha_k^{(m)})$, $m \neq n$, will go to zero. ■

In the space $H^1(\mathbb{R}^N; \mathbb{C})$ one can also consider a set of dislocations of the following form:

$$g_\alpha u(x) = e^{i\Phi(\alpha,x)} u(x + \alpha) \tag{27}$$

with a C^1 -function Φ that is odd in α and has a uniformly bounded gradient. This set is not a group, but it satisfies (I-III), and the concentrated weak convergence in this case implies L^p -convergence as well. This requires only trivial modifications in the proof of Lemma 3.1. Lemma 3.2 will be also true if one assumes that F is dependent on $|u|$ alone.

We now let $H = D^{1,2}(\mathbb{R}^N)$, $N > 2$, and let D be the product of the group of shifts g_α and the group of dilations:

$$h_t u(x) := t^{\frac{N-2}{2}} u(tx), \quad t \in (0, \infty). \quad (28)$$

Note that every element of D can be represented as $u \mapsto h_t u(\cdot + \alpha)$ with some $\alpha \in \mathbb{R}^N$ and $t > 0$ and that it is a unitary operator. Verification of the property (IIa) is similar to the case of H^1 with shifts, if one takes into account that

$$h_{t_k}(\cdot + \alpha_k) \rightarrow 0 \Leftrightarrow t_k \rightarrow 0 \text{ or } t_k \rightarrow \infty \text{ or } |\alpha_k| \rightarrow \infty. \quad (29)$$

Therefore Theorem 2.3 applies also in this case.

In many applications one can benefit from using Theorem 2.3 in a larger space and then study the restriction to the original space. We apply this approach below to restrictions from $D^{1,2}$ to H^1 , while in the next section we will study restrictions to $H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$.

Lemma 3.3. *Let H and D be as above and assume the conditions of Theorem 2.3 and moreover, that*

$$\|u_k\|_{L^2(\mathbb{R}^N)} \leq C, \quad (30)$$

with $C > 0$. Then to the assertions of Theorem 2.3 one can add that in the terms $g_k^{(n)} w^{(n)} := h_{t_k^{(n)}} w^{(n)}(\cdot + \alpha_k^{(n)})$ none of the dilation parameters $t_k^{(n)}$ will converge to zero and that for every n such that $t_k^{(n)}$ is bounded, $w^{(n)} \in H^1(\mathbb{R}^N)$.

Proof. When $t_k^{(n)}$ is bounded, due to Theorem 2.3,

$$w^{(n)} := \text{w-lim}(h_{1/t_k^{(n)}} u_k)(\cdot - \alpha_k^{(n)}), \quad (31)$$

and since the right hand side is bounded in L^2 , the function $w^{(n)}$ is a weak limit also in the sense of H^1 . Without loss of generality assume

that $w^{(n)} \neq 0$. If $t_k^{(n)} \rightarrow 0$, the L^2 norm of the sequence in the right hand side of (31) will converge to zero, which implies (by Fatou lemma) that $\|w^{(n)}\|_{L^2(\mathbb{R}^N)} = 0$, a contradiction. ■

When the set of dislocations includes dilations, concentrated weak convergence implies also convergence in L^{2^*} .

Lemma 3.4. *Let D be as above and let u_k be a bounded sequence in $H^1(\mathbb{R}^N)$, $N > 2$. If $u_k \xrightarrow{cw} 0$, then $\|u_k\|_{L^p} \rightarrow 0$ for $2 < p \leq 2^*$.*

Proof. For $2 < p < 2^*$ this follows from Lemma 3.1. Consider now the case $p = 2^*$. Let $t_k \rightarrow \infty$ be such that

$$\int_{|u_k| > t_k^{\frac{N-2}{2}}} |u_k|^{2^*} \rightarrow 0. \tag{32}$$

Let v_k be defined by

$$u_k = t_k^{\frac{N-2}{2}} v_k(t_k x). \tag{33}$$

Then

$$\int_{|u_k| < t_k^{\frac{N-2}{2}}} |u_k|^{2^*} = \int_{v_k < 1} |v_k|^{2^*} \leq \int_{\mathbb{R}^N} |v_k|^{2+\frac{1}{N}} \rightarrow 0, \tag{34}$$

since $v_k \xrightarrow{cw} 0$ and $2 < 2 + \frac{1}{N} < 2^*$. Relation (32) combined with (34) verifies the lemma. ■

Remark 3.5. We would like to remark that all terms in (5) with their dilation parameters $t_k^{(n)}$ and their reciprocal values $t_k^{(n)-1}$ bounded, can be replaced, on a renumbered subsequence, with terms of the form $w^{(n)}(\cdot + \alpha_k^n)$.

