REVISTA MATEMÁTICA COMPLUTENSE
(2001) vol. XIV, num. 2, 505-521

ISSN 1139-1138

EXISTENCE OF SOLUTIONS OF STRONGLY NONLINEAR ELLIPTIC EQUATIONS IN $\mathbf{R}^N$

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Abstract

The paper is dedicated to the existence of local solutions of strongly nonlinear equations in $\mathbb{R}^N$ and the Orlicz spaces framework is used.

1 Introduction

Let A be the Leray-Lions operator given by $A(u) =$ - div$a(., u, \nabla u)$ where $p \in]1, N]$, $\frac{1}{p} + \frac{1}{p'} = 1$ and where $a : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \mathbf{R}^N$, for all $s \in \mathbf{R}, \xi, \xi^* \in \mathbf{R}^N$ with $\xi \neq \xi^*$:

$$| a(x, s, \xi) | \leq \beta(c(x) + b(x) | s |^{p-1} + d(x) | \xi |^{p-1}), \tag{1.1}$$

$$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] > 0, \tag{1.2}$$

$$\nu | \xi |^p \leq a(x, s, \xi)\xi, \tag{1.3}$$

where $c(x) \in L_{loc}{}^{p'}(\mathbf{R}^N), c \geq 0; b(x), d(x)$ are locally bounded functions, $\beta \in \mathbf{R}^+$ and $\nu > 0$. On the other hand, $g : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that for a certain constant $\delta > 0$:

$$g(x, s) signs \geq | s |^{\delta}, \tag{1.4}$$

holds for all $s \in \mathbf{R}$, a.e $x \in \mathbf{R}^N$. Moreover, for every $t > 0$,

$$\sup_{|s| \leq t} | g(., s) | \in L^1_{loc}(\mathbf{R}^N) \tag{1.5}$$

2000 Mathematics Subject Classification: 35J60.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2001

a.e. $x \in \mathbf{R}^N$.

We consider the following nonlinear equation

$$A(u) + g(.,u) = f \text{ in } \mathbf{R}^N, \; f \in L^1_{loc}(\mathbf{R}^N). \tag{1.6}$$

We say that u is a weak solution of (1.6) if it satisfies:

$$\begin{cases} u \in W^{1,1}_{loc}(\mathbf{R}^N), a(.,u,\nabla u) \in L^1_{loc}(\mathbf{R}^N), g(.,u) \in L^1_{loc}(\mathbf{R}^N) \\ A(u) + g(.,u) = f \text{ in } \mathcal{D}'(\mathbf{R}^N) \end{cases} \tag{1.7}$$

By combining [10] and [11], we can deduce the existence of weak solutions u for (1.6), with the following regularity:

$u \in W^{1,q}_{loc}(\mathbf{R}^N)$ for every $q < \overline{q} = \frac{N(p-1)}{N-1}$ if $\delta > p-1$ and $p_0 = 2 - \frac{1}{N} < p \leq N$.
$u \in W^{1,q}_{loc}(\mathbf{R}^N)$ for every $q < q_1 = \frac{p\delta}{\delta+1}$ if $\delta > \overline{q}^*$ and $p_0 < p < N$ or $\delta(p-1) > 1$ and $1 < p \leq p_0$.

It is our purpose in this paper to prove the limiting regularity $W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ and $W^{1,q_1}_{loc}(\mathbf{R}^N)$ (which is not reached in general) of weak solutions of (1.6), when we replace the conditions (1.1) and (1.3) by the following:

$$\begin{aligned} | a(x,s,\xi) | \leq \beta(c(x) + b(x) | s |^{p-1} \log^{\alpha p}(e+ | s |) \\ +d(x) | \xi |^{p-1} \log^{\alpha p}(e+ | \xi |)) \end{aligned} \tag{1.1$'$}$$

$$\nu | \xi |^p \log^{\alpha p}(e+ | \xi |) \leq a(x,s,\xi)\xi \tag{1.3$'$}$$

a.e. $x \in \mathbf{R}^N, \forall s \in \mathbf{R}$ and $\xi, \xi^* \in \mathbf{R}^N, \xi \neq \xi^*$. With $c, b(x), d(x), \beta, \nu$ as above, $1 < p \leq N$, and $\alpha > \frac{1}{p}$.
It is to be noticed that our study covers the regularity $W^{1,N}_{loc}(\mathbf{R}^N)$.
We prove also that the condition $\alpha > 1/p$ is necessary to obtain the limiting regularity $W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ for the solutions given by theorem 2.1.
The case $g(.,s) sign s \geq 0$ was studied in [8] with second member measure and Ω bounded.

