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# DECOMPOSITION AND MOSER'S LEMMA

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**Abstract**

Using the idea of the optimal decomposition developed in recent papers [EK2] by the same authors and in [CUK] we study the boundedness of the operator $Tg(x) = \int_x^1 g(u)\, du/u$, $x \in (0,1)$, and its logarithmic variant between Lorentz spaces and exponential Orlicz and Lorentz-Orlicz spaces. These operators are naturally linked with Moser's lemma, O'Neil's convolution inequality, and estimates for functions with prescribed rearrangement. We give sufficient conditions for and very simple proofs of uniform boundedness of exponential and double exponential integrals in the spirit of the celebrated lemma due to Moser [Mo].

## 1 Introduction

One of the most awkward features of Orlicz spaces is the definition of the norm. As a rule of thumb this often prevents a straightforward usage of the $L^p$-spaces techniques and approach. The problem sometimes lies in the very analytic hardware for these spaces because the calculus with general Young functions is much more difficult than handling powers. Some of the Orlicz spaces frequently used in applications, such as logarithmic Lebesgue spaces, exponential spaces and Zygmund spaces, permit an alternative extrapolation approach. This part of Orlicz space theory has been developing after the quantitative behaviour of norms of various operators between function spaces near $L^1$ and $L^\infty$ had become known (this concerns especially classical operators of harmonic analysis). Let us recall that special extrapolation techniques, making it possible to extrapolate the Lebesgue spaces towards exponential Orlicz spaces and logarithmic Lebesgue spaces, are the subject of the well-known paper Yano [Y], to list at least one of the first basic works in this field. Among recent papers dealing with extrapolation constructions for special Orlicz

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spaces let us recall e.g. [ET1] (also [ET2]). The abstract extrapolation theory (Jawerth and Milman [JM], Milman [Mi]) stems from classical extrapolation theorems and on this abstract level, the identification of the result of the so-called $\Sigma$-method applied to $L^p$-spaces goes via plugging the $K$-functional into the considerations and then employing the fact that the Zygmund spaces can be interpreted also as special Lorentz-Zygmund spaces (see [BR], [BS], Chapter 4). One usually works with "global" norms here. In [EK2] and [CUK] we have suggested a decomposition method, making it possible to extrapolate towards Orlicz spaces of exponential type just using suitable little pieces of the extrapolating Lebesgue norms. It turns out that this makes things often much easier as we shall see also later here.

Orlicz spaces of exponential type appear naturally as target spaces for Sobolev imbeddings in the limiting case. We shall not try to trace out the history here, starting in the 1960s and connected with works by Trudinger, Pokhozhaev, and others. One of the basic contributions, triggering extensive research in this area is Moser's paper [Mo] with an elegant reduction of the $n$-dimensional imbedding problem to a one-dimensional question (see next section for more details). Moser's proof actually uses elementary means but in a very sophisticated way. A less complicated proof was given later in Adams [A] and Jodeit [J]. Basically, after the above reduction, the problem can be formulated as follows: Given $f \in L^{p'}(0,\infty)$, $\|f\|_{L^{p'}} \le 1$, we consider the operator

$$Ff(x) = \int_0^x f(\tau)\,d\tau, \qquad x \in (0,\infty), \tag{1.1}$$

and we look for a uniform bound for $\int_0^\infty \exp[(\alpha_n Ff(x))^p - x]\,dx$ with some $\alpha_n$ independent of $f$. Alternatively, considering

$$Tg(x) = \int_x^1 \frac{g(t)}{t}\,dt, \qquad x \in (0,1), \tag{1.2}$$

we pose a similar question for $\int_0^1 \exp[\beta_n Tg(x)]^p\,dx$. The operator $T$ naturally arises from the Hardy operator $F$ through the substitution $\tau = \log(1/t)$ and the original condition $\|f\|_{L^{p'}} \le 1$ in [Mo] turns into $\int_0^1 (g(t)^{p'}/t)\,dt \le 1$. This setting is from [J], where the restriction $p \ge 2$ appearing in Moser's famous lemma [Mo] was removed. Let us recall the main result from [J]:

**Theorem 1.1.** *Given a non-negative measurable function $g$ on $(0,1)$ we put $Tg(t) = \int_t^1 g(u)\,du/u$. Let $1 < p < \infty$. Then there is $c_p > 0$ such that*

$$\int_0^1 \exp(Tg(t))^p\,dt \le c_p$$

*for all $g$ such that $\int_0^1 g(u)^{p'}\,du/u \le 1$.*

Here we shall tackle the operator $T$ from (1.2) under more general assumptions on $g$ in terms of Lorentz spaces. We give extremely simple proofs of the existence of uniform constants for exponential norms of $Tg$, using decomposition theorems from [EK2] and [CUK]. We shall not pursue the problem of the best constant $\beta_n$, although this would be possible.