Lemma 3.6. *Let u_k satisfy the assumptions of Lemma 3.3 and let $w^{(n)}$, $\alpha_k^{(n)}$, and $t_k^{(n)}$ be as provided by the lemma. If $F : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying (23) then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k) + |u_k|^{2^*} = \sum_{n \in \mathbb{N}_1} \int F(w^{(n)}) + \sum \int |w^{(n)}|^{2^*} \tag{35}$$

where \mathbb{N}_1 consists of all indices n such that $t_k^{(n)}$ is bounded.

Proof. When $n \notin \mathbb{N}_1$, $t_k^{\frac{N-2}{2}} w^{(n)}(t_k^{(n)}(\cdot + \alpha_k^{(n)})) \rightarrow 0$ in L^p , $2 < p < 2^*$. Thus due to Lemma 3.4 and using Remark 3.5,

$$u_k - \sum_{n \in \mathbb{N}_1} w^{(n)}(\cdot + \alpha_k^{(n)}) \rightarrow 0 \text{ in } L^p, 2 \leq p < 2^* \quad (36)$$

with $|\alpha_k^{(n)} - \alpha_k^{(m)}| \rightarrow \infty$ for $n \neq m$; $n, m \in \mathbb{N}_1$. Thus, by Lemma 3.2 (once one notices that the proof of the lemma uses the asymptotics of (2.3) as L^p -asymptotics),

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k) = \sum_{n \in \mathbb{N}_1} \int_{\mathbb{R}^N} F(w^{(n)}). \quad (37)$$

The proof that

$$\int |u_k|^{2^*} \rightarrow \sum_{n \in \mathbb{N}} \int |w^{(n)}|^{2^*}. \quad (38)$$

is based on the conclusion of Lemma 3.4:

$$u_k - \sum_n t_k^{(n) \frac{N-2}{2}} w^{(n)}(t_k^{(n)}(\cdot + \alpha_k^{(n)})) \rightarrow 0 \text{ in } L^{2^*} \quad (39)$$

with either $t_k^{(n)}/t_k^{(m)} \rightarrow \infty$ or $t_k^{(m)}/t_k^{(n)} \rightarrow \infty$ or $|\alpha_k^{(n)} - \alpha_k^{(m)}| \rightarrow \infty$ for all $m \neq n$, which leads to separation of the contributions of different terms to the integral. The details are left to the reader. ■

4 Traces of dislocated weak limits on $H_0^1(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open set. In this section we consider the question when, for a bounded sequence $u_k \in H_0^1(\Omega)$, extended to $H(\mathbb{R}^N)$ one gets dislocated weak limits $w^{(n)}$ in Theorem 2.3, that can be identified as elements of $H_0^1(\Omega)$. We start with directly defining a class of open sets by this property (*asymptotically contractive sets*) and then give two sufficient geometric conditions of asymptotic contractiveness.

At the end of the section we give two examples of compactness in Sobolev embeddings.

4.1 Asymptotically contractive sets

Definition 4.1. An open set $\Omega \subset \mathbb{R}^N$ will be called asymptotically contractive if for every sequence $u_k \in H_0^1(\Omega)$ and every sequence $\alpha_k \in \mathbb{R}^N$ such that $u_k(\cdot + \alpha_k)$ converges weakly in $H^1(\mathbb{R}^N)$ to some w , there exists a $\gamma \in \mathbb{R}^N$ such that $w(\cdot - \gamma) \in H_0^1(\Omega)$.

An open set $\Omega \subset \mathbb{R}^N$ will be called asymptotically null if for every sequence $u_k \in H_0^1(\Omega)$ and every sequence $\alpha_k \in \mathbb{R}^N$, $|\alpha_k| \rightarrow \infty$, $u_k(\cdot + \alpha_k) \rightarrow 0$.

This definition allows to formulate following statements:

Proposition 4.2. Let $H = H^1(\mathbb{R}^N)$, let D be the group of shifts. If $\Omega \subset \mathbb{R}^N$ is an asymptotically contractive open set and $u_k \in H_0^1(\Omega)$, then the conclusions of Theorem 2.3 hold with $w^{(n)} \in H_0^1(\Omega)$, $n \in \mathbb{N}$.

