TORUS KNOTS THAT CANNOT BE UNTIED BY TWISTING

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Abstract

We give a necessary condition for a torus knot to be untied by a single twisting. By using this result, we give infinitely many torus knots that cannot be untied by a single twisting.

1 Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular all knots are oriented. For an oriented manifold M, $-M$ denotes M with the opposite orientation.

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let $\omega = |\mathrm{lk}(\partial D^2, L)|$ and n an integer. A $-1/n$-Dehn surgery along ∂D^2 changes K into a new knot K' in S^3. We say that K' is obtained from K by (n, ω)-*twisting* (or simply *twisting*). (The second author calls an (n, ω)-twisting a $(-n, \omega)$-twisting in his prior papers [3], [8] and [21].) Then we write $K \overset{(n,\omega)}{\to} K'$. Let $\mathcal{T}$ denote the set of knots that are obtained from a trivial knot by a single twisting. Y. Ohyama [12] showed that any knot can be untied by two twistings. This implies that any knot is obtained from a knot in $\mathcal{T}$ by a single twisting.

A (p, q)-*torus knot* $T(p, q)$ is a knot that wraps around the standard solid torus in the longitudinal direction p times and the meridional direction q times, where the linking number of the meridian and longitude is equal to 1. Note that p and q are coprime. A torus knot $T(p, q)$ $(0 < p < q)$ is *exceptional* if $q \equiv \pm 1 \pmod p$, and *non-exceptional* if it is not exceptional.

Let $p(\geq 2)$ be an integer. It is not hard to see that $T(p, \pm 1) \overset{(k,p)}{\to} T(p, kp \pm 1)$. Since $T(p, \pm 1)$ is a trivial knot, $T(p, kp \pm 1)$ belongs to

$\mathcal{T}$. (In [10], K. Motegi calls $T(p, kp \pm 1)$ a *trivial example* of torus knots that belong to $\mathcal{T}$.) This implies that any exceptional torus knot belongs to $\mathcal{T}$. In particular, all of the knots $T(2, q)$, $T(3, q)$, $T(4, q)$ and $T(6, q)$ belong to $\mathcal{T}$. In contrast with this fact, a non-exceptional torus knot that belongs to $\mathcal{T}$ is not known so far. These facts let us hit on the following.

Conjecture. *No non-exceptional torus knot belongs to $\mathcal{T}$.*

This conjecture seems likely to be true. However a non-exceptional torus knot that is not contained in $\mathcal{T}$ is not known. So we are faced with the following problem before this conjecture.

Problem. *Is there a torus knot that is not contained in $\mathcal{T}$?*

In this paper we give a necessary condition for a non-exceptional torus knot to belong to $\mathcal{T}$, and by using this condition, we give infinitely many non-exceptional torus knots that are not contained in $\mathcal{T}$. K. Miyazaki and the second author [8] gave a sufficient condition for a knot not to be contained in $\mathcal{T}$ and showed that there are infinitely many knots that are not contained in $\mathcal{T}$. The sufficient condition given in [8] cannot be applied to torus knots since it contains the condition that the value of the signature is equal to 0. (It is known that the signature of a nontrivial torus knot does not vanish; see Corollary 2.2 and also see [14] for example.)

For a prime integer d, let $\sigma_d(K)$ be the *Tristram's d-signature* of a knot K [16]. Note that $\sigma_2(K)$ is the same as the signature $\sigma(K)$ in the usual sense [17], [11].

Theorem 1.1. *Let $T(p, q)$ $(0 < p < q)$ be a non-exceptional torus knot. If $T(p, q)$ is obtained from a trivial knot by a single (n, ω)-twisting, then (i) $n = 1$, (ii) $\omega < q$, (iii) $\omega > p$ if ω is even, and (iv) if ω is divisible by a prime integer d, then*

$$\frac{2[d/2](d - [d/2])}{d^2}\omega^2 = -\sigma_d(T(p, q)) \ or \ = 2 - \sigma_d(T(p, q)),$$

where $[x]$ is the greatest integer not exceeding x.

