ON THE SINGULAR NUMBERS FOR SOME INTEGRAL OPERATORS

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Abstract

Two–sided estimates of Schatten–von Neumann norms for weighted Volterra integral operators are established. Analogous problems for some potential-type operators defined on $\mathbb{R}^n$ are solved.

Let H be a separable Hilbert space and let $\sigma_\infty(H)$ be the class of all compact operators $T : H \to H$, which forms an ideal in the normed algebra $\mathbb{B}$ of all bounded linear operators on H. To construct a Schatten-von Neumann ideal $\sigma_p(H)$ $(0 < p \leq \infty)$ in $\sigma_\infty(H)$, the sequence of singular numbers $s_j(T) \equiv \lambda_j(|T|)$ is used, where the eigenvalues $\lambda_j(|T|)$ ($|T| \equiv (T^*T)^{1/2}$) are non-negative and are repeated according to their multiplicity and arranged in decreasing order. A Schatten-von Neumann quasinorm (norm if $1 \leq p \leq \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \Big(\sum_j s_j^p(T) \Big)^{1/p}, \ \ 0 < p < \infty,$$

with the usual modification if $p = \infty$. Thus we have $\|T\|_{\sigma_\infty(H)} = \|T\|$ and $\|T\|_{\sigma_2(H)}$ is the Hilbert- Schmidt norm given by the formula

$$\|T\|_{\sigma_2(H)} = \Big(\int \int |T_1(x,y)|^2 dxdy \Big)^{1/2} \tag{1}$$

for an integral operator

$$Tf(x) = \int T_1(x,y)f(y)dy.$$

We refer, for example, to [2], [6], [7] for more information concerning Schatten-von Neumann ideals.

In this paper necessary and sufficient conditions for the weighted Volterra integral operator

$$K_v f(x) = v(x) \int_0^x f(y) k(x,y) dy, \quad x \in (0,a),$$

to belong to Schatten-von Neumann ideals are established, where v is a measurable function on $(0,a)$ $(0 < a \leq \infty)$.

Two-sided estimates of Schatten-von Neumann p-norms for the weighted Riemann–Liouville operator

$$R_{\alpha,v} f(t) = v(x) \int_0^x f(t)(x-t)^{\alpha-1} dt,$$

when $\alpha > 1/2$ and $p > 1/\alpha$, were established in [13] (for $\alpha = 1$ and $p > 1$ see [14]). Analogous results for the weighted Hardy operator

$$H_{v,u} f(x) = v(x) \int_0^x u(y) f(y) dy$$

were obtained in [3]. Similar problems for the Riemann-Liouville operator with two weights

$$R_{\alpha,v,u} f(x) = v(x) \int_0^x u(t) f(t)(x-t)^{\alpha-1} dt,$$

when $\alpha \in \mathbb{N}$ and $p \geq 1$, were solved in [4]. Further, upper and lower bounds for Schatten–von Neumann p-norms $(p \geq 2)$ of certain Volterra integral operators, involving $R_{\alpha,v,u}$ only for $\alpha \geq 1$, were proved in [4] and [18].

Our main goal is to generalize the results of [13] and [14] for integral transforms with kernels and to give two-sided estimates of the above-mentioned norms for these operators in terms of their kernels.

We denote by $L_w^p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, a weighted Lebesgue space with respect to the weight w defined on Ω.

Throughout the paper the expression $A \approx B$ is interpreted as $c_1 A \leq B \leq c_2 A$ with some positive constants c_1 and c_2.

Let us recall some definitions from [10] (see also [8]).

We say that a kernel $k : \{(x,y) : 0 < y < x < a\} \to \mathbb{R}_+$ belongs to V $(k \in V)$ if there exists a positive constant d_1 such that for all x, y, z with $0 < y < z < x < a$ the inequality

$$k(x,y) \leq d_1 k(x,z)$$

holds. Further, $k \in V_\lambda$ $(1 < \lambda < \infty)$ if there exists a positive constant d_2 such that for all x, $x \in (0, a)$, the inequality

$$\int_{x/2}^{x} k^{\lambda'}(x, y) dy \leq d_2 x k^{\lambda'}(x, x/2), \quad \lambda' = \frac{\lambda}{\lambda - 1}.$$

is fulfilled.