Proposition 4.3. Let $H = H^1(\mathbb{R}^N)$, let D be the product group of shifts and dilations as defined in Section 3. If $\Omega \subset \mathbb{R}^N$ is an asymptotically contractive open set and $u_k \in H_0^1(\Omega)$, then the conclusions of Theorem 2.3 hold with $w^{(n)} \in H_0^1(\Omega)$ whenever $n \in \mathbb{N}_1$.

4.2 Geometric characterization of asymptotically contractive sets

Let $\alpha_k \in \mathbb{R}^N$, $u_k \in D^{1,2}(\mathbb{R}^N)$ and let $u_k(\cdot - \alpha_k) \rightharpoonup w$. Then there exists a set $Y \subset \mathbb{R}^N$ of measure zero such that for all $x \notin Y$, $u_k(x - \alpha_k) \rightarrow w(x)$. Then, if for a given $x \notin Y$ there is a subsequence where $u_k(x - \alpha_k) = 0$, then $w(x) = 0$. In particular, if $u_k \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set, then whenever $x \notin \Omega + \alpha_k$ holds on a subsequence, $w(x) = 0$. We have to note only that one has to replace u_k by their C_0^∞ -approximations in the gradient norm, so that questions about trace on the boundary will not arise. We conclude that except for points in Y , w is non-zero only for such x that $x \in \Omega + \alpha_k$ for all k sufficiently large, that is, for

$$x \in \liminf(\Omega + \alpha_k) := \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} (\Omega + \alpha_k). \quad (40)$$

Lemma 4.4. Let $u_k \in H_0^1(\Omega)$, $\alpha_k \in \mathbb{R}^N$ and let $u_k(\cdot + \alpha_k) \rightharpoonup w$. Then there exist a set C of zero capacity and a set Y of zero measure such that

$$x \notin \liminf(\Omega + \alpha_k) \cup Y \Rightarrow w(x) = 0, \quad (41)$$

$$x \notin \limsup(\Omega + \alpha_k) \cup C \Rightarrow w(x) = 0 \quad (42)$$

Proof. The first assertion of the lemma follows from (40). The proof of the second assertion is similar if one notices that there are convex combinations of $u_k(\cdot - \alpha_k)$ that converge in norm and thus quasi-everywhere, so that $w(x) \neq 0$ if x , apart from a set of zero capacity, is not in the union of translated Ω over any tail of the sequence, i.e. not in $\limsup(\Omega + \alpha_k) := \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} (\Omega + \alpha_k)$. Subadditivity of capacity implies (42). ■

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $u_k \in H_0^1(\mathbb{R}^N)$, $u_k(\cdot - \alpha_k) \xrightarrow{H^1(\mathbb{R}^N)} w$. If any of the following two conditions holds,*

(i) *there is a set Y of measure 0 and a sequence $\gamma_k \in \mathbb{R}^N$, $\gamma_k \rightarrow 0$, such that $\liminf(\Omega + \alpha_k - \gamma_k) \subset \Omega \cup Y$ and $\partial\Omega = \partial(\mathbb{R}^N \setminus \bar{\Omega})$,*
or

(ii) *there exists a subsequence $\{\beta_k\} \subset \{\alpha_k\}$, a sequence $\gamma_k \in \mathbb{R}^N$, $\gamma_k \rightarrow 0$, and a set C of capacity 0 such that $\limsup(\Omega + \beta_k - \gamma_k) \subset \Omega \cup C$,*

then $w \in H_0^1(\Omega)$.

Proof. Without loss of generality we can assume that $\gamma_k = 0$ since $z_k(\cdot - \gamma_k) - z_k \rightarrow 0$ for any bounded sequence γ_k .

Indeed, let $v \in C_0^\infty(\mathbb{R}^N)$. Then

$$|(z_k(\cdot - \gamma_k) - z_k, v)| \leq \|z_k\| \|v(\cdot + \gamma_k) - v\| \rightarrow 0. \quad (43)$$

We consider w refined and thus defined quasi-everywhere (i.e., modulo sets of capacity zero. For definitions and statements from potential theory used in this proof we refer the reader to [1], cf. also [14].) Under condition (i), $w = 0$ a.e. on the open set $\mathbb{R}^N \setminus \bar{\Omega}$ implies that $w = 0$ q.e. on the closure of that set. Under the geometric assumption of (i) $w = 0$ q.e. in the complement of Ω . Same conclusion results from (ii) directly. The assertion of the theorem follows immediately from the Hedberg trace theorem. ■

Remark 4.6. We would like to formulate a conjecture that both conditions (i) and (ii) of Lemma 4.5 can be replaced by a milder sufficient condition

$$\exists \gamma_k \rightarrow 0, \liminf(\Omega + \alpha_k - \gamma_k) \subset \Omega \cup Y. \quad (44)$$

This would be true if the weak convergence in H^1 implied convergence quasi-everywhere. However, the latter statement to our best knowledge is itself an open conjecture.