Remark 1.2. In [7], K. Miyazaki and K. Motegi showed that if a non-exceptional torus knot $T(p, q)$ $(0 < p < q)$ is obtained from a trivial

knot by a single (n, ω)-twisting, then $|n| = 1$. Thus we eliminate the possibility $n = -1$.

By using this theorem, we have the following three results.

Theorem 1.3. *Let p be an odd integer. If $p \geq 9$, $p \equiv 1$ or $\equiv 3$ (mod 8), then $T(p, p+4)$ does not belong to $\mathcal{T}$*

Remark 1.4. Let p be an odd integer. By the argument similar to that in the proof Theorem 1.3, we see that if $p \geq 7$, $p \equiv 5$ or $\equiv 7$ (mod 8), and if $T(p, p+4)$ is obtained from a trivial knot by a single (n, ω)-twisting, then $n = 1$ and $\omega = p + 2$.

Theorem 1.5. *Let r be an even integer.*
(1) If $r \geq 4$, $2p \geq (r/2 + 1)^2 - r/2$, and $p \equiv 1$ (mod $2r$), then $T(p, p+r)$ does not belong to $\mathcal{T}$.
(2) If $r \geq 8$, $2p \geq (r/2 - 1)^2 - r/2$, and $p \equiv -1$ (mod $2r$), then $T(p, p+r)$ does not belong to $\mathcal{T}$.

Example 1.6. Let r be an even integer and n a positive integer. By the theorem above, we have the following: If $4 \leq r \leq 14$ and $p = 2nr + 1$, or if $8 \leq r \leq 20$ and $p = 2nr - 1$, then $T(p, p+r)$ does not belong to $\mathcal{T}$. Note that this contains the case $p \equiv 1$ (mod 8) of Theorem 1.3.

Remark 1.7. Let $p(\geq 5)$ be an odd integer. By the argument similar to that in the proof of Theorem 1.5, we see that if $T(p, p+2)$ is obtained from a trivial knot by a single (n, ω)-twisting, then $n = 1$ and $\omega = p + 1$.

Since the knots $T(2, q)$, $T(3, q)$, $T(4, q)$ and $T(6, q)$ are exceptional, $T(5, 7)$ is the 'minimum' non-exceptional torus knot, i.e., the crossing number of $T(5, 7)$ is minimum in the crossing numbers of non-exceptional torus knots. By Remark 1.7, if $T(5, 7)$ is obtained from a trivial knot by a single (n, ω)-twisting, then $n = 1$ and $\omega = 6$. The authors cannot eliminate the possibility $(n, \omega) = (1, 6)$. So it is still open if $T(5, 7)$ belongs to $\mathcal{T}$ or not. Concerning $T(5, 8)$, which is the minimum one except for $T(5, 7)$, we have the following.

Proposition 1.8. $T(5, 8)$ *does not belong to $\mathcal{T}$.*

2   Signatures of torus knots

In this section, we calculate the signatures of torus knots.

Proposition 2.1. *Let $T(p,q)$ $(0 < p < q)$ be a torus knot. Then*

$$\sigma(T(p,q)) = 2 \sum_{i=1}^{[(p-1)/2]} \left(\left[\frac{(p-2i)q}{2p} \right] - \left[\frac{(3p - 2[p/2] - 2i)q}{2p} \right] \right) + (p - 1 - 2[p/2])(q - 1).$$

Proof. By [6, Proposition 1], we have $\sigma(T(p,q)) = \sigma^+ - \sigma^-$, where

$$\sigma^+ = \# \left\{ (i,j) \in \mathbb{Z} \times \mathbb{Z} \left| 0 < i < p, 0 < j < q, 0 < \frac{i}{p} + \frac{j}{q} < \frac{1}{2} \right. \right\}$$
$$+ \# \left\{ (i,j) \in \mathbb{Z} \times \mathbb{Z} \left| 0 < i < p, 0 < j < q, \frac{3}{2} < \frac{i}{p} + \frac{j}{q} < 2 \right. \right\}$$

and

$$\sigma^- = \# \left\{ (i,j) \in \mathbb{Z} \times \mathbb{Z} \left| 0 < i < p, 0 < j < q, \frac{1}{2} < \frac{i}{p} + \frac{j}{q} < \frac{3}{2} \right. \right\}.$$

