REVISTA MATEMÁTICA COMPLUTENSE (2001) vol. XIV, num. 2, 463-478
ISSN 1139-1138

# LOW-DIMENSIONAL FILIFORM LIE SUPERALGEBRAS

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**Abstract**

The aim of this paper is to give a classification up to isomorphism of low dimension filiform Lie superalgebras.

## 1 Introduction

There exists a lot of work concerning Lie superalgebras. But less of them are interested in nilpotent Lie superalgebras, although, the case of nilpotent Lie algebras has been well studied, see for example [1]. This fact is a consequence of the development of deformation theory. In this paper, we will focus a particular class of nilpotent Lie superalgebras. We recall some definitions from [3]. We point out the definition of filiform Lie superalgebras. This definition is analogue to the definition of filiform Lie algebras given by Vranceanu [10] and Vergne [9]. The classification of this super-algebras is still a problem, as the classification of filiform Lie algebras over an algebraically closed field of characteristic zero is only done up to dimension 11 [4]. In this paper, we will give a first classification for filiform Lie superalgebras in low dimensions.

## 2 Filiform Lie superalgebras

A $\mathbb{Z}_2$-graded vector space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ over an algebraically closed field is a Lie superalgebra if there exists a bilinear product $[,]$ over $\mathcal{G}$ such that

$$[G_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta \mod 2},$$
$$[g_\alpha, g_\beta] = -(-1)^{\alpha.\beta}[g_\beta, g_\alpha]$$

2000 Mathematics Subject Classification: 17B30, 17B70.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2001

for all $g_\alpha \in \mathcal{G}_\alpha$ and $g_\beta \in \mathcal{G}_\beta$ and satisfying Jacobi's super-relation:

$$(-1)^{\gamma.\alpha}[A,[B,C]] + (-1)^{\alpha.\beta}[B,[C,A]] + (-1)^{\beta.\gamma}[C,[A,B]] = 0$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$.

For such a Lie superalgebra we define the lower central series by

$$\begin{cases} C^0(\mathcal{G}) = \mathcal{G}, \\ C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})] \end{cases}$$

A Lie superalgebra $\mathcal{G}$ is nilpotent if there exists an integer $n$ such that $C^n(\mathcal{G}) = \{0\}$.

This definition is not easy to use. Therefore we define the following sequences for a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$:

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \qquad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \qquad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)]$$

**Theorem 2.1.** *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebras. Then $\mathcal{G}$ is nilpotent if and only if there exist $(p, q)$ such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.*

**Proof.** If the Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is nilpotent it is easy to prove that there exist $(p, q)$ such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.

For the converse, we use the classical Engel's theorem. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebra. Assume that there exist $(p, q)$ such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, then $ad(X)$ with $X \in \mathcal{G}_0$ is nilpotent. We have for $Y \in \mathcal{G}_1$:

$$ad(Y) \circ ad(Y) = \frac{1}{2} ad([Y, Y])$$

As $[Y, Y]$ is an element of $\mathcal{G}_0$, then $ad([Y, Y])$ is nilpotent. This implies that $ad(Y)$ is nilpotent for every $Y \in \mathcal{G}_1$. Hence $ad(X)$ and $ad(Y)$ is nilpotent, using Engel's theorem given in [6], $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is a nilpotent Lie superalgebra. ∎

**Definition 2.1.** *Let $\mathcal{G}$ be a nilpotent Lie superalgebra, the super-nilindex of $\mathcal{G}$ is the pair $(p, q)$ such that: $C^p(\mathcal{G}_0) = \{0\}$, $C^{p-1}(\mathcal{G}_0) \neq \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, $C^{q-1}(\mathcal{G}_1) \neq \{0\}$. It is and invariant up to isomorphism.*

**Definition 2.2 (Filiform Lie superalgebras).** *Let* $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ *be a nilpotent Lie superalgebra with* $\dim \mathcal{G}_0 = n+1$ *and* $\dim \mathcal{G}_1 = m$. $\mathcal{G}$ *is called filiform if it's super-nilindex is* $(n, m)$. *We will note* $\mathcal{F}_{n,m}$ *the set of filiform Lie superalgebras.*

Let's define the set $\mathcal{N}_{n,m}^{p,q}$ of Lie superalgebras with $\dim \mathcal{G}_0 = n+1$, $\dim \mathcal{G}_1 = m$ and with super-nilindex $(k_0, k_1)$ such that $k_0 \leq p$ and $k_1 \leq q$. It is obvious that this set can be define by polynomial relations given by the Jacobi relations and the nilpotency relations. This prove that it is a Zariski-closed set of the algebraic variety of nilpotent Lie superalgebras denoted by $\mathcal{N}_{n,m}$.