## 2 The decomposition technique

Let $\Omega \subset \mathbb{R}^n$ be measurable and denote its Lebesgue $n$-measure by $|\Omega|$. For $1 \le p \le \infty$, $L_p = L_p(\Omega)$ is the usual Lebesgue space with norm denoted by $\|\cdot\|_p$ or $\|\cdot\|_{p,\Omega}$, depending on whether we need to emphasize the domain. The *non-increasing rearrangement* of a measurable function on $\Omega$ is defined as

$$f^*(t) = \inf\{s > 0 : |\{x : |f(x)| > s\}| \le t\}, \qquad t > 0.$$

For $1 \le p, q \le \infty$, the Lorentz space $L_{p,q} = L_{p,q}(\Omega)$ is defined as the linear set of all functions $f$ for which $\|f\|_{p,q,\Omega} < \infty$. Here the norm $\|f\|_{p,q,\Omega}$ is given by

$$\|f\|_{p,q,\Omega} = \left(\frac{q}{p}\int_0^{|\Omega|} t^{q/p} f^*(t)^q \frac{dt}{t}\right)^{1/q}$$

if $1 \le p, q < \infty$, by

$$\|f\|_{p,\infty,\Omega} = \sup_{t\in(0,|\Omega|)} t^{1/p} f^*(t)$$

if $1 \le p < \infty$ and $q = \infty$, and finally we put

$$\|f\|_{\infty,\infty,\Omega} = \|f\|_{\infty,\Omega}.$$

If $\Phi$ is a Young function, that is, an even, convex function on $\mathbb{R}^1$, increasing on $(0,\infty)$ and such that $\Phi(0)=0$, we define the Orlicz class $\widetilde{L}_\Phi(\Omega)$ as the set of all functions $f$ for which $m(f,\Phi,\Omega)=\int_\Omega \Phi(f(x))\,dx<\infty$. The Orlicz space $L_\Phi(\Omega)$ is the linear hull of $\widetilde{L}_\Phi(\Omega)$ endowed with the (Luxemburg) norm $\|f\|_{\Phi,\Omega}=\|f\|_{L_{\Phi(\Omega)}}=\inf\{\lambda>0 : m(f/\lambda,\Phi,\Omega)\le 1)\}$ (i.e. Minkowski's functional of the mapping $f\mapsto m(f,\Phi,\Omega)$). The symbol $\Omega$ can be omitted if no confusion arises. For the special case in which $\Phi(t)=\exp t^\alpha-1$ for some $\alpha>0$, we shall use the brief notation $L^{\exp t^\alpha}$.

In the sequel we shall largely work with real measurable functions defined and/or supported on the interval $(0,1)$. This is merely a technical assumption corresponding to $|\Omega|=1$; any bounded interval can be considered. If $I\subset(0,1)$, then we define the localized Lorentz norm $\|f\|_{p,q,I}$ similarly as above, where the integration and/or taking the sup is restricted to the interval $I$.

As all the function spaces in the following are rearrangement-invariant, we shall assume without loss of generality that all functions are non-negative.

Let $\alpha>0$. It is well known that the condition

$$\int_0^1 \exp\left(|\lambda f(x)|^\alpha\right)dx<\infty$$

for some $\lambda>0$ is equivalent to the condition

$$\sup_{k\in\mathbb{N}} k^{-1/\alpha}\|f\|_{k,(0,1)}<\infty. \tag{2.1}$$

This follows by the ratio test for the convergence of the corresponding Taylor series for the exponential function and it is one of the most prominent examples of extrapolation of norms. In [EK2] we have proved that a condition much weaker at first sight is necessary and sufficient; this results from replacing the norms $\|f\|_{k,(0,1)}$ by $\|f\|_{k,I_k}$ in the extrapolation condition (2.1). Here and in the following we use the notation from [EK2] and [CUK]: We decompose the interval $(0,1)$ (up to a set of measure zero) into the sequence of intervals

$$I_k=(e^{-k},e^{-k+1}),\qquad k=1,2,\ldots.$$

This characterization has been extended in [CUK] to include the extrapolation of $\|f\|_{k,\infty,I_k}$ and the scales of Lorentz spaces have been employed to obtain an "optimal" decomposition of functions in the Orlicz-Lorentz spaces. Let us also observe that the decomposition idea from [EK2] has been used in the context of more general logarithmic Lorentz-Zygmund spaces by Neves [N].