We note that if $p > 0$ and $q > 0$, then

$$\begin{aligned}
0 < i/p + j/q < 1/2 &\Leftrightarrow -qi/p < j < (p - 2i)q/2p, \\
3/2 < i/p + j/q < 2 &\Leftrightarrow (3p - 2i)q/2p < j < (2p - i)q/p, \\
1/2 < i/p + j/q < 3/2 &\Leftrightarrow (p - 2i)q/2p < j < (3p - 2i)q/2p, \\
(p - 2i)q/2p \leq 0 &\Leftrightarrow i \geq p/2, \\
(3p - 2i)q/2p \geq q &\Leftrightarrow i \leq p/2,
\end{aligned}$$

and $-qi/p < 0$, $(p - 2i)q/2p < q$ and $(2p - i)q/p \geq q$ for $0 < i < p$. So we have

$$\sigma^+ = \# \left\{ j \left| 0 < i < \frac{p}{2}, 0 < j < \frac{(p - 2i)q}{2p} \right. \right\}$$
$$+ \# \left\{ j \left| \frac{p}{2} < i < p, \frac{(3p - 2i)q}{2p} < j < q \right. \right\}$$

and

$$\sigma^- = \# \left\{ j \left| 0 < i \leq \frac{p}{2}, \frac{(p - 2i)q}{2p} < j < q \right. \right\}$$
$$+ \# \left\{ j \left| \frac{p}{2} < i < p, 0 < j < \frac{(3p - 2i)q}{2p} \right. \right\}.$$

Since $0 < (p - 2i)/p < 1$ for $0 < i < p/2$, and p and q are coprime, $(p - 2i)q/2p$ is not an integer for $0 < i < p/2$. Suppose $(3p - 2i)q/2p$ is

an integer for some i ($p/2 < i < p$). If q is odd, then $(3p - 2i)/2p$ is an integer. Since $1/2 < (3p - 2i)/2p < 1$ for $p/2 < i < p$, this is absurd. Therefore q is even. Then p is odd, and hence i/p is an integer. This is a contradiction. So $(3p - 2i)q/2p$ is not an integer for $p/2 < i < p$. It follows from that

$$
\begin{aligned}
\sigma^+ = \# & \left\{ j \left| 0 < i < \frac{p}{2}, 0 < j \le \left[\frac{(p - 2i)q}{2p} \right] \right. \right\} \\
& + \# \left\{ j \left| \frac{p}{2} < i < p, \left[\frac{(3p - 2i)q}{2p} \right] < j < q \right. \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma^- = \# & \left\{ j \left| 0 < i \le \frac{p}{2}, \left[\frac{(p - 2i)q}{2p} \right] < j < q \right. \right\} \\
& + \# \left\{ j \left| \frac{p}{2} < i < p, 0 < j \le \left[\frac{(3p - 2i)q}{2p} \right] \right. \right\}.
\end{aligned}
$$

This implies that

$$
\sigma^+ = \sum_{i=1}^{[(p-1)/2]} \left[\frac{(p - 2i)q}{2p} \right] + \sum_{i=[p/2]+1}^{p-1} \left(q - 1 - \left[\frac{(3p - 2i)q}{2p} \right] \right)
$$

and

$$
\sigma^- = \sum_{i=1}^{[p/2]} \left(q - 1 - \left[\frac{(p - 2i)q}{2p} \right] \right) + \sum_{i=[p/2]+1}^{p-1} \left[\frac{(3p - 2i)q}{2p} \right].
$$

Thus it is not hard to see that $\sigma(T(p, q))$ is equal to

$$
2 \sum_{i=1}^{[(p-1)/2]} \left[\frac{(p - 2i)q}{2p} \right] - 2 \sum_{i=[p/2]+1}^{p-1} \left[\frac{(3p - 2i)q}{2p} \right] + (p - 1 - 2[p/2])(q - 1).
$$