For example, if $k_1(x) = x^{\alpha-1}$, where $\frac{1}{\lambda} < \alpha \leq 1$, then $k(x, y) = k_1(x - y)$ belongs to $V \cap V_\lambda$ (for other examples of kernel k see [10], [8]).

First we investigate the mapping properties of K_v in Lebesgue spaces.

The following statements in equivalent form were proved in [10] (see also [8], [11]).

Theorem A. *Let $1 < p \leq q < \infty$, $a = \infty$ and let $k \in V \cap V_p$. Then*
  (a) *K_v is bounded from $L^p(0, \infty)$ into $L^q(0, \infty)$ if and only if*

$$D_\infty \equiv \sup_{j \in \mathbb{Z}} D_\infty(j) \equiv \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} k^q(x, x/2) x^{q/p'} |v(x)|^q dx \right)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_v\| \approx D_\infty$.
  (b) *K_v acts compactly from $L^p(0, a)$ into $L^q(0, a)$ if and only if $D_\infty < \infty$ and $\lim_{j \to +\infty} D_\infty(j) = \lim_{j \to -\infty} D_\infty(j) = 0$.*

Theorem B. *Let $1 < p \leq q < \infty$, $a < \infty$ and let $k \in V \cap V_p$. Then*
  (a) *K_v is bounded from $L^p(0, a)$ to $L^q(0, a)$ if and only if*

$$D_a \equiv \sup_{j \geq 0} D_a(j) \equiv \sup_{j \geq 0} \left(\int_{2^{-(j+1)}a}^{2^{-j}a} |v(x)|^q k^q(x, x/2) x^{q/p'} dx \right)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_v\| \approx D_a$.
  (b) *K_v acts compactly from $L^p(0, a)$ into $L^q(0, a)$ if and only if $D_a < \infty$ and $\lim_{j \to +\infty} D_a(j) = 0$;*

Analogous problems for the Riemann-Liouville operator for $\alpha > 1/p$ were solved in [9] (For boundedness two-weight criteria of general integral operators with positive kernels see [5], Chapter 3).

Let $0 < a \leq \infty$, $k : \{(x, y) : 0 < y < x < a\} \to \mathbb{R}^1_+$ be a kernel and let $k_0(x) \equiv x k^2(x, x/2)$.

We denote by $l^p(L^2_{k_0}(0, a))$ the set of all measurable functions $g : (0, a) \to \mathbb{R}^1$ for which

$$\|g\|_{l^p(L^2_{k_0}(0,\infty))} = \left(\sum_{n \in \mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} |g(x)|^2 k_0(x) dx \right)^{p/2} \right)^{1/p} < \infty$$

if $a = \infty$ and

$$\|g\|_{l^p(L^2_{k_0}(0,a))} = \left(\sum_{n=0}^{+\infty} \left(\int_{2^{-(n+1)}a}^{2^{-n}a} |g(x)|^2 k_0(x) dx \right)^{p/2} \right)^{1/p} < \infty$$

if $a < \infty$, with the usual modification for $p = \infty$.

We shall need the following interpolation result (see, e.g., [19], p. 147 for the interpolation properties of the Schatten classes, and p. 127 for the corresponding properties of the sequence spaces. See also [1], Theorem 5.1.2):

Proposition A. *Let* $0 < a \leq \infty$, $1 \leq p_0, p_1 \leq \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *If* T *is a bounded operator from* $lp_i(L^2_{k_0}(0, a))$ *into* $\sigma_{p_i}(L^2(0, a))$, *where* $i = 0, 1$, *then it is also bounded from* $l^p(L^2_{k_0}(0, a))$ *into* $\sigma_p(L^2(0, a))$. *Moreover,*

$$\|T\|_{l^p(L^2_{k_0}) \to \sigma_p(L^2)} \leq \|T\|_{l^{p_0}(L^2_{k_0}) \to \sigma_{p_0}(L^2)}^{1-\theta} \|T\|_{l^{p_1}(L^2_{k_0}) \to \sigma_{p_1}(L^2)}^{\theta}.$$

The next statement is obvious when $p = \infty$; and when $1 \leq p < \infty$ it follows from Lemma 2.11.12 of [15].