For the reader's convenience we recall the characterizations of $L^{\exp t^a}$ from [EK2] and [CUK].

**Theorem 2.1. ([CUK], [EK2]).** *Given a function $f$ and $a > 0$, the following are equivalent:*

(i) $f \in L^{\exp t^a}$;

(ii) $\displaystyle\sup_k \frac{\|f\|_{k,(0,1)}}{k^{1/a}} < \infty$;

(iii) $\displaystyle\sup_k \frac{\|f\|_{k,I_k}}{k^{1/a}} < \infty$.

(iv) $\displaystyle\sup_k \frac{\|f\|_{k,\infty,(0,1)}}{k^{1/a}} < \infty$;

(v) $\displaystyle\sup_k \frac{\|f\|_{k,\infty,I_k}}{k^{1/a}} < \infty$.

**Remark 2.2.** The conditions (ii) and (iii) were considered in [EK2] and their weak versions (iv) and (v) in [CUK]. For the equivalence of (iv) and (i) in the last theorem see also the abstract framework of the $\Sigma$-method ([Mi]).

Given a function $f$ on $\Omega \subset \mathbb{R}^n$ we shall also use its *non-increasing radially symmetric rearrangement* $f^\#$ (*symmetric rearrangement* in the sequel) defined by

$$f^\#(x) = f^*(\omega_n |x|^n), \qquad x \in \mathbb{R}^n,$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Note that $\|f^\#\|_{L_q(\mathbb{R}^n)} = \|f^*\|_{L_q(0,\infty)}$ for all $q$, $1 \le q \le \infty$. Plainly

$$f^*(t) = f^\#\left(\left(\frac{t}{\omega_n}\right)^{1/n} \mathbf{e_1}\right),$$

where $\mathbf{e_1}$ denotes the unit vector in the direction of the $x_1$-axis.

Recall the (non-trivial) fact that if $f$ is smooth (Lipschitz, for instance), then $f^{\#}$ has partial derivatives and if $\Phi$ is a convex increasing function on $[0,\infty)$, $\Phi(0)=0$, then

$$\int_{\mathbb{R}^n} \Phi(|\nabla f^{\#}(x)|)\,dx \le \int_{\mathbb{R}^n} \Phi(|\nabla f(x)|)\,dx. \tag{2.2}$$

The inequality (2.2) has found many applications in analysis and has been treated in numerous papers; we refer to a comprehensive survey paper of Talenti in [T] with detailed proofs, references, exposition of the history etc.

Now suppose that $|\Omega| = 1$ and consider the change of variables $t = e^{-y}$. We get the function

$$w(y) = f^*(e^{-y}) = f^{\#}\left(\left(\frac{e^{-y}}{\omega_n}\right)^{1/n}\mathbf{e_1}\right), \qquad y \in (0,\infty).$$

We have

$$w'(y) = -\frac{e^{-y/n}}{n\omega_n^{1/n}}\frac{\partial f^{\#}}{\partial x_1}\left(\left(\frac{e^{-y}}{\omega_n}\right)^{1/n}\mathbf{e_1}\right)$$

and

$$\|w'\|^q_{L_q(0,\infty)} = \frac{1}{n^q\omega_n}\int_{\mathbb{R}^n}\left\|\nabla f^{\#}(x)\right\|^q |x|^{q-n}\,dx.$$

Observe that if $q = n$, then the power weight on the right hand side equals 1; this is the core of Moser's reduction of the original imbedding problem to an $\mathbb{R}^1$-problem.

Later we shall need a formula for the Lorentz norm after the substitution $t = \log(1/\sigma)$. This is just a plain computation: Given a non-increasing, non-negative $f$ on $(0,\infty)$ supported in $[0,1]$, and $\alpha,\beta > 0$, then

$$\int_0^\infty [t^{1/\beta} f(t)]^\alpha \frac{dt}{t} = \int_0^1 [g(\sigma)(\log(1/\sigma)^{1/\beta}]^\alpha \frac{d\sigma}{\sigma\log(1/\sigma)},$$

where

$$g(\sigma) = f(\log(1/\sigma)), \qquad \sigma \in (0,1). \tag{2.3}$$

All positive constants whose exact value is not important for our purposes will be denoted by $c$.