We note that

$$
\sum_{i=[p/2]+1}^{p-1} \left[\frac{(3p - 2i)q}{2p} \right] = \sum_{i=1}^{[(p-1)/2]} \left[\frac{(3p - 2[p/2] - 2i)q}{2p} \right].
$$

This completes the proof. ∎

Corollary 2.2. *Let* $T(p,q)$ $(0 < p < q)$ *be a torus knot. Then*

$$\sigma(T(p,q)) \leq -2 \left[\frac{p}{2}\right] \left[\frac{q}{2}\right].$$

Proof. Suppose p is odd. Then, by Proposition 2.1,

$$\sigma(T(p,q)) = 2 \sum_{i=1}^{(p-1)/2} \left(\left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(2p-2i+1)q}{2p}\right] \right).$$

Since

$$
\begin{aligned}
\left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(2p-2i+1)q}{2p}\right] &= \left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(p-2i)q}{2p} + \frac{(p+1)q}{2p}\right] \\
&\leq -\left[\frac{(p+1)q}{2p}\right],
\end{aligned}
$$

we have

$$\sigma(T(p,q)) \leq -(p-1)\left[\frac{(p+1)q}{2p}\right] \leq -(p-1)\left[\frac{q}{2}\right] = -2\left[\frac{p}{2}\right]\left[\frac{q}{2}\right].$$

Suppose p is even. Note that q is odd. Then, by Proposition 2.1,

$$\sigma(T(p,q)) = 2 \sum_{i=1}^{p/2-1} \left(\left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(2p-2i)q}{2p}\right] \right) - q + 1.$$

Since

$$\left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(2p-2i)q}{2p}\right] = \left[\frac{(p-2i)q}{2p}\right] - \left[\frac{(p-2i)q}{2p} + \frac{pq}{2p}\right] \leq -\left[\frac{q}{2}\right],$$

we have

$$\sigma(T(p,q)) \leq -2(p/2-1)\left[\frac{q}{2}\right] - q + 1 = -\frac{(p-2)(q-1)}{2} - q + 1 = -2\left[\frac{p}{2}\right]\left[\frac{q}{2}\right].$$

This completes the proof. ∎

Proposition 2.3. *Let $p(> 0)$ be an odd integer and r $(2 \leq r < p)$ an even integer, and $T(p, p + r)$ a torus knot. Then $\sigma(T(p, p + r))$ is equal to*

$$-\frac{(p-1)(p+r+1)}{2} + 2\sum_{i=1}^{r/2}\left(\left[\frac{(2i-1)p}{2r}\right] - \left[\frac{(2i-1)p+r}{2r}\right]\right)$$

Proof. By Proposition 2.1, we have

$$\sigma(T(p, p+r)) = 2\sum_{i=1}^{(p-1)/2}\left(\left[\frac{(p-2i)(p+r)}{2p}\right] - \left[\frac{(2p-2i+1)(p+r)}{2p}\right]\right).$$

Note that

$$\left[\frac{(p-2i)(p+r)}{2p}\right] = \frac{(p-2i-1)}{2} + \left[\frac{(r+1)}{2} - \frac{ri}{p}\right].$$

Since

$$\left[\frac{(r+1)}{2} - \frac{ri}{p}\right] = \begin{cases} \dfrac{r}{2} & 0 < i \leq \dfrac{p}{2r}, \\ \dfrac{r}{2} - 1 & \dfrac{p}{2r} < i \leq \dfrac{3p}{2r}, \\ \vdots & \vdots \\ 1 & \dfrac{(r-3)p}{2r} < i \leq \dfrac{(r-1)p}{2r}, \\ 0 & \dfrac{(r-1)p}{2r} < i \leq \dfrac{p-1}{2}, \end{cases}$$

we have

$$\sum_{i=1}^{(p-1)/2}\left[\frac{(r+1)}{2} - \frac{ri}{p}\right] = \left[\frac{p}{2r}\right] + \left[\frac{3p}{2r}\right] + \cdots + \left[\frac{(r-1)p}{2r}\right].$$