For other strongly nonlinear equations in Orlicz spaces see [7], [5], [6], [15], [16]

2 Preliminaries

We list some well known results about Orlicz and Orlicz-Sobolev spaces.

2.1. Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, i. e. M is continuous, convex with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation:$M(t) = \int_0^t a(s)\, ds$ where $a : \mathbf{R}^+ \to \mathbf{R}^+$ is nondecreasing, right continuous, with $a(0) = 0, a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)\, ds, \overline{a} : \mathbf{R}^+ \to \mathbf{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1], [17]).

The N-function is said to satisfy the Δ_2 condition if, for some $k > 0$:

$$M(2t) \leq kM(t) \qquad \forall t \geq 0, \tag{2.1}$$

when holds only for $t \geq t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity .

We will extend these N -functions into even functions on all $\mathbf{R}$.

2.2. Let Ω be an open subset of $\mathbf{R}^N$.The Orlicz class $K_M(\Omega)$ (resp.the Orlicz space $L_M(\Omega)$)is defined as the set of (equivalences classes of)real valued measurable functions u on Ω such that:

$\int_\Omega M(u(x))\, dx < +\infty$ (resp.$\int_\Omega M(\frac{u(x)}{\lambda})\, dx < +\infty$ for some $\lambda > 0$).

$L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})\, dx \leq 1\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not .

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of pairing $\int_\Omega u(x)v(x)\, dx$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalently to $\|u\|_{\overline{M}}, \Omega$.

The space $L_M(\Omega)$ is *reflexive* if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not .

2.3. We now turn to the Orlicz-Sobolev space.$W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$).

It is a Banach space under the norm :

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of
N + 1 copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$)denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$(resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [13], [14]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

The following abstract lemma will be applied in the following.

Lemma 2.1. (see [6]) *Let $F : \mathbf{R} \to \mathbf{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an N- function and let $u \in W_0^1L_M(\Omega)$ (resp. $W_0^1E_M(\Omega)$).*

Then $F(u) \in W_0^1L_M(\Omega)$ (resp.$W_0^1E_M(\Omega)$). Moreover, if the set of discontinuity points of F' is finite, then:

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\} \end{cases}$$

3 Main results

Theorem 3.1. *Under the hypotheses (1.1)', (1.2), (1.3)', (1.4), (1.5) $f \in L^1_{loc}(\mathbf{R}^N), \delta > p-1, p_0 < p \leq N$ and $\alpha > \frac{1}{p}$, there exists at least one weak solution $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ for (1.7).*

In the following, we put $M(t) = t^p \log^{\alpha p}(e+t)$, where $p \in]1, N]$ and $\alpha > \frac{1}{p}$.
Recall that the N-function M and $\overline{M}$ satisfy the Δ_2 condition on $\mathbf{R}^+$ (see [17]).

Proof of Theorem.

For $n \geq 1,$ we set $f_n = \inf(| f |, n) sign(f), B_n = \{x \in \mathbf{R}^N, | x |< n\}$. Consider the approximate problem :

$$\begin{cases} A(u_n) + g(., u_n) = f_n \\ u_n \in W_0^1 L_M(B_n), g(., u_n) \in L^1(B_n), u_n g(., u_n) \in L^1(B_n). \end{cases} \tag{3.1}$$

The problem (1.3) has a solution by theorem 6 of [15].

Estimation
Case $p = N$
Step 1. In the following, all constants C, C_i and C'_i depends only on the data.
We follow the same argument as in lemma 2.1 of [11].
Let K be the N-function defined by $K(t) = \exp t^{N'} \_1$. We recall that $W^{1,N}(B_r) \hookrightarrow L_K(B_r)$
Let $r > 0, n > 2r$, we claim there exists a constants C which does not depend on n such that:

$$\|u_n\|_{\delta,B_r} \leq \|g(., u_n)\|_{1,B_r} \leq C. \tag{3.2}$$

$$\int_{B_r} \frac{M(| \nabla u_n |)}{(1 + L_n)\log^{\alpha N}(e + L_n)} \leq C\|u_n\|_{K,B_r}$$

(3.3) if $\|u_n\|_{K,B_r} > 1$, then

where $L_n = \frac{|u_n|}{\|u_n\|_{K,B_r}}$.