The set $\mathcal{F}_{n,m}$ of filiform Lie superalgebras can be written as the complementary of a close set:

$$\mathcal{F}_{n,m} = \mathcal{N}_{n,m} \setminus (\mathcal{N}_{n,m}^{n-1,m} \cup \mathcal{N}_{n,m}^{n,m-1})$$

Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras (see [5]).

# 3 Classifications of filiforms over $\mathbb{C}$ in low dimensions

## 3.1 Adapted basis

Like for the filiform Lie algebras [9], there exists an adapted bases of a filiform Lie superalgebra:

**Proposition 3.1.** *Let* $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ *be a filiform Lie superalgebra with* $\dim \mathcal{G}_0 = n+1$ *and* $\dim \mathcal{G}_1 = m$. *Then there exists a bases* $\{X_0, X_1, \dots X_n, Y_1, Y_2, \dots Y_m\}$ *of* $\mathcal{G}$ *with* $X_i \in \mathcal{G}_0$ *and* $Y_i \in \mathcal{G}_1$ *such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1} \quad 1 \leq i \leq n-1, \quad [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \cdots + \mathbb{C}.X_n; \\ [X_0, Y_i] = Y_{i+1} \quad 1 \leq i \leq m-1, \quad [X_0, Y_m] = 0. \end{cases}$$

*Such basis are called adapted.*

**Proof.** Let $gr\mathcal{G} \in \mathcal{F}_{n,m}$ be a graded filiform lie superalgebra. The lower central series implies the following graduations:

$$gr\mathcal{G}_0 = W_1^0 \oplus W_2^0 \oplus \cdots \oplus W_{n-1}^0$$
$$gr\mathcal{G}_1 = W_1^1 \oplus W_2^1 \oplus \cdots \oplus W_{m-1}^1$$

with $\dim W_1^0 = 2$, $\dim W_i^0 = 1$ for $2 \leq i \leq n-1$ and $\dim W_j^1 = 1$ for $1 \leq j \leq m-1$. Then we have $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.

Let $x_i \in W_i^0$ and $y_j \in W_j^1$ be non zero elements, then if $w \in W_1^0$,

$$[w, x_i] = \lambda_i^0(w)\ x_{i+1}$$
$$[w, y_j] = \lambda_j^1(w)\ y_{j+1}$$

the maps $w \to \lambda_i^0(w)$ and $w \to \lambda_j^1(w)$ are non zero linear forms on $W_1^0$, because $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.
As the field is $\mathbb{C}$, there exists $x_0$ in $W_1^0$ such that $\lambda_i^0(w) \neq 0$ for $1 \leq i \leq n-1$ and $\lambda_j^1(w) \neq 0$ for $1 \leq j \leq m-1$. Hence it exist a bases $X_0, X_1, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ with $X_i$ ( resp. $Y_j$) multiple of $x_i$ (resp. $y_j$) such that:

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = a\ X_3 \text{ for } a \in \mathbb{C}$$

Substitute $X_1$ by $X_1 - a\ X_0$ we get the adapted bases of $gr\mathcal{G}$:

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = 0$$

If the Lie superalgebras is not graded, then we introduce a graded Lie superalgebra, as it was done for Lie algebras in [9]. ■

### 3.2 Adapted changes of basis

**Definition 3.1.** *Let $f$ be a graded change of bases of a filiform Lie superalgebra $\mathcal{G}$. $f$ is an adapted changes of bases if $f$ is Lie superalgebra homomorphism and if the image of an adapted bases of $\mathcal{G}$ is an adapted bases.*

**Proposition 3.2.** *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra. Let $f = f_0 + f_1$ be an adapted change of basis of $\mathcal{G}$. Then $f_0$ is an adapted change of bases of the filiform Lie algebra $\mathcal{G}_0$ and $f_1$ satisfies:*

$$\begin{cases} f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \cdots + d_m\ Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

*with the condition:*

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

*where the $A_j$'s are defined by:*

$$A_j = a_0 + \sum_{k=0}^{j-1} (-1)^k C_{j-1}^k \left( \sum_{i=1}^{n-k} a_i . r_{i+k,2+k} \right)$$