We refer to [KR], and [RR] for the theory of Orlicz spaces and integral operators acting on them.

# 3 Moser's lemma in Lebesgue spaces via decomposition

**Theorem 3.1.** *Let $1 < p < \infty$, $p' = p/(p-1)$. Then there exists $c > 0$ such that*

$$\sup_{k\in\mathbb{N}} \frac{\|Tg\|_{k,\infty,I_k}}{k^{1/p}} \le c$$

*for every non-negative measurable function $g$ such that*

$$J(g) = \int_0^1 \frac{g(s)^{p'}}{s}\, ds \le 1. \tag{3.1}$$

**Proof.** Since $Tg$ is non-increasing on $(0,1)$ we have, by Hölder's inequality,

$$\begin{aligned}\|Tg\|_{k,\infty,I_k} &= \sup_{t\in I_k} t^{1/k}(Tg)^*(t) \le (e^{-k+1})^{1/k} \int_{e^{-k}}^1 \frac{g(s)}{s}\, ds \\ &\le c\bigg(\int_{e^{-k}}^1 \frac{g(s)^{p'}}{s}\, ds\bigg)^{1/p'} \Big(\log e^k\Big)^{1/p} \le \text{const.}\, k^{1/p}.\end{aligned}$$

■

**Remark 3.2.** The condition (3.1) is actually the original assumption from Moser's lemma—after the substitution mentioned following the introduction of the operators $T$ and $F$ in (1.2) and (1.1), respectively. According to the condition (v) of Theorem 2.1 the claim in the preceding theorem is equivalent to $Tg \in L^{\exp t^p}$. Hence the above proof (together with the proof of (v) in Theorem 2.1, which is straightforward and uses the same decomposition of $(0,1)$) (see [CUK]), gives an extremely simplified proof of the exponential integrability of $Tg$ established in [J].

**Remark 3.3.** Theorem 3.1 can be alternatively proved by invoking the extrapolation theorem from [EK2]. The proof is then just two lines longer. We have, by Minkowski's and Hölder's inequalities,

$$\begin{aligned}
k^{-1/p}\|Tg\|_{k,I_k} &= k^{-1/p}\Big(\int_{I_k}\Big(\int_x^1 g(t)\,\frac{dt}{t}\Big)^k dx\Big)^{1/k}\\
&\le k^{-1/p}\Big(\int_{I_k}\Big(\int_{e^{-k}}^1 g(t)\,\frac{dt}{t}\Big)^k dx\Big)^{1/k}\\
&\le k^{-1/p}\int_{e^{-k}}^1\Big(\int_{I_k}\Big(\frac{g(t)}{t}\Big)^k dx\Big)^{1/k} dt\\
&\le k^{-1/p}\int_{e^{-k}}^1 \frac{g(t)}{t}\big[e^{-k}(e-1)\big]^{1/k} dt\\
&\le \frac{(e-1)^{1/k}k^{-1/p}}{e}\int_{e^{-k}}^1 \frac{g(t)}{t^{(1/p)+(1/p')}}\,dt\\
&\le k^{-1/p}\Big(\int_0^1 \frac{g(t)^{p'}}{t}\,dt\Big)^{1/p'}\Big(\int_{e^{-k}}^1\frac{dt}{t}\Big)^{1/p} \le 1.
\end{aligned}$$

Let us now consider alternative sufficient conditions for the same claim as in the preceding theorem, passing to endpoints of the Lebesgue spaces scale.

**Theorem 3.4.** *Let $1 < p < \infty$. Then there exists $c > 0$ such that*

$$\sup_k \frac{\|Tg\|_{k,\infty,I_k}}{k^{1/p}} \le c. \tag{3.2}$$

*for all non-negative functions $g$ such that*

$$L(g) = \int_0^1 \frac{g(s)}{(\log(1/s))^{1/p}}\,\frac{ds}{s} \le 1. \tag{3.3}$$

**Proof.** Since the function $t \mapsto Tg(t)$ is non-increasing on $(0,1)$ it coincides with its non-increasing rearrangement. Hence

$$\begin{aligned}
\frac{\|Tg\|_{k,\infty,I_k}}{k^{1/p}} &\le \frac{1}{k^{1/p}}\sup_{t\in I_k} t^{1/k}(Tg)^*(t) = \frac{1}{k^{1/p}}\sup_{t\in I_k} t^{1/k}\int_t^1 \frac{g(s)}{s}\,ds\\
&\le \frac{1}{k^{1/p}}\int_{e^{-k}}^1 \frac{g(s)}{s}\,ds \le c\int_{e^{-k}}^1 \frac{g(s)}{s\,(\log(1/s))^{1/p}}\,ds.
\end{aligned}$$

■

Now we have a third sufficient condition for the exponential boundedness of $T$.