Meanwhile, we have

$$\left[\frac{(2p-2i+1)(p+r)}{2p}\right] = p - i + \left[r + \frac{1}{2} - \frac{(2i-1)r}{2p}\right].$$

Note that if $p > 2r$, then

$$
\left[r + \frac{1}{2} - \frac{(2i-1)r}{2p} \right] =
\begin{cases}
r & 0 < i \le \dfrac{p+r}{2r}, \\
r-1 & \dfrac{p+r}{2r} < i \le \dfrac{3p+r}{2r}, \\
\vdots & \vdots \\
\dfrac{r}{2}+1 & \dfrac{(r-3)p+r}{2r} < i \le \dfrac{(r-1)p+r}{2r}, \\
\dfrac{r}{2} & \dfrac{(r-1)p+r}{2r} < i \le \dfrac{p-1}{2},
\end{cases}
$$

and if $(r <)p < 2r$, then

$$
\left[r + \frac{1}{2} - \frac{(2i-1)r}{2p} \right] =
\begin{cases}
r & 0 < i \le \dfrac{p+r}{2r}, \\
r-1 & \dfrac{p+r}{2r} < i \le \dfrac{3p+r}{2r}, \\
\vdots & \vdots \\
\dfrac{r}{2}+2 & \dfrac{(r-5)p+r}{2r} < i \le \dfrac{(r-3)p+r}{2r}, \\
\dfrac{r}{2}+1 & \dfrac{(r-3)p+r}{2r} < i \le \dfrac{p-1}{2}.
\end{cases}
$$

This implies

$$
\sum_{i=1}^{(p-1)/2} \left[r + \frac{1}{2} - \frac{(2i-1)r}{2p} \right]
$$
$$
= \left[\frac{p+r}{2r} \right] + \left[\frac{3p+r}{2r} \right] + \cdots + \left[\frac{(r-1)p+r}{2r} \right] + \frac{r(p-1)}{4},
$$

if $p > 2r$, and

$$
\sum_{i=1}^{(p-1)/2} \left[r + \frac{1}{2} - \frac{(2i-1)r}{2p} \right]
$$
$$
= \left[\frac{p+r}{2r} \right] + \left[\frac{3p+r}{2r} \right] + \cdots + \left[\frac{(r-3)p+r}{2r} \right] + \frac{(r+2)(p-1)}{4},
$$

if $p < 2r$. If $p < 2r$, then since p is odd,

$$
\frac{(r+2)(p-1)}{4} - \frac{r(p-1)}{4} = \frac{p-1}{2} = \left[\frac{p+1}{2} - \frac{p}{2r} \right] = \left[\frac{(r-1)p+r}{2r} \right].
$$

Thus we have the required equation.   ∎

Proposition 2.4. *Let $p = 2nr \pm 1(> 0)$ be an integer, r $(2 \leq r < p)$ an even integer, and $T(p, p + r)$ a torus knot. Then*

$$\sigma(T(p, p+r)) = \begin{cases} -\dfrac{(p-1)(p+r+1)}{2} & \text{if } p = 2nr+1, \\ -\dfrac{(p-1)(p+r+1)}{2} - r & \text{if } p = 2nr-1. \end{cases}$$

Proof. We note that $[(2i-1)p/2r] = (2i-1)n$ and $[((2i-1)p+r)/2r] = (2i-1)n$ for $0 < i \leq r/2$ if $p = 2nr+1$, and $[(2i-1)p/2r] = (2i-1)n-1$ and $[((2i-1)p+r)/2r] = (2i-1)n$ for $0 < i \leq r/2$ if $p = 2nr-1$. This and Proposition 2.3 complete the proof.   ∎