In the following, we omit the index n for simplicity.
Let $\phi : \mathbf{R} \to \mathbf{R}$ the function defined by :

$$\phi(s) = \begin{cases} \int_0^s \frac{dt}{(1+t)\log^{\alpha N}(e+t)} & \text{if } s \geq 0 \\ -\phi(-s) & \text{if } s < 0 \end{cases}$$

Let $\zeta \in \mathcal{D}(B_{2r}), 0 \leq \zeta \leq 1, \zeta = 1$ in B_r and $| \nabla\zeta |\leq \frac{2}{r}$.
Let $\eta > 0$ and $0 < \epsilon < \frac{1}{2}$ (the choice of ϵ and η will be fixed in the following).
We take $v = \phi(u)\zeta^\eta$ as test function in (3.1) (see lemma 2.1), we have

$$\begin{aligned}\int a(.,u,\nabla u)\phi'(u)\nabla u\zeta^\eta \quad + \quad & \int g(.,u)\phi(u)\zeta^\eta \leq \int f\phi(u)\,dx + \frac{2}{r}\int_{B_{2r}} c(x)\,dx \\ & + \frac{2\eta}{r}C\int | \nabla u |^{N-1}\log^{\alpha N}(e+ | \nabla u |)\phi(u)\zeta^{\eta-1} \\ & + \frac{2\eta}{r}C\int | u |^{N-1}\log^{\alpha N}(e+ | u |)\phi(u)\zeta^{\eta-1}\end{aligned}$$

in the other hand, we have

$$\begin{aligned}& | \nabla u |^{N-1}\log^{\alpha N}(e+ | \nabla u |)\zeta^{\eta-1} = \\ & | \nabla u |^{N-1}\log^{\alpha(N-1+\epsilon)}(e+ | \nabla u |)\log^{\alpha(1-\epsilon)}(e+ | \nabla u |)\zeta^{\eta-1} \\ & \leq C(\alpha,\epsilon)| \nabla u |^{N-1+\epsilon}\log^{\alpha(N-1+\epsilon)}(e+ | \nabla u |)\zeta^{\eta-1}.\end{aligned}$$

Let $h > 0, l = \frac{N}{N-1+\epsilon}$, by Young inequality with exponents $l, l' = \frac{N}{N-\epsilon}$ we obtain

$$\begin{aligned}& | \nabla u |^{N-1}\log^{\alpha N}(e+ | \nabla u |)\zeta^{\eta-1} \\ & \leq C(\alpha,\epsilon,N)\zeta^\eta h\frac{M(| \nabla u |)}{(1+ | u |)\log^{\alpha N}(e+ | u |)} \\ & +(1+ | u |)^{(l'-1)}\log^{\alpha N(l'-1)}(e+ | u |)\zeta^{\eta-l'}\frac{C(\alpha,\epsilon,)}{h^{l'-1}}.\end{aligned} \tag{3.4}$$

Hence, by (1.3)’

$$\begin{aligned}& \nu\int M(| \nabla u |)\phi'(u)\zeta^\eta + \int g(.,u)\phi(u)\zeta^\eta \\ & \leq C_1 + \frac{2\eta}{r}Ch\int M(| \nabla u |)\phi'(u)\zeta^\eta \\ & + \frac{2\eta C}{rh^{l'-1}}\int (1+ | u |)^{(l'-1)}\log^{\alpha N(l'-1)}(e+ | u |)\zeta^{\eta-l'} \\ & + \frac{2\eta}{r}C\int | u |^{N-1}\log^{\alpha N}(e+ | u |)\phi(u)\zeta^{\eta-1}\,dx,\end{aligned}$$

where $C = C(\alpha,\epsilon,N,r)$.
We shall fix $\epsilon > 0$ such that $l' - 1 < \delta$ and $\eta > l'\frac{\delta}{\delta-\mu}$.