Lemma 4.5 provides geometric sufficient conditions for an open set to be asymptotically contractive:

Corollary 4.7. *An open set $\Omega \subset \mathbb{R}^N$ is asymptotically contractive, if for every sequence $\alpha_k \in \mathbb{R}^N$ there is a renumbered subsequence and a $\gamma \in \mathbb{R}^N$ such that any of two conditions (i), (ii) of Lemma 4.5, with $\gamma_k \rightarrow 0$ replaced by $\gamma_k \rightarrow \gamma$, holds.*

Corollary 4.8. *The following sets are asymptotically contractive:*

- (i) *An asymptotically null set (in particular any open set Ω such that whenever $|\alpha_k| \rightarrow \infty$, $|\liminf(\Omega + \alpha_k)| = 0$, e.g. a bounded open set);*
- (ii) *An open set $\Omega \subset \mathbb{R}^N$ such that for every $\alpha \in \mathbb{Z}^N$, $\Omega + \alpha \subset \Omega$ (and in particular, \mathbb{R}^N);*
- (iii) *An open set $\Omega \subset \mathbb{R}^N$ such that for certain $M \in \mathbb{N}$, $M < N$, and for every $\alpha \in \mathbb{Z}^M$, $\Omega + \alpha = \Omega$, and, moreover, that there is a bounded set $\omega \subset \mathbb{R}^{N-M}$ such that $\Omega \subset \omega \times \mathbb{R}^M$ (this includes any cylindrical domain);*
- (iv) *A union of an asymptotically contractive set and an asymptotically null set;*
- (v) *A finite union of cylindrical domains.*

The proof in all the cases is elementary. We provide only the sketches for two least trivial statements. In the case (i), by Lemma 4.4, all dislocated limits are equal zero almost everywhere on \mathbb{R}^N , and thus their trace on $H_0^1(\Omega)$ is zero. The case (iv) is proved by a partition of unity.

Corollary 4.9. *The following sets are not asymptotically contractive:*

- (i) *An open set $\Omega \subset \mathbb{R}^N$ (other than \mathbb{R}^N itself) which for every $R > 0$ contains a ball of radius R ;*
- (ii) *An open cylinder from which one has subtracted a compact subset of positive capacity;*

(iii) A product $\omega \times (0, \infty)$, where ω is an open set (in particular, a half-space).

The proof is elementary and is based on finding a sequence α_k such that $\liminf(\Omega + \alpha_k)$ contains a closed set Z with $\text{cap}(Z \setminus \Omega) > 0$ taking a function with support in Z and translating shifting it back while multiplying it with an appropriate cut-off function.

4.3 Examples of compactness in Sobolev imbeddings

Example 1. Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be an open set. Due to Theorem 2.3, the imbedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, $p \in (2, 2^*)$, is compact if and only if Ω is asymptotically null. Indeed, if Ω is asymptotically null, for every bounded sequence $u_k \in H_0^1(\Omega)$, the dislocated limits $w^{(n)}$ will be zero for all $n > 1$. In other words, for an appropriate subsequence, $u_k \xrightarrow{cw} w^{(1)} := w\text{-lim } u_k$, which by Lemma 3.1 implies convergence in L^p . Conversely, if Ω is not asymptotically null, there is a sequence α_k , $|\alpha_k| \rightarrow \infty$, and a sequence u_k , such that

$$u_k(\cdot + \alpha_k) \rightharpoonup w \neq 0 \text{ in } H^1(\mathbb{R}^N). \quad (45)$$

Without loss of generality, we can assume that $u_k \rightharpoonup 0$, since we can always subtract from u_k its weak limit while preserving (45). Then by Lemma 3.2, the L^p -norm of u_k for k large will be bounded away from zero, implying lack of compactness.