**Theorem 3.5.** *Let $1 < p < \infty$. Then there exists $c > 0$ such that*

$$\sup_{k\in\mathbb{N}} \frac{\|Tg\|_{k,\infty,I_k}}{k^{1/p}} \le c.$$

*for all non-negative functions $g$ such that*

$$S(g) = \sup_{0<t<1} g(t)\left(\log\frac{1}{t}\right)^{1/p'} \le 1. \tag{3.4}$$

**Proof.** In Remark 3.3 we saw that

$$k^{-1/p}\|Tg\|_{k,I_k} \le k^{-1/p}\int_{e^{-k}}^{1}\frac{g(t)}{t}\,dt.$$

Hence

$$\begin{aligned}
k^{-1/p}\|Tg\|_{k,I_k} &\le k^{-1/p}\int_{e^{-k}}^{1}\frac{g(t)(\log(1/t))^{1/p'}}{t(\log(1/t))^{1/p'}}\,dt\\
&\le k^{-1/p}\sup_{0<t<1} g(t)(\log(1/t))^{1/p'}\int_{e^{-k}}^{1}\frac{dt}{t(\log(1/t))^{1/p'}}\\
&= k^{-1/p}\sup_{0<t<1} g(t)(\log(1/t))^{1/p'}\left[\frac{(\log(1/t))^{1/p}}{1/p}\right]_1^{e^{-k}}\\
&\le k^{-1/p}\sup_{0<t<1} g(t)(\log(1/t))^{1/p'}\cdot p\cdot k^{1/p}\\
&\le p\sup_{0<t<1} g(t)(\log(1/t))^{1/p'}.
\end{aligned}$$

■

**Remark 3.6.** Let us compare the conditions (3.1), (3.3), and (3.4).
1. First we show that (3.1) does not generally imply (3.3). Put $u(s) = g(s)/s^{1/p'}$. Then (3.1) turns into

$$\int_0^1 u(s)^{p'}\,ds < \infty, \tag{3.5}$$

and (3.3) into

$$\int_0^1 \frac{u(s)\,ds}{s^{1/p}\,(\log(1/s))^{1/p}} < \infty. \tag{3.6}$$

Take

$$u(s) = \frac{1}{s^{1/p'}\,(\log(1/s))^{1/p'}\,(\log\log(1/s))^{(1+\delta)/p'}},$$

where $0 < \delta \leq p' - 1$. Plainly (3.5) holds and at the same time (3.6) does not.

2. Any function $g$ supported in a sufficiently small left neighbourhood of 1 and integrable there satisfies (3.3). But choosing $g$ in order that its $p'$-th power is not integrable, we see that the integral in (3.1) diverges. In other words, (3.3) is not stronger than (3.1) and together with the previous example this shows that these conditions are incomparable.

3. Put $g(s) = (\log(1/s))^{-1/p'}$, $s \in (0,1)$. Then (3.4) is trivially satisfied, whereas the integral in (3.3) diverges. On the other hand there are plainly functions unbounded in a neighbourhood of 1 integrable with the $p'$-th power with respect to $ds/s$. Hence (3.1) and (3.4) are incomparable.

4. The same function $g$ as in the previous item inserted into (3.3) gives the integral $\int_0^1 s^{-1}\,(\log(1/s))^{-1}\,ds$, which is independent of $p$ and diverges. Furthermore, an argument analogous to that given before shows that (3.3) does not generally imply (3.4) (any unbounded integrable function in a neighbourhood of 1 will do).