We choose μ, such that $N-1<\mu<\delta$, and $l'-1<\mu<\delta$, then

$$\begin{aligned}\nu\int M(\mid\nabla u\mid)\phi'(u)\zeta^\eta\,dx\quad &+\quad\int g(.,u)\phi(u)\zeta^\eta\,dx\\ &\le C_1+C_2h\int M(\mid\nabla u\mid)\phi'(u)\zeta^\eta\,dx\\ &+C_2\int(1+\mid u\mid)^\mu\zeta^{\eta-l'}\,dx,\end{aligned}$$

since the logarithm function is less than any power function small enough near infinity.
A convenient h gives:

$$\begin{aligned}\nu'\int M(\mid\nabla u\mid)\phi'(u)\zeta^\eta\,dx\quad &+\quad\int g(.,u)\phi(u)\zeta^\eta\,dx\le\\ &\le C_1+\le C_2\int(1+\mid u\mid)^\mu\zeta^{\eta-l'}\,dx\end{aligned}\tag{3.5}$$

with $\nu'>0$.
Let now $\lambda>0$. By using the Young inequality, with exponents $\frac{\delta}{\mu}$ and $(\frac{\delta}{\mu})'$, we have

$$\zeta^{\eta-l'}(1+\mid u\mid)^\mu\le\frac{\lambda\mu\zeta^\eta}{\delta}(1+\mid u\mid)^\delta+\frac{\delta-\mu}{\delta\lambda^{\frac{\mu}{\delta-\mu}}}\zeta^{\eta-l'\frac{\delta}{\delta-\mu}}.$$

η is chosen such that, $\eta>l'\frac{\delta}{\delta-\mu}$, hence

$$\zeta^{\eta-l'}(1+\mid u\mid)^\mu\le\frac{2^\delta\lambda\mu\zeta^\eta}{\delta}\mid u\mid^\delta+\frac{2^\delta\lambda\mu\zeta^\eta}{\delta}+\frac{\delta-\mu}{\delta\lambda^{\frac{\mu}{\delta-\mu}}}\zeta^{\eta-l'\frac{\delta}{\delta-\mu}}\tag{3.6}$$

We choose λ small such that $\frac{2^\delta\lambda\mu}{\delta}<1$.

We introduce (3.6) in (3.5) and we use the fact that $\mid s\mid^\delta\le g(.,s)\frac{\phi(s)}{\phi(1)}+1$, we obtain

$$\nu'\int_{B_r}M(\mid\nabla u\mid)\phi'(u)\zeta^\eta\,dx+C\int_{B_r}g(.,u)\phi(u)\zeta^\eta\,dx\le C_3$$

Then, we deduce (3.2), and

$$\int_{B_r} M(|\nabla u|)\phi'(u)\zeta^{\eta}\,dx \le C. \tag{3.3$'$}$$

In order to obtain (3.3), we proceed as follow:
In (3.1) we take $v = \phi_0(u)$ as test function, where $\phi_0(u) = \|u\|_{K,B_r}\phi(\frac{u}{\|u\|_{K,B_r}})$ and we suppose $\|u\|_{K,B_r} > 1$.
We follow the same way as before, we have then in place of (3.5), the inequality

$$\nu'\int M(|\nabla u|)\phi'(\frac{|u|}{\|u\|_{K,B_r}})\zeta^{\eta}\,dx + \int g(.,u)\phi_0(u)\zeta^{\eta}dx \le C_1'\|u\|_{K,B_r}$$
$$+C_2'\|u\|_{K,B_r}\int (1+|u|)^{\mu}\zeta^{\eta-l'}\,dx$$

since $\|u\|_{K,B_r} > 1$

We now combine (1.5), (3.2) and (3.6) to get

$$\nu'\int M(|\nabla u|)\phi'(\frac{|u|}{\|u\|_{K,B_r}})\zeta^{\eta}\,dx \le C_1'\|u\|_{K,B_r} + C_2'\|u\|_{K,B_r}.$$

Then (3.3) follows.

Step 2. In this step, we give an estimate of the solution u in $W^{1,N}(B_r)$.
Let $H(t) = t\log^{\alpha N}(e+t), B(t) = t\log(e+t)$.
Let $0 < \epsilon < \frac{1}{2}, 1+\epsilon < N'$ and $\log^{\alpha^N}(e+t) \le \frac{t^{\epsilon}}{\epsilon}$, for all $t \ge 1$.
W e suppose $L \ge 1$, then :

$$\begin{aligned}
\frac{\epsilon}{2} \mid \nabla u \mid^N &= \frac{\epsilon}{2} \frac{\mid \nabla u \mid^N}{(1+L)\log^{\alpha^N}(e+L)}(1+L)\log^{\alpha^N}(e+L) \\
&\leq H(\frac{\mid \nabla u \mid^N}{(1+L)\log^{\alpha^N}(e+L)}) + \overline{H}(\epsilon L \log^{\alpha^N}(e+L)) \\
&\leq C\frac{M(\mid \nabla u \mid)}{(1+L)\log^{\alpha^N}(e+L)} + C + \overline{H}(L^{1+\epsilon}) \\
&\leq C\frac{M(\mid \nabla u \mid)}{(1+L)\log^{\alpha^N}(e+L)} + C + K(L)
\end{aligned}$$

since $\overline{H}(t) < \overline{B}(t)$ and $1+\epsilon < N'$.
Thus

$$\mid \nabla u \mid^N \leq C\frac{M(\mid \nabla u \mid)}{(1+L)\log^{\alpha N}(e+L)} + C + CK(L)$$