*and the product of $\mathcal{G}$ is given in the adapted bases by*

$$[X_i, Y_1] = \sum_{s=2}^{m} r_{i,s}\ Y_s \quad 1 \leq i \leq n$$

**Proof.** The proof can be found in [3]. We give a sketch of it.

First, we establish that $f_0$ is an adapted change of bases for Lie algebras as in [4]. The we set:

$$\begin{cases} f_0(X_0) = a_0\ X_0 + a_1\ X_1 + \ldots a_n\ X_n \\ f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \ldots d_m\ Y_m \end{cases}$$

and assuming $f(Y_t) = [f(X_0), f(Y_{t-1})]$ for $2 \leq t \leq m$ we prove by induction on $t$ that:

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p . Y_t + \sum_{p \geq t+1} d_p . Y_p$$

This implies that $f_1(Y_m) = d_1 \prod_{p=1}^{m-1} A_p\ Y_m$. For this vector to be non zero, we must have $d_1 \prod_{p=1}^{m-1} \neq 0$ This prove that

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p . Y_t + \sum_{p \geq t+1} d_p . Y_p$$

start we a non zero componant on $Y_t$, and then the images of the vectors $Y_t$ by $f$ form a bases.

Conversely, let $f_0$ be an adapted change of bases of filiform Lie algebras. Then the map $f = f_0 + f_1$, with $f_1$ given by:

$$\begin{cases} f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \ldots d_m\ Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

and the condition:

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

is an adapted change of basis. ■

### 3.3 Classification in low dimensions

We have a description of the products of the filiform Lie superalgebras on $\mathcal{F}_{1,m}$. There exists two types:

(1)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1 \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, \text{ if } i+j \text{ even and } 2 \leq i+j \leq m+1 \end{cases}$$

the other products vanish.

(2)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1 \\ [X_1, Y_r] = \sum_{s=2}^{m} r_s Y_{s+r-1} \text{ with } s+r-1 \leq m \end{cases}$$

the other products vanish.

Using adapted changes of bases, like it was done for Lie algebras in [4], we can eliminate the parameters $a_{\frac{i+j}{2}}$.

**Theorem 3.1.** *Every filiform Lie superalgebra of $\mathcal{F}_{1,m}$ which is not a Lie algebra is isomorphic to one of the following filiform Lie superalgebras:*

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k$$

*with $1 \leq k \leq z+1$ if $m = 2z+1$ and $1 \leq k \leq z$ if $m = 2z$.*

**Proof.** We can assume that every non trivial filiform Lie superalgebra with $1 \leq p \leq k-1$ is isomorphic to:

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k$$
$$[Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, \text{ if } i+j \text{ even and } 2 \leq i+j \leq 2(k-p)$$

for $1 \leq p \leq k-1$. The other products vanish.

Using this change of adapted bases:

$$\begin{cases} X_0^1 = X_0 \\ X_1^1 = X_1 \\ Y_1^1 = Y_1 - (-1)^p \dfrac{a_{k-1}}{2} Y_{2p+1} \\ \dots \\ Y_{k-p}^1 = Y_{k-p} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p} \\ Y_{k-p+s}^1 = Y_{k-p+s} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p+s} \end{cases}$$

we can reduce to

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1 \\ [Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} \ X_1, \text{ if } i+j \text{ even and } 2 \leq i+j \leq 2(k-p-1) \end{cases}$$

The other products vanish.

By induction on $p$, we have a complete classification. ■

By using adapted changes of bases, we established the following classifications:

$\mathcal{F}_{1,2}$:

(1) $\{ \ [X_0, Y_1] = Y_2$

(2) $\begin{cases} [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(3) $\begin{cases} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1 \end{cases}$

$\mathcal{F}_{1,3}$:

(1) $\{ \ [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3$

(2) $\begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1 \end{cases}$

$$(3)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_2,Y_2]=X_1, \quad [Y_1,Y_3]=-X_1 \end{cases}$$

$$(4)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3 \end{cases}$$

$$(5)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_3 \end{cases}$$

$$(6)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3 \end{cases}$$

$\mathcal{F}_{1,4}$:

$$(1)\ \{\ [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4$$

$$(2)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [Y_1,Y_1]=X_1 \end{cases}$$

$$(3)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [Y_2,Y_2]=X_1, \quad [Y_1,Y_3]=-X_1 \end{cases}$$

$$(4)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(5)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_3, \quad [X_1,Y_2]=Y_4 \end{cases}$$

$$(6)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_4 \end{cases}$$

$$(7)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3+Y_4, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(8)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_4, \quad [X_1,Y_2]=Y_3, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(9)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_3+2\,Y_4, \quad [X_1,Y_2]=Y_3+Y_4, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$\mathcal{F}_{2,2}$:

$$(1)\ \{\ [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2$$

$$(2)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \dfrac{1}{2} X_2 \end{cases}$$

$$(3)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(4)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(5)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_2 \end{cases}$$