Proposition B. *Let* $1 \leq p \leq \infty$ *and let* $\{f_k\}$, $\{g_k\}$ *be orthonormal systems in a Hilbert space* H. *If* $T \in \sigma_p(H)$, *then*

$$\|T\|_{\sigma_p(H)} \geq \left(\sum_n |\langle T f_n, g_n \rangle|^p \right)^{1/p}.$$

Now we prove the main results.

In the sequel we shall assume that $v \in L^2_{k_0}(2^n, 2^{n+1})$ for all $n \in \mathbb{Z}$.

Theorem 1. *Let* $a = \infty$, $2 \leq p < \infty$ *and let* $k \in V \cap V_2$. *Then* K_v *belongs to* $\sigma_p(L^2(0, \infty))$ *if and only if* $v \in l^p(L^2_{k_0}(0, \infty))$. *Moreover, there exist positive constants* b_1 *and* b_2 *such that*

$$b_1 \|v\|_{l^p(L^2_{k_0}(0,\infty))} \leq \|K_v\|_{\sigma_p(L^2(0,\infty))} \leq b_2 \|v\|_{l^p(L^2_{k_0}(0,\infty))}.$$

Proof. *Sufficiency.* Note that the fact $k \in V \cap V_2$ implies

$$I(x) \equiv \int_0^x k^2(x,y)dy \le ck_0(x) \qquad (2)$$

for some positive constant c independent of x. Indeed, by the condition $k \in V \cap V_2$ we have

$$I(x) = \int_0^{x/2} k^2(x,y)dy + \int_{x/2}^x k^2(x,y)dy \le c_1 k_0(x) + c_2 k_0(x) = c_3 k_0(x).$$

Consequently, using the Hilbert-Schmidt formula (1) and taking into account (2), we find that

$$\|K_v\|_{\sigma_2(L^2(0,\infty))} = \left(\int_0^\infty \int_0^x k^2(x,y)v^2(x)dxdy \right)^{1/2}$$

$$= \left(\int_0^\infty v^2(x) \left(\int_0^x k^2(x,y)dy \right) dx \right)^{1/2} \le c_4 \left(\int_0^\infty v^2(x)k_0(x)dx \right)^{1/2}$$

$$= c_4 \left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx \right)^{1/2} = c_4 \|v\|_{l^2(L^2_{k_0}(0,\infty))}.$$

On the other hand, in view of Theorem A we see that there exist positive constants c_5 and c_6 such that

$$c_5 \|v\|_{l^\infty(L^2_{k_0}(0,\infty))} \le \|K_v\|_{\sigma_\infty(L^2(0,\infty))} \le c_6 \|v\|_{l^\infty(L^2_{k_0}(0,\infty))}.$$

Further, Proposition A yields

$$\|K_v\|_{\sigma_p(L^2(0,\infty))} \le c_7 \|v\|_{l^p((L^2_{k_0}(0,\infty))},$$

where $2 \le p < \infty$.