In the other case, where $L < 1$, we obtain a similar inequality.

If $\|u\|_{K,B_r} > 1$, the first step gives

$$\int_{B_r} \mid \nabla u \mid^N dx \leq C\|u\|_{K,B_r} + C$$

If $\|u\|_{K,B_r} \leq 1$, then by a similair argument as before, we have

$$\mid \nabla u \mid^N \leq C\frac{M(\mid \nabla u \mid)}{(1+\mid u \mid)\log^{\alpha N}(e+\mid u \mid)} + C + CK(\mid u \mid).$$

Also

$$\int_{B_r} \mid \nabla u \mid^N dx \leq C\|u\|_{K,B_r} + C.$$

In the both cases, we obtain

$$\|\nabla u\|^N_{N,B_r} \leq C\|u\|_{K,B_r} + C. \tag{3.7}$$

In the other hand, we have $\|u\_\overline{u}_r\|_{K,B_r} \leq C\|\nabla u\|_{N,B_r}$ (see [12]), where $\overline{u}_r = \frac{1}{|B_r|}\int_{B_r} u\,dx$
Then

$$\|u\|_{K,B_r} \leq C\|\nabla u\|_{N,B_r} + \frac{1}{K^{-1}(\frac{1}{|B_r|})} \mid \overline{u}_r \mid .$$

Holder inequality gives

$$| \overline{u}_r | \leq \frac{1}{| B_r |^{\frac{1}{N'}}} \|u\|_{N',B_r}.$$

We suppose $\|u\|_{K,B_r} > 1$ (the other case is obvious by (3.7)), then

$$\|u\|^{N'}_{N',B_r} \leq C\|u\|_{K,B_r}.$$

Indeed:
If $N > 2$, then $N' < 2 < \delta$ and $\|u\|_{N',B_r} \leq C\|u\|_{\delta,B_r} \leq C$, hence

$$\|u\|^{N'}_{N',B_r} \leq C\|u\|_{K,B_r}$$

If $N = 2$, $\int_{B_r} | u |^2 \leq \|u\|_{\delta,B_r}\|u\|_{\delta',B_r} \leq C\|u\|_{K,B_r}$

By (3.2) and since $L_K(B_r) \hookrightarrow L^{\delta'}(B_r)$ (see [1])

Then, $\|u\|^{N'}_{N',B_r} \leq C\|u\|_{K,B_r}$, so

$$\|u\|_{K,B_r} \leq C\|u\|^{\frac{1}{N}}_{K,B_r} + \frac{1}{K^{-1}(\frac{1}{|B_r|})} \frac{1}{| B_r |^{\frac{1}{N'}}} \|u\|^{\frac{1}{N'}}_{K,B_r}$$

We deduce then

$\|u\|_{K,B_r} \leq C$, also $\|\nabla u\|_{N,B_r} \leq C.$

hence

$$\|g(.,u_n)\|_{1,B_r} \leq C.$$

$$\|u_n\|_{W^{1,N}(B_r)} \leq C.$$

Case $p_0 < p < N$.

As the same proof as in step 1, we have (3.2) and (3.3)' by taking $l = \frac{p}{(p-1)+\epsilon}$ with an appropriate ϵ.