**Remark 3.7.** The claim in Theorem 3.1 is known to hold for $g \in L^{p',\infty}(0,1;dt/t)$; see [JM], p. 61. Let us give an example of a $g$ satisfying (3.4), but outside of $L^{p',\infty}(0,1;dt/t)$. Put

$$g(t) = \frac{1}{(\log(1/t))^{1/(p'-\varepsilon)}}$$

with $0 < \varepsilon < p'$. Then clearly $g$ satisfies (3.4). On the other hand, $m(g,\lambda)$, the distribution function of $g$, that is the measure of the set $\{g(t) > \lambda\}$, $\lambda > 0$, is

$$m(g,\lambda) = 1 - \exp(-\lambda^{\varepsilon-p'}),$$

and

$$g(t) > \lambda \quad \text{iff} \quad t > \exp(-\lambda^{\varepsilon-p'}).$$

We have

$$\sup_{\lambda>0} \lambda \biggl( \int_{\exp(-\lambda^{\varepsilon-p'})}^{1} \frac{dt}{t} \biggr)^{1/p'} = \sup_{\lambda>0} \lambda (\lambda^{-p'+\varepsilon})^{1/p'} = \infty.$$

As was observed earlier the operator $T$ is naturally linked with Sobolev imbeddings in the critical case. In the last ten years further tuning of the scale of the Sobolev type spaces imbedded into exponential spaces has been tackled, leading to target spaces of multiple exponential type (see e.g. [FLS], [EK1]).

The decomposition technique was used in [KSchm] for an easy proof of a characterization of the double exponential spaces analogous to those in Theorem 2.1 (see also [EGO] for the "classical" global condition on $(0,1)$).

**Theorem 3.8 ([KSchm]).** *Let $J_\tau = (e^{-e\tau}, e^{-\tau})$, $\tau \geq 1$, and $1 < p < \infty$. Then there is $\lambda$ such that $\int_0^1 \exp\exp(\lambda h(t))^p < \infty$ if and only if*

$$\sup_{\tau\geq 1} \frac{\|h^*\|_{L_\tau(J_\tau)}}{\log^{1/p}(e+\tau)} < \infty.$$

Having in mind this characterization we prove

**Theorem 3.9.** *Let*

$$\varphi(u) = u \log \frac{e}{u}, \qquad u > 0,$$

*and*

$$T_1 g(t) = \int_t^1 \frac{g(u)}{\varphi(u)}\, du, \qquad t \in (0,1).$$

*Let $1 < p < \infty$. Then there exists $c > 0$ such that*

$$\sup_{\tau\geq 1} \frac{\|T_1 g\|_{L_\tau(I_\tau)}}{\log^{1/p}(e+\tau)} \leq c \tag{3.7}$$

*for all $g$ satisfying*

$$\int_0^1 \frac{(g(t))^{p'}}{\varphi(t)}\, dt \leq 1. \tag{3.8}$$

**Proof.** Denote

$$A_\tau = \log^{-1/p}(e+\tau)\Big(\int_{J_\tau} T_1 g(u)^\tau\,du\Big)^{1/\tau}.$$

Then by Minkowski's and Hölder's inequalities,

$$\begin{aligned}
A_\tau &= \log^{-1/p}(e+\tau)\Big(\int_{J_\tau}\Big[\int_x^1 \frac{g(u)}{\varphi(u)}\,du\Big]^\tau dx\Big)^{1/\tau}\\
&\le \log^{-1/p}(e+\tau)\int_{e^{-e\tau}}^1\Big(\int_{J_\tau}\Big(\frac{g(u)}{\varphi(u)}\Big)^\tau dx\Big)^{1/\tau} du\\
&\le \log^{-1/p}(e+\tau)\int_{e^{-e\tau}}^1 \frac{g(u)}{\varphi(u)}\,du\,|J_\tau|^{1/\tau}\\
&\le \log^{-1/p}(e+\tau)\,|J_\tau|^{1/\tau}\Big(\int_0^1\frac{(g(u))^{p'}}{\varphi(u)}\,du\Big)^{1/p'}\Big(\int_{e^{-e\tau}}^1\frac{du}{\varphi(u)}\Big)^{1/p}.
\end{aligned}$$

Further,

$$|J_\tau|^{1/\tau} = \Big(\frac{1}{e^\tau}-\frac{1}{e^{e\tau}}\Big)^{1/\tau} = \Big(\frac{1}{e^\tau}-\frac{1}{(e^\tau)^e}\Big)^{1/\tau}$$

and it is not difficult to check that

$$\lim_{\tau\to\infty}\Big(\frac{1}{e^\tau}-\frac{1}{(e^\tau)^e}\Big)^{1/\tau} = e^{-1}.$$

It remains to estimate $\Big(\int_{e^{-e\tau}}^1 du/\varphi(u)\Big)^{1/p}$. We have

$$\Big(\int_{e^{-e\tau}}^1\frac{du}{\varphi(u)}\Big)^{1/p} = \Big(\int_{e^{-e\tau}}^1\frac{du}{u\log(e/u)}\Big)^{1/p}.$$