Necessity. Let $K_v \in \sigma_p(L^2(0,\infty))$ and let

$$f_n(x) = \chi_{[2^n,2^{n+1})}(x)2^{-n/2},$$

$$g_n(x) = v(x)x^{1/2}\chi_{[3\cdot2^{n-1},2^{n+1})}(x)k(x,x/2)\alpha_n^{-1/2},$$

where

$$\alpha_n = \int_{3\cdot 2^{n-1}}^{2^{n+1}} v^2(y)k_0(y)dy.$$

Then it is easy to verify that $\{f_n\}$ and $\{g_n\}$ are orthonormal systems. Further, by virtue of Proposition B (for $p \geq 1$) we have

$$\infty > \|K_v\|_{\sigma_p(L^2(0,\infty))} \geq \left(\sum_{n\in\mathbb{Z}} |\langle K_v f_n, g_n\rangle|^p \right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}} \left(\int_{3\cdot 2^{n-1}}^{2^{n+1}} \left(\int_{2^n}^{x} 2^{-n/2}k(x,y)dy \right) v^2(x)x^{1/2}k(x,x/2)\alpha_n^{-1/2}dx \right)^p \right)^{1/p}$$

$$\geq c_8 \left(\sum_{n\in\mathbb{Z}} \left(\alpha_n^{-1/2} \int_{3\cdot 2^{n-1}}^{2^{n+1}} 2^{-n/2}k(x,x/2)v^2(x)(x-2^n)x^{1/2}dx \right)^p \right)^{1/p}$$

$$\geq c_9 \left(\sum_{n\in\mathbb{Z}} \left(\alpha_n^{-1/2} \int_{3\cdot 2^{n-1}}^{2^{n+1}} k_0(x)v^2(x)dx \right)^p \right)^{1/p} = c_9 \left(\sum_{n\in\mathbb{Z}} \alpha_n^{p/2} \right)^{1/p}.$$

Now let

$$f_n'(x) = \chi_{[3\cdot 2^{n-2},3\cdot 2^{n-1})}(x)(3\cdot 2^{n-2})^{-1/2}$$

and

$$g_n'(x) = v(x)x^{1/2}\chi_{[2^n,3\cdot 2^{n-1})}(x)k(x,x/2)\beta_n^{-1/2},$$

where

$$\beta_n = \int_{2^n}^{3\cdot 2^{n-1}} v^2(y)k_0(y)dy.$$

Then it is easy to verify that $\{f_m'\}$ and $\{g_m'\}$ are orthonormal systems. Further,

$$\infty > \|K_v\|_{\sigma_p(L^2(0,\infty))} \geq \left(\sum_{n\in\mathbb{Z}} |\langle K_v f_n', g_n'\rangle|^p \right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{3\cdot 2^{n-1}} \left(\int_{3\cdot 2^{n-2}}^{x} (3\cdot 2^{n-2})^{-1/2}k(x,y)dy \right) \right. \right.$$

$$\left. \left. \times v^2(x)x^{1/2}k(x,x/2)\beta_n^{-1/2}dx \right)^p \right)^{1/p}$$

$$\geq c_{10}\bigg(\sum_{n\in\mathbb{Z}} \bigg(\beta_n^{-1/2} \int_{2^n}^{3\cdot 2^{n-1}} 2^{-(n-2)/2} k^2(x,x/2)v^2(x)$$

$$\times (x - 3\cdot 2^{n-2})x^{1/2}dx \bigg)^p \bigg)^{1/p}$$

$$\geq c_{11}\bigg(\sum_{n\in\mathbb{Z}} \bigg(\beta_n^{-1/2} \int_{2^n}^{3\cdot 2^{n-1}} k_0(x)v^2(x)dx \bigg)^p \bigg)^{1/p} = c_{11}\bigg(\sum_{n\in\mathbb{Z}} \beta_n^{p/2} \bigg)^{1/p},$$

where $p \geq 1$. Consequently

$$\bigg(\sum_{n\in\mathbb{Z}} \bigg(\int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx \bigg)^{p/2} \bigg)^{1/p} \leq \bigg(\sum_{n\in\mathbb{Z}} (\beta_n + \alpha_n)^{p/2} \bigg)^{1/p}$$

$$\leq c_{12}\|K_v\|_{\sigma_p(L^2(0,\infty))} + c_{12}\|K_v\|_{\sigma_p(L^2(0,\infty))}$$

$$\leq c_{13}\|K_v\|_{\sigma_p(L^2(0,\infty))} < \infty.$$

∎

Let us now consider the case $a < \infty$. We have the following statement:

Theorem 2. *Let $0 < a < \infty$, $2 \leq p < \infty$ and let $k \in V \cap V_2$. Then K_v belongs to $\sigma_p(L^2(0,a))$ if and only if $v \in l^p(L^2_{k_0}(0,a))$. Moreover, there exists positive constants b_1 and b_2 such that*

$$b_1\|v\|_{l^p(L^2_{k_0}(0,a))} \leq \|K_v\|_{\sigma_p(L^2(0,a))} \leq b_2\|v\|_{l^p(L^2_{k_0}(0,a))}.$$

Proof. *Sufficiency.* The Hilbert– Schmidt formula and the condition $k \in V \cap V_2$ yield

$$\|K_v\|_{\sigma_p(L^2(0,a))} = \bigg(\int_0^a v^2(x)\bigg(\int_0^x k^2(x,y)dy \bigg)dx \bigg)^{1/2}$$

$$\leq c_1\bigg(\int_0^a v^2(x)k_0(x)dx \bigg)^{1/2}$$

$$= c_1\bigg(\sum_{n=0}^{\infty} \int_{2^{-(n+1)}a}^{2^{-n}a} v^2(x)k_0(x)dx \bigg)^{1/2} = c_1\|v\|_{l^2(L^2_{k_0}(0,a))}.$$

In view of Theorem B (part (a)) we arrive at

$$\|K_v\|_{\sigma_\infty(L^2(0,a))} \approx \|v\|_{l^\infty(L^2_{k_0}(0,a))}.$$

Using Proposition A we derive

$$\|K_v\|_{\sigma_p(L^2(0,a))} \leq c_2 \|v\|_{l^p(L^2_{k_0}(0,a))}$$

when $p \geq 2$.

To prove necessity we take the orthonormal systems of functions defined on $(0,a)$:

$$f_n(x) = \chi_{[2^{-(n+1)}a,2^{-n}a)}(x)(2^{-(n+1)}a)^{-1/2}$$

and

$$g_n(x) = v(x)x^{1/2}\chi_{[3\cdot2^{-(n+2)}a,2^{-n}a)}(x)k(x,x/2)\alpha_n^{-1/2},$$

where

$$\alpha_n = \int_{3\cdot2^{-(n+2)}a}^{2^{-n}a} v^2(y)k_0(y)dy$$

and $n = 0,1,2,\cdots$. Consequently Proposition B yields

$$\infty > \|K_v\|_{\sigma_p(L^2(0,a))} \geq \left(\sum_{n=0}^{+\infty} |\langle K_v f_n, g_n\rangle|^p\right)^{1/p}$$

$$= \left(\sum_{n=0}^{\infty} \left(\int_{3\cdot2^{-(n+2)}a}^{2^{-n}a} x^{1/2}v^2(x)k(x,x/2)\right.\right.$$

$$\times \left.\left.\left(\int_{2^{-(n+1)}a}^{x} (2^{-(n+1)}a)^{-1/2}k(x,y)dy\right)\alpha_n^{-1/2}dx\right)^p\right)^{1/p}$$

$$\geq c_3\left(\sum_{n=0}^{\infty} \alpha_n^{p/2}\right)^{1/p}.$$

If we take the following orthonormal systems:

$$f_n'(x) = \chi_{[3\cdot2^{-(n+3)}a,3\cdot2^{-(n+2)}a)}(x)(3\cdot2^{-(n+3)}a)^{-1/2},$$

$$g_n'(x) = v(x)x^{1/2}\chi_{[2^{-(n+1)}a,3\cdot2^{-(n+2)}a)}(x)k(x,x/2)\beta_n^{-1/2},$$

where

$$\beta_n = \int_{2^{-(n+1)}a}^{3\cdot 2^{-(n+2)}a} v^2(y)k_0(y)dy,$$

then we arrive at the estimate

$$\|K_v\|_{\sigma_p(L^2(0,a))} \geq c_4 \left(\sum_{n=0}^{\infty} \beta_n^{p/2} \right)^{1/p}.$$

Finally we have the lower estimate for $\|K_v\|_{\sigma_p(L^2(0,a))}$.   ∎

Remark 1. It follows from the proof of Theorems 1 and 2 that the lower estimate of $\|K_v\|_{\sigma_p(L^2(0,a))}$ holds for $1 \leq p \leq \infty$.