Proposition 2.5. *Let $p(> 0)$ be an odd integer and $T(p, p+4)$ a torus knot. Then*

$$\sigma(T(p, p+4)) = \begin{cases} -\dfrac{(p-1)(p+5)}{2} & \text{if } p \equiv 1, \text{ or } \equiv 3 \ (mod \ 8), \\ -\dfrac{(p-1)(p+5)}{2} - 4 & \text{if } p \equiv 5, \text{ or } \equiv 7 \ (mod \ 8). \end{cases}$$

Proof. Since $\sigma(T(1,5)) = 0$, $\sigma(T(3,7)) = -8$, we may assume that $p > 4$. By combining Proposition 2.3 and the following, we complete proof.

$$\left[\frac{p}{8}\right] + \left[\frac{3p}{8}\right] - \left[\frac{p+4}{8}\right] - \left[\frac{3p+4}{8}\right] = \begin{cases} 0 & \text{if } p \equiv 1, \text{ or } \equiv 3 \ (mod \ 8), \\ -2 & \text{if } p \equiv 5, \text{ or } \equiv 7 \ (mod \ 8). \end{cases}$$

∎

3   Proofs of Theorems 1.1, 1.3, 1.5 and Proposition 1.8

Similar results to the following two lemmas, Lemmas 3.1 and 3.2, are mentioned in several articles [19], [20], [21], [8], [9], [3], etc. The first lemma is a spacial case of [3, Lemma 4.4]. The second one is proven by combining [9, Example 2] and the proof of [9, Lemma 2.3].

Lemma 3.1. *Let K_1 and K_2 be knots. Let M be a twice punctured $-\varepsilon \mathbb{C}P^2$. If $K_1 \overset{(\varepsilon,\omega)}{\to} K_2$, then there exists an annulus A in M such that $(\partial M, \partial A) \cong (-S^3, -K_1) \cup (S^3, K_2)$ and A represents a homology element $\omega \gamma$, where $|\varepsilon| = 1$ and γ is a standard generator of $H_2(M, \partial M; \mathbb{Z})$ with the intersection number $\gamma \cdot \gamma = -\varepsilon$.* ∎

Lemma 3.2. *Let K_1 and K_2 be knots. Let M be a twice punctured $S^2 \times S^2$. If $K_1 \overset{(2n,\omega)}{\to} K_2$, then there exists an annulus A in M such that $(\partial M, \partial A) \cong (-S^3, -K_1) \cup (S^3, K_2)$ and A represents a homology element $\omega \alpha - n\omega\beta$, where α, β are standard generators of $H_2(M, \partial M; \mathbb{Z})$ with $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$.* ∎

The following theorem is originally due to O.Ya. Viro [18]. It is also obtained by letting $a = [d/2]$ in the inequality of [2, Remarks(a) on p-371] by P. Gilmer.

Theorem 3.3. (P.M. Gilmer [2], O.Ya. Viro [18]) *Let M be a compact, oriented, once punctured 4-manifold, and K a knot in ∂M. Suppose that K bounds a properly embedded, oriented surface F in M that represents an element $\xi \in H_2(M, \partial M; \mathbb{Z})$. If ξ is divisible by a prime integer d, then we have*

$$\left| \frac{2[d/2](d - [d/2])}{d^2} \xi \cdot \xi - \sigma(M) - \sigma_d(K) \right| \leq \dim H_2(M; \mathbb{Z}_p) + 2\operatorname{genus}(F).$$

∎

The following is a well known result for $d = 2$ [11] [1]. J.H. Przytycki showed it in [13]. Here we show it by using Theorem 3.3.

Lemma 3.4. *Let K_+ and K_- be knots. If K_- is obtained from K_+ by changing a positive crossing into negative one, then for any prime integer d*

$$\sigma_d(K_-) - 2 \leq \sigma_d(K_+) \leq \sigma_d(K_-).$$

Proof. It is not hard to see that $K_+ \overset{(1,0)}{\to} K_-$. This implies that $K_+ \#(-\overline{K_+}) \overset{(1,0)}{\to} K_- \#(-\overline{K_+})$, where $-\overline{K_+}$ is the reflected inverse of K_+. Since $K_+ \#(-\overline{K_+})$ is a slice knot, by Lemma 3.1, there is a 2-disk in once punctured $-\mathbb{C}P^2$ bounded by $K_- \#(-\overline{K_+})$ that represents the

zero element. By Theorem 3.3, we have $|1 - \sigma_d(K_- \# (-\overline{K_+}))| \leq 1$, so we have $|1 - \sigma_d(K_-) + \sigma_d(K_+)| \leq 1$. This completes the proof. ∎