Put $e^{-e\tau}=\xi$, i.e. $\tau = (\log(1/\xi))/e$. Then $\tau\to\infty$ iff $\xi\to 0$. However,

$$\lim_{\xi\to 0}\frac{\displaystyle\int_\xi^1\frac{du}{u\log(e/u)}}{\log\log(1/\xi)} = \lim_{\xi\to 0}\frac{-\dfrac{1}{\xi\log(e/\xi)}}{\dfrac{1}{\log(1/\xi)}\cdot\xi\cdot\Big(\dfrac{-1}{\xi^2}\Big)} = 1,$$

which yields $\sup_{\tau\ge 1} A_\tau < \infty$ and completes the proof. ■

# 4 Moser's lemma in Lorentz spaces via decomposition

Next we present Lorentz-space versions of exponential estimates.

**Theorem 4.1.** *Let $f$ and $g$ be associated with the formula* (2.3) *and assume that $f \in L^{q',p}(0,\infty)$ with $1 < p < \infty$, $1 < q < \infty$, $p \neq q'$. Let*

$$Tg(t) = \int_t^1 \frac{g(s)}{s}\, ds.$$

*Then there exists $c > 0$ such that*

$$\sup_k k^{-1/q} \|Tg\|_{k,\infty,I_k} \leq c$$

*for all $f$ with $\|f\|_{L^{q',p}} \leq 1$.*

**Proof.** We estimate $k^{-1/q}\|Tg\|_{k,\infty,I_k}$ as follows:

$$\begin{aligned}
&k^{-1/q} \int_{e^{-k}}^1 \frac{g(\sigma)}{\sigma}\, d\sigma \\
&\quad = k^{-1/q} \int_{e^{-k}}^1 g(\sigma) \left(\log \frac{1}{\sigma}\right)^{(-1+p/q')/p} \left(\log \frac{1}{\sigma}\right)^{(1-p/q')/p} \frac{d\sigma}{\sigma} \\
&\quad \leq k^{-1/q} \left( \int_{e^{-k}}^1 g(\sigma)^p \left(\log \frac{1}{\sigma}\right)^{-1+p/q'} \frac{d\sigma}{\sigma}\right)^{1/p} \\
&\qquad \times \left( \int_{e^{-k}}^1 \left(\log \frac{1}{\sigma}\right)^{(1/p-1/q')p'} \frac{d\sigma}{\sigma}\right)^{1/p'}.
\end{aligned}$$

We have

$$\left(\frac{1}{p} - \frac{1}{q'}\right) p' = \frac{p(q'-1)}{q'(p-1)} - 1,$$

hence

$$\left(\int_{e^{-k}}^{1}\left(\log\frac{1}{\sigma}\right)^{(1/p-1/q')p'}\frac{d\sigma}{\sigma}\right)^{1/p'}$$
$$=\left(\int_{e^{-k}}^{1}\left(\log\frac{1}{\sigma}\right)^{\frac{p(q'-1)}{q'(p-1)}-1}\frac{d\sigma}{\sigma}\right)^{1/p'}$$
$$=\left(\frac{q'(p-1)}{p(q'-1)}\right)^{1/p'}\left(-\left[\left(\log\frac{1}{\sigma}\right)^{\frac{p(q'-1)}{q'(p-1)}}\right]_{\sigma=e^{-k}}^{\sigma=1}\right)^{1/p'}$$
$$=\left(\frac{q'(p-1)}{p(q'-1)}\right)^{1/p'}k^{\frac{p'}{q}\cdot\frac{1}{p'}}=\text{const.}\,k^{1/q}.$$

■

We tackle the case $p = 1$ separately.

**Theorem 4.2.** *Let $f$ and $g$ be associated with the formula* (2.3) *and assume that $f \in L^{q',1}$ with $1 < q < \infty$. Let*

$$Tg(t)=\int_t^1\frac{g(s)}{s}\,ds.$$

*Then there exists $c > 0$ such that*

$$\sup_k k^{-1/q}\|Tg\|_{k,\infty,I_k}\le c$$

*for all $f$ with $\|f\|_{L^{q',1}} \le 1$.*

**Proof.** As this is similar to what we have done above we proceed briefly. We have

$$k^{-1/q}\int_{e^{-k}}^{1}\frac{g(\sigma)}{\sigma}\,d\sigma$$
$$=k^{-1/q}\int_{e^{-k}}^{1}g(\sigma)\left(\log(1/\sigma)\right)^{1/q'}\left(\log(1/\sigma)\right)^{1/q}\frac{d\sigma}{\sigma\log(1/\sigma)}$$
$$\le c\|f\|_{L^{1,q'}(0,\infty)}.$$

■

# 5 An example of application to imbeddings in the critical case

The decomposition technique turns out to provide very transparent proofs of claims involving exponential Orlicz spaces. As an example we give a simplified proof of the Sobolev imbedding theorem in the critical case due to Adams [A].