Now we formulate and prove the next statement.

Proposition 1. *Let* $1 \leq p < \infty$. *Then*

$$\|v\|_{l^p(L^2_{k_0}(0,\infty))} \approx J(v,p),$$

where

$$J(v,p) = \left(\int_0^{\infty} \left(\int_{x/2}^{2x} v^2(y)k^2(y,y/2)dy \right)^{p/2} x^{p/2-1}dx \right)^{1/p}.$$

Proof. We have

$$\|v\|_{l^p(L^2_{k_0}(0,\infty))} = \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k_0(x)dx \right)^{p/2} \right)^{1/p}$$

$$\leq \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} 2^{(n+1)p/2} \right)^{1/p}$$

$$= c_1 \left(\sum_{n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} 2^{np/2} \right)^{1/p}$$

$$\leq c_2 \left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left(\int_{2^n}^{2^{n+1}} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p}$$

$$\leq c_2 \left(\sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left(\int_{y/2}^{2y} v^2(x)k^2(x,x/2)dx \right)^{p/2} dy \right)^{1/p} = c_2 J(v,p).$$

To prove the reverse inequality we observe that

$$
J(v,p) = \left(\sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} y^{p/2-1} \left(\int_{y/2}^{2y} v^2(x) k^2(x, x/2) dx \right)^{p/2} dy \right)^{1/p}
$$

$$
\leq \left(\sum_{n \in \mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} y^{p/2-1} dy \right) \left(\int_{2^{n-1}}^{2^{n+2}} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p}
$$

$$
\leq c_3 \left(\sum_{n \in \mathbb{Z}} 2^{np/2} \left(\int_{2^{n-1}}^{2^n} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p}
$$

$$
+ c_3 \left(\sum_{n \in \mathbb{Z}} 2^{np/2} \left(\int_{2^n}^{2^{n+1}} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p}
$$

$$
+ c_3 \left(\sum_{n \in \mathbb{Z}} 2^{np/2} \left(\int_{2^{n+1}}^{2^{n+2}} v^2(x) k^2(x, x/2) dx \right)^{p/2} \right)^{1/p} \leq c_4 \|v\|_{l^p(L^2_{k_0}(0,\infty))}.
$$

∎

From Theorem 1 and Proposition 1 we easily derive the following statement:

Theorem 3. *Let $2 \leq p < \infty$ and let $k \in V \cap V_\lambda$. Then*

$$
\|K_v\|_{\sigma_p(L^2(0,\infty))} \approx J(v,p).
$$

A result analogous to Theorem 1 was obtained in [13] for the Riemann-Liouville operator $R_{\alpha,v}$, assuming that $\alpha > 1/2$ and $p > 1/\alpha$ (see [14] for $\alpha = 1$ and $p > 1$).

Let us now consider the multidimensional case. In particular, we shall deal with the operator

$$
B^\alpha_{+,v} f(x) = v(x) \int_{|y|<|x|} \frac{\left(|x|^2 - |y|^2 \right)^\alpha}{|x-y|^n} f(y) dy, \quad \alpha > 0,
$$

where v is a Lebesgue-measurable function on $\mathbb{R}^n$ with $v \in L^2(\{2^n < |y| < 2^{n+1}\})$ for all $n \in \mathbb{Z}$ (for the definition and some properties of $B_{+,v}$, where $v \equiv 1$, see, e.g., [16], Chapter 7, and [17], Section 29).