$\mathcal{F}_{2,3}$:

$$(1)\ \{\ [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3$$

$$(2)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \end{cases}$$

$$(3)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$$

$$(4)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_2] = -Y_3, \quad [X_2, Y_1] = Y_3 \end{cases}$$

$$(5)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, \quad [X_1, Y_2] = Y_3 \end{cases}$$

$$(6)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_2, Y_1] = Y_3 \end{cases}$$

$$(7)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = 2Y_2, \quad [X_1, Y_2] = Y_3, \quad [X_2, Y_1] = Y_3 \end{cases}$$

$$(8)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(9)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(10)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(11)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(12)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$$(13)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3 \\ [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$$(14)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2 \end{cases}$$

$$(15)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_2]=-Y_3, \quad [X_2,Y_1]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2 \end{cases}$$

$$(16)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2, \quad [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$\mathcal{F}_{3,2}$:

$$(1)\quad \big\{\ [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2$$

$$(2)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2, \quad [Y_2,Y_2]=\frac{1}{2}X_3 \end{cases}$$

$$(3)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_2, \quad [Y_1,Y_2]=\frac{1}{2}X_3 \end{cases}$$

$$(4)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_3 \end{cases}$$

$$(5)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(6)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_3 \end{cases}$$

$\mathcal{F}_{4,2}$:

$$(1)\ \{\ [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2$$

$$(2)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2, \quad [Y_1, Y_2] = \frac{1}{2}X_3, \quad [Y_2, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(3)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(4)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_4 \end{cases}$$

$$(5)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(6)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_4 \end{cases}$$

$$(7)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \end{cases}$$

$$(8)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(9)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [Y_1, Y_1] = X_4 \end{cases}$$

$$(10)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, Y_1] = Y_2 \end{cases}$$

$$(11)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_4 \end{cases}$$

$\mathcal{F}_{5,2}$:

$$(1)\ \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \end{cases}$$

$$(2)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\dfrac{1}{2}X_4, \quad [Y_2,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(3)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(4)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(5)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(6)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(7)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5 \end{cases}$$

$$(8)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\dfrac{1}{2}X_4, \quad [Y_2,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(9)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(10)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(11)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,Y_1]=Y_2 \end{cases}$$

$$(12)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(13)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(14)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(15)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(16)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(17)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(18)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(19)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=Y_2 \end{cases}$$

$$(20)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(21)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2 \end{cases}$$

$$(22)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2, \quad [X_1,X_4]=-X_5 \\ [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(23)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(24)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(25)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(26)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(27)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(28)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=-Y_2 \end{cases}$$

$$(29)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(30)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

$$(31)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, \quad [X_1, X_3] = X_5, \quad [X_1, Y_1] = -2Y_2 \end{cases}$$

$$(32)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, [X_1, X_3] = X_5, [X_1, Y_1] = -2Y_2, [Y_1, Y_1] = X_5 \end{cases}$$

$$(33)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2 \end{cases}$$

$$(34)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{cases}$$

$$(35)\quad \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

## References

[1] J.M. Ancochea-Bermúdez, M. Goze, *On the varieties of nilpotent Lie algebras of dimension 7 and 8*, J. Pure Appl. Algebra 77 (1992) 131-140

[2] D. B. Fuks *Cohomology of Infinite-Dimensional Lie Algebras*, Plenum publishing Copr. ISBN 0-306-10990-5

[3] M. Gilg *Super-algèbres de Lie nilpotentes* Thèse, Université de Haute-Alsace, 2000

[4] J.R. Gómez, A. Jimenéz-Merchán, Y. Khakimdjanov *Low-dimensional filiform Lie algebras* J. of pure and app. Algebras 130 (1998) 133-158

[5] M. Goze *Perturbations des super-algèbres de Lie* J. Geom. Phys. vol6, n. 4,1889

[6] M. Scheunert *The Theory of Lie Superalgebras* Lecture Notes in Math. 716 (1979)

[7] V. G. Kac *A Sketch of Lie Superalgebra Theory* Commun. math. phys. 53, 31-64 (1977)

[8] V. G. Kac *Lie Superalgebras* Adv. Math. 26(8), 8-96 (1977)

[9] M. Vergne *Cohomologie des algèbres de Lie nilpotentes. Application à l'etude de la variété des algèbres de Lie nilpotentes* Bull. Soc. Math. France, 98, 1970 p.81 à 116

[10] G. Vrănceanu *Clasificarea grupurilor lui Lie de rang zero* Studii şi cercatări mathematice, 1, 1950, 269-308

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Recibido: 26 de Junio de 2000
Revisado: 8 de Enero de 2001