GLOBAL EXISTENCE AND DECAY OF SOLUTIONS OF A COUPLED SYSTEM OF BBM-BURGERS EQUATIONS

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Abstract

The global well-posedness of the initial-value problem associated to the coupled system of BBM-Burgers equations

$$u_t - u_{xxt} - a_3 v_{xxt} + u^p u_x + a_1 v^p v_x + a_2(u^p v)_x - \varepsilon_1 u_{xx} = 0$$
$$b_1 v_t - v_{xxt} - b_2 a_3 \, u_{xxt} + v^p v_x + b_2 a_2 u^p u_x + b_2 a_1 (u v^p)_x$$
$$-\varepsilon_2 v_{xx} = 0 \qquad\qquad (*)$$

in the classical Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 2$ is studied. Furthermore we find decay estimates of the solutions of (*) in the norm $L^q(\mathbb{R}) \times L^q(\mathbb{R})$, $2 \leq q \leq \infty$ for general initial data provided that $a_3^2 b_2 < 1$ and $p \geq 3$. Model (*) is motivated by a work due to Gear and Grimshaw [10] who considered strong interaction of weakly nonlinear long waves governed by a coupled system of KdV equations.

1   Introduction

We consider the initial value problem for a coupled system of BBM-Burgers equations

$$u_t - u_{xxt} - a_3 v_{xxt} + u^p u_x + a_1 v^p v_x + a_2(u^p v)_x - \varepsilon_1 u_{xx} = 0$$
$$b_1 v_t - v_{xxt} - b_2 a_3 \, u_{xxt} + v^p v_x + b_2 a_2 u^p u_x + b_2 a_1 (u v^p)_x - \varepsilon_2 v_{xx} = 0 \qquad (1.1)$$

with initial conditions

$$u(x,0) = \varphi_1(x), \qquad v(x,0) = \varphi_2(x) \qquad (1.2)$$

where $u = u(x,t)$, $v = v(x,t)$ are real functions of the real variables $-\infty < x < \infty$, $t \geq 0$. All coefficients in (1.1) are real constants with b_1, b_2, ε_1, ε_2 positive and the power p is an integer bigger than or equal to one.

To motivate our study let us mention some works related with (1.1). The following coupled system

$$u_t + u_{xxx} + a_3\,v_{xxx} + u^p u_x + a_1\,v^p v_x + a_2(u^p v)_x - \varepsilon\,u_{xx} = 0$$
$$b_1 v_t + r v_x + v_{xxx} + b_2\,a_3\,u_{xxx} + v^p v_x + b_2\,a_2\,u^p u_x + b_2 a_1 (u v^p)_x$$
$$-\varepsilon\,v_{xx} = 0 \qquad\qquad\qquad (1.3)$$

was originally derived in the case $p = 1$ and $\varepsilon = 0$ by J. A. Gear and R. Grimshaw in 1984 [10] as a model to describe the strong interaction of weakly nonlinear, long waves. The Cauchy problem for (1.3) with $p = 1$ and $\varepsilon = 0$ was studied by Bona, Ponce, Saut and Tom [8], and by Marshall Ash, Cohen and Wang [12].

The asymptotic behaviour in time of solutions of (1.3) was studied by E. Bisognin, V. Bisognin and G. Perla Menzala in [3]. In particular, for $\varepsilon > 0$, they obtained the following decay estimate

$$\|u(\,\cdot\,,t)\|_{L^q} + \|(v\,\cdot\,,t)\|_{L^q} \leq C\,t^{\frac{1-q}{2q}}, \quad \text{as } t \to \infty$$

for any $2 \leq q \leq \infty$, provided that the initial data are sufficiently small, $p > 4$ and $a_3^2\,b_2 < 1$.

Concerning model (1.1), V. Bisognin [5] proved global existence and the asymptotic behaviour of solutions when $\varepsilon_1 = \varepsilon_2 = 0$ and $b_1 = 1$, where, under suitable smallness assumptions on the initial data and $p > 4$ the following L^∞ decay estimates are obtained

$$\|u(\,\cdot\,,t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{3}}, \quad \|v(\,\cdot\,,t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{3}} \text{ as } t \to \infty.$$

Observe that if we consider $a_1 = a_2 = a_3 = 0$ in (1.1) and (1.3) we obtain, respectively, BBM-Burgers and KdV-Burgers type equations, which are well known model equations for long waves in nonlinear dispersive media where dissipative effects are considered. The existence and decay of solutions for these (and related) equations has been intensively studied by several authors in recent years (see [1], [2], [6], [7] for example and the references therein).