By Corollary 4.8 the condition $\forall \alpha_k \in \mathbb{R}^N$, $|\alpha_k| \rightarrow \infty$, $|\liminf(\Omega + \alpha_k)| = 0$ is a sufficient condition for compactness. Related conditions have been known in literature for decades (e.g. [2],[4]; see [14] for further references and a necessary and sufficient condition of compactness in terms of capacity).

Example 2. The following result was proved (for less general domains) in [6].

Let G_V be a group of rotations in a subspace V of \mathbb{R}^N of dimension at least 2 (extended to \mathbb{R}^N by the identity on V^\perp). Let $\Omega \subset \mathbb{R}^N$ satisfy $g\Omega = \Omega$ for all $g \in G_V$ and be such that for every sequence α with $\alpha_k \in \mathbb{R}^N$ satisfying $\sup_k |\pi_V \alpha_k| < \infty$ and $\alpha_k \rightarrow \infty$, the asymptotic set $\liminf(\Omega + \alpha_k)$ has measure zero. Let $H = \{u \in H_0^1(\Omega) : u \circ g = u\}$. Then H is compactly imbedded into L^p , $2 < p < 2^*$.

This statement can be proved as follows. We will consider the functions u_k extended by zero to elements of $H^1(\mathbb{R}^N)$. Without loss of generality assume that u_k weakly converges to zero. Let $\alpha_k \in \mathbb{R}^N$, $|\alpha_k| \rightarrow \infty$ and assume that

$$0 \neq w = \text{w-lim } u_k(\cdot + \alpha_k). \quad (46)$$

If $\sup_k |\pi_V \alpha_k| < \infty$, then, by assumption, $|\liminf(\Omega + \alpha_k)| = 0$ and by Lemma 4.4, $w = 0$ a.e., so from (46) follows that, on a renamed subsequence, $|\pi_V \alpha_k| \rightarrow \infty$. This implies in turn that if $g \in G_V$, $g \neq id$, then $|g\alpha_k - \alpha_k| = |\pi_V(g - id)\alpha_k| \rightarrow \infty$. Using invariance of the functions w and u_k , one obtains $w = w(g^{-1}\cdot) = \text{w-lim } u_k(g^{-1}\cdot + \alpha_k) = \text{w-lim } u_k(\cdot + g\alpha_k)$.

Now, taking infinitely many distinct g 's in Theorem 2.3 we get

$$+\infty = \sum_g \|w \circ g^*\|^2 \leq \liminf \|u_k\|^2, \quad (47)$$

a contradiction. ■

5 Applications to elliptic problems

5.1 Existence of the ground state solution on asymptotically contractive domains

We consider the following variational problem.

$$C_\Omega := \sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega |u|^{p+1} dx \quad (48)$$

with an unbounded open set $\Omega \subset \mathbb{R}^N, N > 2$.

Theorem 5.1. *If the set Ω is asymptotically contractive and $2 < p+1 < 2^*$, then the supremum in (48) is attained and the maximizer is, modulo a constant multiple, a solution to*

$$-\Delta u + u = u^p, u > 0 \text{ in } \Omega. \quad (49)$$

Proof. Let u_k be a maximizing sequence for (48). We consider the decomposition of Theorem 2.3 for u_k as a sequence in $H^1(\mathbb{R}^N)$. Since Ω is asymptotically contractive, $w^{(n)} \in H_0^1(\Omega)$ by Proposition 4.2.

Every $w^{(n)}$ has to be, modulo a constant multiple, a maximizer for (48): otherwise we replace $w^{(n)}$ in the decomposition of Theorem 2.3 by a function that gives a higher value of $\int_{\Omega} |u_k|^{p+1} dx$ with the same Sobolev norm. It is easy to see that the Sobolev norm of the new decomposition (in the limit, modulo an extraction) does not increase, while the value of $\int_{\Omega} |u_k|^{p+1} dx$ on the new sequence becomes greater than and bounded away from C_{Ω} , a contradiction. ■

5.2 A problem with a critical exponent on unbounded domains

Let

$$\lambda_0 := \inf_{\int_{\Omega} u^2=1, u \in C_0^{\infty}(\Omega)} \int_{\Omega} |\nabla u|^2. \quad (50)$$

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$, $N > 3$, be asymptotically contractive and assume that $\lambda_0 > 0$. If $\lambda \in (0, \lambda_0)$, then the problem*

$$-\Delta u - \lambda u = u^{2^*-1}, u \in H_0^1(\Omega), \quad (51)$$

has a positive solution.