**Theorem 5.1 ([A]).** *Let $0 < \alpha < n \in \mathbb{N}$, put $p = n/\alpha$ and suppose that $f \in L_p(\mathbb{R}^n)$ is such that $\|f\|_{p,\mathbb{R}^n} = 1$ and $\operatorname{supp} f \subset B = B(0,r)$, the open ball in $\mathbb{R}^n$ with centre 0 and radius $r$. Then there is a constant $A = A(n,p)$ such that*

$$\int_B \exp\left(\beta_0 |I_\alpha * f|^{p'}\right) dx \le Ar^n.$$

*Here $I_\alpha(x) = \gamma_\alpha / |x|^{n-\alpha}$, $\gamma_\alpha = \Gamma\left(\frac{n-\alpha}{2}\right) / \left(\pi^{n/2} 2^\alpha \Gamma(\alpha/2)\right)$ and $\beta_0 = \gamma_\alpha^{-p'} n/\omega_{n-1}$, $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$.*

**Proof.** Put $R_\alpha(x) = \beta_0^{1/p'} I_\alpha(x) = (n/\omega_{n-1})^{1/p'} / |x|^{n-\alpha}$, $h = R_\alpha * f$ and assume without loss of generality that $f \ge 0$. Then

$$R_\alpha^*(t) = t^{-1/p'}, \quad R_\alpha^{**}(t) = pR_\alpha^*(t),$$

and by O'Neil's convolution inequality

$$h^*(t) \le h^{**}(t) \le pt^{-1/p'} \int_0^t f^*(s)\, ds + \int_t^{|B|} f^*(s) s^{-1/p'}\, ds.$$

Hence

$$\int_B \exp\left(h(x)^{p'}\right) dx \le \int_0^{|B|} \exp\left(F(t)^{p'}\right) dt,$$

where

$$F(t) = pt^{-1/p'} \int_0^t f^*(s)\, ds + \int_t^{|B|} f^*(s) s^{-1/p'}\, ds.$$

Put

$$g(y) = f^*(|B|\, y)\, (|B|\, y)^{1/p}, \qquad 0 < y < 1.$$

Then

$$\int_0^1 g(y)^p \frac{dy}{y} = \int_0^{|B|} f^*(t)^p\, dt = 1$$

and

$$F(t) = p\,(t/\,|B|)^{-1/p'} \int_0^{t/|B|} \frac{g(u)}{u^{1/p}}\, du + \int_{t/|B|}^1 g(u)\, \frac{du}{u}.$$

It follows that

$$\int_B \exp\left(h\,(x)^{p'}\right)\, dx \le |B| \int_0^1 \exp\left\{(Sg)(t) + (Tg)(t)\right\}^{p'}\, dt, \tag{5.1}$$

where

$$(Sg)(t) = pt^{-1/p'} \int_0^t \frac{g(u)}{u^{1/p}}\, du, \qquad (Tg)(t) = \int_t^1 g(u)\, \frac{du}{u}.$$

By Theorem 2.1, to prove that the integral on the right-hand side of (5.1) is finite it is enough to show that

$$\sup_{k\in\mathbb{N}} k^{-1/p'} \|Sg + Tg\|_{k,I_k} < \infty,$$

where $I_k = \left(e^{-k-1}, e^{-k}\right)$. However, $Sg$ is bounded, since by Hölder's inequality,

$$t^{-1/p'} \int_0^t \frac{g(u)}{u^{1/p}}\, du \le \left(\int_0^1 g(u)^p\, \frac{du}{u}\right)^{1/p} \le 1.$$

Thus it is enough to check that

$$\sup_{k\in\mathbb{N}} k^{-1/p'} \|Tg\|_{k,I_k} < \infty,$$

and this is guaranteed by Theorems 2.1 and 3.1. ■

**Remark 5.2.** We observe that the operator $T$ and questions about its exponential integrability appears in various applications; let us refer for instance to [F], dealing with qualitative properties of elliptic PDEs. Our decomposition might be considered as an alternative easy and powerful method for handling operators of this type.

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