In this paper we study the global well-posedness and decay of solutions of the initial-value problem associated to the system (1.1) in the case $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. We rewrite (1.1)–(1.2) as follows

$$b_1 U_t - A U_{xxt} + F(U)_x = B U_{xx} \tag{1.4}$$

$$U(x,0) = \varphi(x) \tag{1.5}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$A = \begin{bmatrix} b_1 & b_1 a_3 \\ b_2 a_3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} \varepsilon_1 b_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \tag{1.6}$$

and the components of $F(U)$ are given by

$$F_1(U) = \tfrac{b_1}{p+1} u^{p+1} + b_1 a_2 u^p v + \tfrac{b_1 a_1}{p+1} v^{p+1}$$
$$\tag{1.7}$$
$$F_2(U) = \tfrac{b_2 a_2}{p+1} u^{p+1} + b_2 a_1 u v^p + \tfrac{1}{p+1} v^{p+1}$$

A local solution of (1.4)–(1.5) with initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 2$ and $a_3^2 b_2 < 1$ can easily be obtained by inverting the operator $M = b_1 I - A \frac{\partial^2}{\partial x^2}$ in (1.4) and applying Banach Fixed Point Theorem to the integral equation associated with (1.4)–(1.5). Here I denotes the 2×2 identity matrix. To extend the local solution to a global solution we use the regularization property of the operator $M^{-1} \circ \frac{\partial}{\partial x}$ together with an $H^1 \times H^1$ a priori estimate (energy estimate). See sections 2 and 3.

Due to the dissipative terms $\varepsilon_1 u_{xx}$, $\varepsilon_2 v_{xx}$ the energy associated to the linear system, that is (1.1) without the nonlinear terms, decays to zero as $t \to \infty$, so that we could expect the same behaviour for the solutions of (1.1)–(1.2) (or equivalently (1.4)–(1.5)). In section 4 we prove that this in fact happens provided $p \geq 3$. More precisely, if $\varphi \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \geq 2$, $a_3^2 b_2 < 1$ and $p \geq 3$, then using our previous analysis on the linear part studied in section 2 we obtain the following decay estimates for the solution of (1.4)–(1.5):

(i) $\|U(\cdot,t)\|_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{4}} \|\varphi\|_{H^1 \times H^1}$

(ii) $\|U(\cdot,t)\|_{L^q \times L^q} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})} \left(\|\varphi\|_{H^1 \times H^1} + \|\varphi\|_{L^1 \times L^1} \right)$

for all $2 \leq q \leq \infty$, where in (ii) we also assume that $\varphi \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$.

We remark that the above results are proved without any restriction on smallness of the initial data. However the technique used in this paper to obtain (i)-(ii) does not apply to the case of $1 \leq p < 3$. Recently, for the case $p = 1$, the decay in time of space-periodic solutions of a generalization of system (1.1) was studied by E. Bisognin, V. Bisognin and G. Perla Menzala in [4].

Some notations used in this paper are as follows. In general, for X and Y Banach spaces, the norm of f in X will be denoted by $||f||_X$ and in the product $X \times Y$ we always consider the norm $||(f,g)||^2_{X \times Y} = ||f||^2_X + ||g||^2_Y$. For $1 \leq p \leq \infty$ we denote by $L^p = L^p(\mathbb{R})$ the Banach space of real-valued functions defined on $\mathbb{R}$ whose p-th power is Lebesgue integrable (essentially bounded in the case $p = \infty$) with the usual norm. In the particular case $p = 2$ we will denote by $(\ ,\ )_{L^2}$ the inner product of L^2. We denote by $\hat{f}$ the Fourier transform of a function f. If $f \in L^2(\mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R})$ and the Fourier transform can be extended to an isomorphism from $L^2(\mathbb{R})$ onto itself. For any real number s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of order s. We will denote by C a generic constant which value can be vary from line to line.