Proposition 3.5. *Let $T(p, q)$ $(0 < p < q)$ be a torus knot. If $T(p, q)$ is neither a trivial knot nor $T(2, 3)$, then $\sigma_d(T(p, q)) \leq -4$ for any prime integer d.*

Proof. In [15], J.H. Przytycki and K. Taniyama showed that, except for connected sums of pretzel knots $P(p_1, p_2, p_3)$ ($p_1 p_2 p_3$ is odd), a *positive knot* can be   deformed into $T(2, 5)$ by changing some positive crossings to be negative, where a positive knot is a knot that has a diagram with all crossings positive. Since $T(p, q)$ is a prime, positive knot and genus$(T(p, q)) \neq$ genus$(P(p_1, p_2, p_3)) = 1$, $T(2, 5)$ is obtained from $T(p, q)$ by changing some positive crossings. Since $\sigma_d(T(2, 5)) = -4$ for any prime integer d ([16, Lemma 3.5]), by Lemma 3.4, we have the conclusion. ∎

Proof of Theorem 1.1. Note that $p \geq 5$ since $T(p, q)$ is non-exceptional. In [7], K. Miyazaki and K. Motegi showed that if a non-exceptional torus knot is obtained from a trivial knot by a single (n, ω)-twisting, then $|n| = 1$. We may assume that $T(p, q)$ is obtained from a trivial knot by a single (ε, ω)-twisting, where $|\varepsilon| = 1$. By Lemma 3.1, there is a 2-disk Δ in a punctured $-\varepsilon \mathbb{C}P^2$, M, such that $(\partial M, \partial \Delta) \cong (S^3, T(p, q))$ and Δ represents $\omega \gamma \in H_2(M, \partial M; \mathbb{Z})$. If ω is divisible by a prime integer d, by Theorem 3.3,

$$\left| -\frac{2\varepsilon[d/2](d - [d/2])}{d^2} \omega^2 + \varepsilon - \sigma_d(T(p, q)) \right| \leq 1.$$

By Proposition 3.5, $\sigma_d(T(p, q)) \leq -4$. This gives condition (i), i.e., $\varepsilon = 1$. So we have

$$\left| -\frac{2[d/2](d - [d/2])}{d^2} \omega^2 + 1 - \sigma_d(T(p, q)) \right| \leq 1.$$

This implies

$$-\frac{\sigma_d(T(p, q))}{2} \leq \frac{[d/2](d - [d/2])}{d^2} \omega^2 \leq 1 - \frac{\sigma_d(T(p, q))}{2}.$$

Since ω is divisible by d, $[d/2](d - [d/2])\omega^2/d^2$ is an integer. This and the fact that $\sigma_d(T(p, q))$ is even ([16, Lemma 2.16]) give condition (iv).

It is known that $T(p, q)$ bounds an orientable surface in S^3 with genus $(p-1)(q-1)/2$. Therefore we have a closed, orientable surface with genus $(p-1)(q-1)/2$ in $-\mathbb{C}P^2$ that represents $\omega\gamma$. Thom Conjecture, which is solved by P. Kronheimer and T. Mrowka [5], implies

$$\frac{(\omega - 1)(\omega - 2)}{2} \leq \frac{(p-1)(q-1)}{2}.$$

Since $p - 1 < q - 2$, we have (ii) $\omega < q$. Suppose that ω is even. By condition (iv), $\omega^2 \geq -2\sigma(T(p,q))$. Since $q - 1 \geq p + 1$, by Corollary 2.2,

$$\omega^2 \geq \begin{cases} (p-1)(p+1) > (p-1)^2 & \text{if } p \text{ is odd,} \\ p(p+1) > p^2 & \text{if } p \text{ is even.} \end{cases}$$

Thus we have $\omega > p - 1$ if p is odd, and $\omega > p$ if p is even. Since ω is even, we have condition (iii). ∎

To prove Theorems 1.3, 1.5 and Proposition 1.8, we need the following theorem.