Let w be a measurable a.e. positive function on $\mathbb{R}^n$. We denote by $l^p(L^2_w(\mathbb{R}^n))$ a set of all measurable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^1$ for which

$$\|\varphi\|_{l^p(L^2_w(\mathbb{R}))} = \left(\sum_{k \in \mathbb{Z}} \left(\int_{2^k < |x| < 2^{k+1}} \varphi^2(x) w(x) dx \right)^{p/2} \right)^{1/p} < \infty.$$

The next result is from [19] (pp. 127, 147).

Proposition C. *Let* $1 \leq p_0,\ p_1 \leq \infty,\ 0 \leq \theta \leq 1,\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$ *If* T *is a bounded operator from* $l^{p_i}(L^2_w(\mathbb{R}^n))$ *into* $\sigma_{p_i}(L^2_w(\mathbb{R}^n))$, *where* $i = 0, 1$, *then it is also bounded from* $l^p(L^2_w(\mathbb{R}^n))$ *into* $\sigma_p(L^2(\mathbb{R}^n))$.

In the sequel we shall use the notation $l^p(L^2_{|x|^\beta}(\mathbb{R}^n)) \equiv l^p(L^2_\beta(\mathbb{R}^n))$.

First we formulate some statements concerning the mapping properties of $B^\alpha_{+,v}$.

Theorem C ([12]). *Let* $1 < p \leq q < \infty,\ \alpha > \frac{n}{p}.$ *Then* $B^\alpha_{+,v}$ *acts boundedly from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *if and only if*

$$F \equiv \sup_{j \in \mathbb{Z}} F(j) \equiv \sup_{j \in \mathbb{Z}} \left(\int_{2^j < |x| < 2^{j+1}} |v(x)|^q |x|^{q(2\alpha - n/p)} dx \right)^{1/q} < \infty.$$

Moreover, $\left\| B^\alpha_{+,v} \right\| \approx F.$

The following result can be obtained in the same as Theorem 5 from [12], therefore we omit the proof (see also [11]).

Theorem D. *Let* $1 < p \leq q < \infty$ *and let* $\alpha > \frac{n}{p}.$ *Then* $B^\alpha_{+,v}$ *acts compactly from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *if and only if* $F < \infty$ *and* $\lim_{j \to -\infty} F(j) = \lim_{j \to +\infty} F(j) = 0.$

Now we state and prove the following Theorem:

Theorem 4. *Let* $2 \leq p < \infty$ *and let* $\alpha > n/2.$ *Then* $B^\alpha_{+,v} \in \sigma_p(L^2(\mathbb{R}^n))$ *if and only if* $v \in l^p(L^2_{4\alpha-n}(\mathbb{R}^n)).$ *Moreover, there exist positive constants* b_1 *and* b_2 *such that*

$$b_1 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \|B^\alpha_{+,v}\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq b_2 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}.$$

Proof. For sufficiency, we use the Hilbert-Schmidt formula (1) and the condition $\alpha > \frac{n}{2}$. Thus,

$$\left\| B_{+,v}^{\alpha} \right\|_{\sigma_2(L^2(\mathbb{R}^n))} = \left(\int\limits_{\mathbb{R}^n} v^2(x) \left(\int\limits_{|y|<|x|} \frac{\left(|x|^2 - |y|^2 \right)^{2\alpha}}{|x-y|^{2n}} dy \right) dx \right)^{\frac{1}{2}}$$

$$\leq c_1 \left(\int\limits_{\mathbb{R}^n} |x|^{2\alpha} v^2(x) \left(\int\limits_{|y|<|x|} |x-y|^{(\alpha-n)2} dy \right) dx \right)^{\frac{1}{2}}$$

$$\leq c_2 \left(\int\limits_{\mathbb{R}^n} |x|^{4\alpha-n} v^2(x) dx \right)^{\frac{1}{2}} = c_2 \left(\sum_{k=-\infty}^{+\infty} a_k^2 \right)^{\frac{1}{2}},$$

where

$$a_k = \left(\int\limits_{2^k < |y| < 2^{k+1}} |x|^{4\alpha-n} v^2(x) dx \right)^{1/2}.$$