We have with $\overline{q}^* = \frac{\overline{q}}{p-\overline{q}}$

$$\begin{aligned}
\int_{B_r} |\nabla u_n|^{\overline{q}}\,dx &= \int_{B_r} \frac{|\nabla u_n|^{\overline{q}}}{(1+|u_n|)^{\frac{\overline{q}}{p}}}(1+|u_n|)^{\frac{\overline{q}}{p}}\,dx \\
&\leq \Big(\int_{B_r} \frac{|\nabla u_n|^{p}}{(1+|u_n|)}\,dx\Big)^{\frac{\overline{q}}{p}}\Big(\int_{B_r}(1+|u_n|)^{\overline{q}^*}dx\Big)^{1-\frac{\overline{q}}{p}} \\
&\leq \Big(\int_{B_r\cap\{|u_n|<|\nabla u_n|\}} \frac{M(|\nabla u_n|)}{(1+|u_n|)\log^{\alpha p}(e+|\nabla u_n|)} \\
&+ \int_{B_r} |\nabla u_n|^{p-1}\Big)^{\frac{\overline{q}}{p}}\Big(\int_{B_r}(1+|u_n|)^{\overline{q}^*}\Big)^{1-\frac{\overline{q}}{p}} \\
&\leq C + C\Big(\int_{B_r}|u_n|^{\overline{q}^*}\Big)^{1-\frac{\overline{q}}{p}}
\end{aligned}$$

by (3.3)' and since $\|\nabla u_n\|_{p-1,B_r} \leq C$ (lemma 2.2 of [11]).
Then, we continue the proof as in lemma 2.2 ([11] part i)) to have the boundedness of (u_n) in $W^{1,\overline{q}}(B_r)$.

Passage to the limit.

By the estimation's step we have, u_n is bounded in $W^{1,\overline{q}}(B_r)$, then there exists a subsequence noted (u_n) such that:

$$\begin{cases} u_n \to u & \text{weakly in } W^{1,\overline{q}}(B_r), \\ u_n \to u & \text{strongly in } L^{\overline{q}}(B_r). \end{cases}$$

By the same technique as in lemma 1 of [10], we deduce that:

$$\nabla u_n \to \nabla u \text{ in measure and also a.e.}$$

We have $g(.,u_n) \to g(.,u)$ a.e. and $(g(.,u_n))_{n\geq 0} \subset L^1_{loc}(\mathbf{R}^N)$, then

$$g(.,u_n) \to g(.,u) \text{ strongly in } L^1_{loc}(\mathbf{R}^N). \text{ (see [11])}$$

We combine the fact that a is a Carathéodory function, the boundedness of (u_n) in $W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ and the simple inequality: $\log(e+t) \leq C\frac{t^\epsilon}{\epsilon}$ for $t \geq 1, \epsilon \in]0, \frac{1}{2}[$, we deduce the existence for some $\beta > 1$ such that:

$$a(., u_n, \nabla u_n) \to a(., u, \nabla u) \text{ strongly in } L^\beta_{loc}(\mathbf{R}^N).$$

By passage to the limit, we obtain that u is a weak solution of (1.6) and $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$.

In the following, we suppose a (resp. g) satisfies (1.1)', (1.2), (1.3)' (resp. (1.4), (1.5)).

Remark 3.1. In the last theorem, if f is nonnegative, so is u (see [7]).

Remark 3.2. The theorem 3.1 can be formulated as follow:
Under the hypotheses (1.1)', (1.2), (1.3)', (1.4), (1.5), $f \in L^1_{loc}(\mathbf{R}^N), \delta > p-1, \alpha > \frac{1}{p}$, there exists at least one weak solution $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ for (1.6) satisfying, for every $r > 0$

$$\begin{cases} \| u\|_{K,B_r} \leq C\|f\|_{1,B_{2r}} + C(\text{ in the case } p = N) \\ \| \nabla u\|_{\overline{q},B_r} \leq C\|f\|_{1,B_{2r}} + C \end{cases} \tag{H}$$

Remark 3.3. The condition $\alpha > \frac{1}{p}$ is necessary to obtain the regularity $W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ for the solutions satisfying (H).

Counterexample. We suppose $p = N = 2, 0 < \alpha \leq \frac{1}{2}$ and for every $f \in L^1_{loc}(\mathbf{R}^N)$, the problem (1.7) has a solution $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ satisfying (H).
Let $\mu \in M_b(\mathbf{R}^N), A(u) = -\text{div}(\nabla u \log^{2\alpha}(e+ \mid \nabla u \mid)), g(., s) = |s|^{\delta-1}s, \delta > 1$ and $K(t) = \exp t^{N'} - 1$.
Consider the problem :

$$\begin{cases} u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N), a(., \nabla u) \in L^1_{loc}(\mathbf{R}^N), g(., u) \in L^1_{loc}(\mathbf{R}^N) \\ A(u) + g(., u) = \mu \text{ in } \mathcal{D}'(\mathbf{R}^N) \end{cases} \tag{3.8}$$