Theorem 3.6. (K. Kikuchi [4]) *Let M be a closed, oriented simply connected 4-manifold with $b_2^+(M) \leq 3$ and $b_2^-(M) \leq 3$. Let ξ be a characteristic element of $H_2(M; \mathbb{Z})$. If ξ is represented by a 2-sphere, then $\xi \cdot \xi = \sigma(M)$. Here $b_2^+(M)$ (resp. $b_2^-(M)$) is the rank of positive (resp. negative) part of the intersection form of M.* ∎

Proof of Theorem 1.5. (1) If $T(p, p + r)$ is obtained from a trivial knot by an (n, ω)-twisting, then by Theorem 1.1, $n = 1$ and $\omega > p$ if ω is even.

Suppose ω is even. Then by Theorem 1.1 and Proposition 2.4, $(p-1)(p+r+1) = \omega^2, \omega^2 - 4$. Hence we have $(p+r/2+\omega)(p+r/2-\omega) = (r/2+1)^2$ or $(r/2+1)^2 - 4$. Since $r \geq 4$,

$$0 < \left(\frac{r}{2} + 1\right)^2 - 4 \leq \left(p + \frac{r}{2} + \omega\right)\left(p + \frac{r}{2} - \omega\right) \leq \left(\frac{r}{2} + 1\right)^2.$$

This is absurd because $p + r/2 + \omega > 2p + r/2 \geq (r/2+1)^2 - r/2 + r/2 = (r/2+1)^2$.

Suppose ω is odd. Set $p = 2nr + 1$ ($n \geq 1$). Then we have $O \overset{(1,\omega)}{\to} T(p, p+r) \overset{(-1,p)}{\to} T(p, r) \cong T(r, 2nr+1) \overset{(-2n,r)}{\to} T(r, 1) \cong O$. By Lemmas 3.1 and 3.2, there is a 2-sphere in $-\mathbb{C}P^2 \# \mathbb{C}P^2 \# S^2 \times S^2$ that represents

a characteristic element $\omega\overline{\gamma} + p\gamma + r\alpha + nr\beta$. By Theorem 3.6, $-\omega^2 + p^2 + 2nr^2 = 0$. (Note that $\omega^2 = p^2 + 2nr^2 > p^2$.) Hence we have $(p + r/2 + \omega)(p + r/2 - \omega) = (r/2 + 1)^2 - 1$. This is absurd because $r \geq 4$, $\omega > p$ and $2p \geq (r/2 + 1)^2 - r/2$

(2) By the argument similar to above, we have the conclusion. ■

Proof of Theorem 1.3. The case that $p \equiv 1 \pmod 8$ is a special case of Theorem 1.5; see Example 1.6. Suppose that $p \equiv 3 \pmod 8$. Set $p = 8n + 3$ ($n \geq 1$). Then we have $O \stackrel{(1,\omega)}{\to} T(p, p+4) \stackrel{(-1,p)}{\to} T(p, 4) \cong T(4, 8n+3) \stackrel{(-2n,4)}{\to} T(4, 3) \cong T(3, 4) \stackrel{(-1,3)}{\to} T(3, 1) \cong O$. By the argument similar to that in the proof of Theorem 1.5, we have the conclusion. (Here we use Proposition 2.5 instead of Proposition 2.4.) ■

Proof of Proposition 1.8. Since $\sigma(T(5, 8)) = -20$ and $T(5, 8) \stackrel{(-1,5)}{\to} T(5, 3) \stackrel{(-1,5)}{\to} T(5, -2) \cong T(2, -5) \stackrel{(2,2)}{\to} T(2, -1) \cong O$. By the argument similar to that in the proof of Theorem 1.5, we have the conclusion. ■

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