Moreover, using Theorem C we arrive at the following two-sided inequality:

$$c_3 \|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \left\| B_{+,v}^{\alpha} \right\|_{\sigma_\infty(L^2(\mathbb{R}^n))} \leq c_4 \|v\|_{l^\infty(L^2_{4\alpha-n}(\mathbb{R}^n))}.$$

By Proposition C we conclude that

$$\left\| B_{+,v}^{\alpha} \right\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_5 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}, \quad 2 \leq p < \infty.$$

Now we prove necessity. For this we take the orthonormal systems $\{f_k\}$ and $\{g_k\}$, where

$$f_k(x) = \chi_{\left\{ 2^{k-2} < |y| < 2^{k-1} \right\}}(x) 2^{-(k-2)n/2} \cdot \lambda_n^{-\frac{1}{2}},$$

$$g_k(x) = \chi_{\left\{ 2^k \leq |y| < 2^{k+1} \right\}}(x) |x|^{2\alpha - \frac{n}{2}} v(x) \alpha_k^{-\frac{1}{2}},$$

$\lambda_n = (2^n - 1)\pi^{n/2}/\Gamma(n/2+1)$ and

$$\alpha_k = \int\limits_{2^k \leq |x| < 2^{k+1}} v^2(x) |x|^{4\alpha-n} dx.$$

Then in view of Proposition B we have

$$\infty > \|B_{+,v}^\alpha\|_{\sigma_p(L^2(\mathbb{R}^n))} \geq c_6 \bigg(\sum_{k\in\mathbb{Z}} \bigg(\alpha_k^{-1/2} \int\limits_{2^k<|x|<2^{k+1}} v^2(x)|x|^{2\alpha-\frac{n}{2}}$$

$$\times \bigg(\int\limits_{2^{k-2}<|y|<2^{k-1}} \frac{(|x|^2-|y|^2)^\alpha}{|x-y|^n} 2^{-(k-2)n/2} dy \bigg) dx \bigg)^p \bigg)^{\frac{1}{p}}$$

$$\geq c_7 \bigg(\sum_{k\in\mathbb{Z}} \alpha_k^{p/2} \bigg)^{1/p} = c_7 \|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))}$$

which completes the proof.   ∎

The following result is also true:

Theorem 5. *Let $2 \leq p < \infty$ and let $\alpha > n/2$. Then $B_{+,v}^\alpha \in \sigma_p(L^2(\mathbb{R}^n))$ if and only if*

$$I(v,p,\alpha) \equiv \bigg(\int\limits_{\mathbb{R}^n} \bigg(\int\limits_{\frac{|x|}{2}<|y|<2|x|} v^2(y)|y|^{4\alpha-2n} dy \bigg)^{p/2} |x|^{np/2-n} dx \bigg)^{\frac{1}{p}} < \infty.$$

Moreover,

$$c_1 I(v,p,\alpha) \leq \big\|B_{+,v}^\alpha\big\|_{\sigma_p(L^2(\mathbb{R}^n))} \leq c_2 I(v,p,\alpha)$$

for some positive constants c_1 and c_2.

Proof. Taking into account Theorem 4, the statement will be proved if we show that

$$\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \approx I(v,p,\alpha).$$

Indeed, we have

$$\|v\|_{l^p(L^2_{4\alpha-n}(\mathbb{R}^n))} \leq \bigg(\sum_{k\in\mathbb{Z}} \bigg(\int\limits_{2^k<|x|<2^{k+1}} v^2(x)|x|^{4\alpha-2n} dx \bigg)^{p/2} 2^{(k+1)np/2} \bigg)^{\frac{1}{p}}$$

$$= b_1 \bigg(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|y|<2^{k+1}} |y|^{np/2-n} \bigg(\int\limits_{\frac{|y|}{2}<|x|<2|y|} v^2(x)|x|^{4\alpha-2n} dx \bigg)^{p/2} dy \bigg)^{1/p}$$

$$= b_1 I(v, p, \alpha).$$

The reverse inequality follows similarly.   ■

Remark 2. Some results of this paper were announced in [11].

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