Let $(f_n) \subset W^{-1}L_{\overline{M}}(B_{2r}) \cap L^1(B_{2r})$ such that $\|f_n\|_1 \leq \|\mu\|_{M_b(B_{2r})}$ and $f_n \to \mu$ in $\mathcal{D}'(B_{2r})$, for every $r > 0$. There exists a solution $u_n \in$

$W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$ satisfying (H) solution of the problem:

$$\begin{cases} u_n \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N), a(.,\nabla u_n) \in L^1_{loc}(\mathbf{R}^N), g(.,u_n) \in L^1_{loc}(\mathbf{R}^N) \\ A(u_n) + g(.,u_n) = f_n \text{ in } \mathcal{D}'(\mathbf{R}^N) \end{cases}$$

(u_n) is bounded in $L_K(B_r)$ and $(|\nabla u_n|)$ is bounded in $L^{\overline{q}}(B_r)$, so (u_n) is in $W^{1,\overline{q}}(B_r)$.
A subsequence (u_n) and $u \in W^{1,\overline{q}}(B_r)$ exist such that:

$$\begin{cases} u_n \to u \text{ in } W^{1,\overline{q}}(B_r), \text{ weakly and a.e. in } B_r \\ g(.,u_n) \to g(.,u) \text{ a.e in } B_r. \end{cases}$$

(u_n) is bounded in $L_K(B_r)$, so $(|u_n|^{\delta-1}u_n)$ is bounded in $L_{K'}(B_r)$, with $K(t) = \exp t^{N'} - 1$ and $K'(t) = \exp t^{\frac{\delta}{N'}} - 1$. Then the subsequence $(g(.,u_n))$ is equiintegrable on B_r, so

$$g(.,u_n) \to g(.,u) \,\text{strongly in}\, L^1(B_r).$$

Then, we deduce $A(u) = \mu - g(.,u) = \mu - \nu$ in $\mathcal{D}'(B_r)$ and $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$. ν is a Radon measure associated to $g(.,u)$ which does not charge the sets of zero C_{1,N_α}, where C_{1,N_α} is the N_α - capacity defined in [3].
Let $r \in \mathbf{R}^*_+$ small enough $(0 < r < 1/2)$, $f(x) = |\, x \,|^{-r}$, then:

$$f \in L_{N_\alpha}(B_r), N_\alpha(t) = t^2 \log^{4\alpha}(e+t). \tag{3.9}$$
$$G_1 * f(0) = +\infty, G_1 \text{ is the Bessel potential.} \tag{3.10}$$

Indeed, it is clear that $f \in L_{N_\alpha}(B_r)$, since r is small.

$G_1 * f(0) > \int_{B(0,1)} G_1(y) f(-y)\, dy \geq C \int_{B(0,l)} |\, y \,|^{-r-1}\, dy = +\infty$, where l is small enough, since G_1 is equivalent to $|\, y \,|^{-1}$ near zero.
By (3.9), (3.10) and theorem 3 of [3], we have

$$C_{1,N_\alpha}\{0\} = 0.$$

We have also $A(u) \in W^{-1}L_{\overline{N_\alpha}}(B_r)$ and $J(N_\alpha, 2) \geq 1$ (where $J(N_\alpha, 2)$ is the well known Donaldson-Trudinger's indice defined in [1] chap.8)
Indeed:
$\overline{N_\alpha}(t) \leq Ct^2 \log^{-4\alpha}(e+t)$ (see [17]), hence
$\overline{N_\alpha}(|\, a(.,\nabla u)\, |) \leq C \,|\, \nabla u \,|^2\, \log^{4\alpha}(e + |\, \nabla u \,|) \log^{-4\alpha}(e + |\, \nabla u \,|$

$\log^{2\alpha}(e+\mid \nabla u \mid)) \leq C \mid \nabla u \mid^2$.
On the other hand, since $N_\alpha^{-1}(t) \geq Ct^{\frac{1}{2}} \log^{-2\alpha}(e+t)$, then $\int_.^\infty \frac{N_\alpha{}^{-1}(t)}{t^{1+\frac{1}{2}}}\,dt = +\infty$ (because $\alpha \leq \frac{1}{2}$).
Then, we conclude : $C_{1,N_\alpha}\{0\} = 0, A(u) \in W^{-1}L_{\overline{N_\alpha}}(B_r) \cap M_b(B_r)$ and $J(N_\alpha, 2) \geq 1$.
By lemma 2 of [4] (see also [2]), $A(u)$ defines a bounded measure which is absolutely continuous with respect to C_{1,N_α}, then if we take $\mu = \delta_0$ we deduce the result.