This result is an immediate extension of the well known result [3] of Brezis and Nirenberg to the case of unbounded domain. There exists an earlier generalization by [16] which, in addition to asymptotic contractiveness, also requires that domains are asymptotically cylindrical.

Proof. Consider the following variational problem:

$$c(\Omega, \lambda) = \sup_{\int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} u^2 = 1} \int_{\Omega} |u|^{2^*}, \lambda > 0. \quad (52)$$

The constant $c(\Omega, \lambda)$ is finite only for $\lambda < \lambda_0$. To verify solvability of (51) we will show that the maximum in (52) is attained. It is known ([3], Lemma 1.1, cf. also [12]) that

$$c(\Omega, \lambda) \geq c(B_{\epsilon}, \lambda) > c(\mathbb{R}^N, 0) \text{ for all } \lambda > 0, \quad (53)$$

where B_{ϵ} is an open ball of radius $\epsilon > 0$ contained in Ω . We will use this relation to exclude maximizing sequences for $c(\Omega, \lambda)$ that involve dilations of a maximizer for $c(\mathbb{R}^N, 0)$. The argument is as follows. Let u_k

be a maximizing sequence for $c(\Omega, \lambda)$ which we consider as a sequence of elements in $H_0^1(\mathbb{R}^N)$. We apply Proposition 4.3. Let $Q_\lambda(u) := \int (|\nabla u|^2 - \lambda u^2)$. Then we will have, on a renamed subsequence, with \mathbb{N}_1 as in Lemma 3.6,

$$c(\Omega, \lambda) = \sum_n \int |w^{(n)}|^{2^*}, \tag{54}$$

and

$$\sum_{n \in \mathbb{N}_1} Q_\lambda(w^{(n)}) + \sum_{n \notin \mathbb{N}_1} Q_0(w^{(n)}) \leq 1. \tag{55}$$

The latter equation requires to reproduce the argument of (16)-(19) with the square $D^{1,2}$ -norm replaced by Q_λ . If for any $n \notin \mathbb{N}_1$, $w^{(n)}$ is not a maximizer (up to a constant multiple) for $c(\mathbb{R}^N, 0)$, then u_k is not a maximizing sequence, since subtracting this $w^{(n)}$ from u_k and adding instead a sequence of suitable dilations of the Talenti function, multiplied by a smooth cut-off function supported on Ω , will preserve the bound (55) while increasing the value of $\int |u_k|^{2^*}$ above $c(\Omega, \lambda)$. A simple subadditivity argument shows that one would also increase the value of $\int |u_k|^{2^*}$ if one had more than one non-zero $w^{(n)}, n \notin \mathbb{N}_1$, which we will denote as $w^{(n_0)}$. If for all $n \in \mathbb{N}_1$, $w^{(n)} = 0$, then we would have $c(\Omega, \lambda) = c(\mathbb{R}^N, 0)$, a contradiction to (53). A similar subadditivity argument would provide that $w^{(n)}$ will be non-zero for at most one $n \in \mathbb{N}_1$, say n_1 , and that $\int |u|^{2^*}$ will not attain its maximal value unless $w^{(n_0)} = 0$ and $Q_\lambda(w^{(n_1)}) = 1$. Thus $u_k(\cdot - \alpha_k^{(n_1)}) \rightarrow w^{(n_1)}$ in H^1 . Consequently, $w^{(n_1)}$ is the maximizer for $c(\Omega, \lambda)$. ■

5.3 Symmetric ground state solution on an exterior domain

The following result is proved in [17], using Theorem 2.3 to verify the (PS)-condition; dislocated terms in the expansion for the Palais-Smale sequence are excluded by energy estimates.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^N, N > 2$, be a complement of a star-shaped closed set and let $f \in C(\mathbb{R})$ satisfy*

$$\forall \epsilon > 0 \exists C_\epsilon > 0 : |f(s)| \leq \epsilon |s| + C_\epsilon |s|^{2^*-1}, \tag{56}$$

$$\lim_{s \rightarrow \infty} f(s)/s = +\infty. \tag{57}$$

Let

$$T_n = \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) & 0 & \cdots & 0 \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (58)$$

Let $H_n = \{u \in H_0^1(\Omega) : u \circ T_n = u\}$. Then there exists a $n_0 \in \mathbb{N}$ dependent on N and f such that for every Ω satisfying $T_n\Omega = \Omega$ with a $n > n_0$, the problem

$$-\Delta u + u = f(u); u \in H_n \setminus \{0\} \quad (59)$$

has a nontrivial solution.

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