Remark 3.4. In the case $p = N, \alpha > \frac{1}{N}, g(.,s) = \mid s \mid^{\delta-1} s$ and $\delta > N-1$, (3.8) has a weak solution $u \in W^{1,\overline{q}}_{loc}(\mathbf{R}^N)$.

Indeed:
Let r, f_n, u_n as in the counterexample. The subsequence $(g(.,u_n))$ is equiintegrable on B_r, so
$g(.,u_n) \to g(.,u)$ strongly in $L^1(B_r)$.
We then conclude the result.

Theorem 3.2. *Let* $1 < p \leq p_0, \delta(p-1) > 1$, *and* $\alpha > \frac{1}{p}$. *Then, for every* $f \in L^1_{loc}(\mathbf{R}^N)$, *there exists at least a weak solution* u *of (1.6), which belongs to* $W^{1,q_1}_{loc}(\mathbf{R}^N)$.

Proof. We take the same constructions of the solutions u_n as in theorem 3.1.
By the same technique as in step 1 (theorem 3.1), we have (3.2), (3.3)', which we combine with lemma 2.3 of [11], Young inequality and using the decomposition:
$B_r = (\{|u_n| < |\nabla u_n|\} \cap B_r) \cup (\{|u_n| |\nabla u_n|\} \cap B_r)$, we obtain

$$\begin{aligned}
\int_{B_r} \mid \nabla u_n \mid^{q_1} dx &\leq \int_{B_r} \frac{\mid \nabla u_n \mid^p}{1+\mid u_n \mid}\,dx + C\int_{B_r} (1+\mid u_n \mid)^\delta\,dx \\
&\leq \int_{B_r} \frac{M(\mid \nabla u_n \mid)}{(1+\mid u_n \mid)\log^{\alpha p}(e+\mid u_n \mid)}\,dx \\
&+ \int_{B_r} \mid \nabla u_n \mid^{p-1} + C\int_{B_r} \mid u_n \mid^\delta dx \leq C
\end{aligned}$$

Since $\delta > p-1$, we have $q_1 < \delta$, then $\|u_n\|_{q_1,B_r} \leq C$.
Then, we deduce:

$$\begin{cases} u_n \to u \text{ weakly in } W^{1,q_1}(B_r) \\ u_n \to u \text{ strongly in } L^{q_1}(B_r) \\ g(.,u_n) \to g(.,u) \text{ strongly in } L^1_{loc}(\mathbf{R}^N) \text{ (as in theorem 3,[11])}. \end{cases}$$

Following the same way as the third step (theorem 3.1), we conclude the result.

Theorem 3.3. *Let $f \in L^1_{loc}(\mathbf{R}^N)$, and assume that: $p_0 < p < N, \delta > \overline{q}^*$ and $\alpha > \frac{1}{p}$, then the solution constructed in theorem 3.1 satisfies $\mid \nabla u \mid \in L^{q_1}_{loc}(\mathbf{R}^N)$.*

Proof. As in theorem 3.1, we obtain

$$\begin{aligned} \int_{B_r} \mid \nabla u_n \mid^{q_1} dx \quad &\leq \quad \int_{B_r} \frac{M(\mid \nabla u_n \mid)}{(1+ \mid u_n \mid)\log^{\alpha p}(e+ \mid \nabla u_n \mid)} dx \\ &+ \quad \int_{B_r} \mid \nabla u_n \mid^{p-1} + C \int_{B_r} \mid u_n \mid^{\delta} dx + C \end{aligned}$$

Combining (3.2), (3.3)', lemma 2.2 of [11] and using the same decomposition of B_n as in theorem 3.2, we conclude the result.

Remark 3.5. It is to be noticed that the results in the last theorems can be proved in the case where we replace $\mathbf{R}^N$ by a bounded open subset Ω. The proof is the same as in [11]. In this case, our existence result in theorem 3.1 is in some sense the "dual" of the following one (due to L. Boccardo and T. Gallouet in [10]): If the right hand side belongs to the Orlicz space $LLogL(\Omega)$, then the solutions belong to $W_0^{1,\overline{q}}(\Omega)$.

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Recibido: 15 de Septiembre de 1998
Revisado: 16 de Febrero de 2000