2   Linear estimates

We consider the linear system

$$u_t - u_{xxt} - a_3\, v_{xxt} - \varepsilon_1\, u_{xx} = 0$$

$$b_1\, v_t - b_2 a_3\, u_{xxt} - v_{xxt} - \varepsilon_2\, v_{xx} = 0$$

(2.1)

with initial conditions

$$u(x,0) = \varphi_1(x), \qquad v(x,0) = \varphi_2(x).$$

(2.2)

Our aim in this section is to study some decay estimates of the solutions of (2.1)–(2.2) and prove some facts which will be used in the next sections.

With the notation of the introduction we rewrite (2.1)–(2.2) as

$$b_1\, U_t - AU_{xxt} = BU_{xx}, \qquad U(x,0) = \varphi(x)$$

(2.3)

where A and B are as in (1.6), and $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$.

Let us consider equation (2.3) in the Sobolev space $H^s \times H^s$ for $s \geq 2$. Recall that M denotes the operator $M = b_1 I - A \frac{\partial^2}{\partial x^2}$.

Lemma 2.1. *Let $a_3^2 b_2 < 1$ and $s \geq 2$. Then the operator $M: H^s \times H^s \to H^{s-2} \times H^{s-2}$ is invertible and its inverse M^{-1} is given by $M^{-1} g = K * g$, where $K = (k_{j\ell})$, with $k_{j\ell} \in L^1 \cap L^\infty$, $j, \ell = 1, 2$, and $*$ denotes spatial convolution.*

Proof. The hypothesis $a_3^2 b_2 < 1$ implies that the matrix A has two positive eigenvalues λ_1, λ_2. Let P a nonsingular matrix such that $P^{-1} A P$ is equal to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Given $g \in H^{s-2} \times H^{s-2}$ consider the equation $b_1 V - D V_{xx} = P^{-1} g$. Using the Fourier transform we see that this equation has a unique solution $V \in H^s \times H^s$ for $s \geq 2$, which is given by

$$V = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} * P^{-1} g, \quad \text{where} \quad L_j(x) = \frac{1}{2 b_1} \sqrt{\frac{b_1}{\lambda_j}} \exp\left(-|x| \sqrt{\frac{b_1}{\lambda_j}} \right),$$

$j = 1, 2$. Consequently, $U = PV$ is the unique solution of $MU = g$. This shows that M is invertible. To find M^{-1}, let $P = (p_{j\ell})$, $P^{-1} = (q_{j\ell})$, $U = \binom{u_1}{u_2}$ and $g = \binom{g_1}{g_2}$. From $U = PV$ we obtain

$$u_j = p_{j1} L_1 * (q_{11} g_1 + q_{12} g_2) + p_{j2} L_2 * (q_{21} g_1 + q_{22} g_2)$$
$$= (p_{j1} q_{11} L_1 + p_{j2} q_{21} L_2) * g_1 + (p_{j1} q_{12} L_1 + p_{j2} q_{22} L_2) * g_2,$$

$j = 1, 2$. This shows that $M^{-1} g = K * g$, with $K = (k_{j\ell})$, where $k_{j\ell} = p_{j1} q_{1\ell} L_1 + p_{j2} q_{2\ell} L_2$ are all in $L^1 \cap L^\infty$.

■

Corollary 2.2. *For any $g \in H^s \times H^s$ with $s \geq 0$, there exists a positive constant C_0 such that $\|M^{-1} g\|_{H^{s+1} \times H^{s+1}} \leq C_0 \|g\|_{H^{s-1} \times H^{s-1}}$.*

Proof. The components of the vector $M^{-1} g$ are

$$k_{j1} * g_1 + k_{j2} * g_2 = p_{j1} q_{11} L_1 * g_1 + p_{j2} q_{21} L_2 * g_1$$
$$+ p_{j1} q_{12} L_1 * g_2 + p_{j2} q_{22} L_2 * g_2, \quad j = 1, 2$$

where $g = \binom{g_1}{g_2}$. Thus,

$$\|M^{-1} g\|_{H^{s+1} \times H^{s+1}} \leq \sum_{j=1}^{2} \|k_{j1} * g_1 + k_{j2} * g_2\|_{H^{s+1}}$$

$$= \sum_{j=1}^{2} ||p_{j1}q_{11}L_1 * g_1 + p_{j2}q_{21}L_2 * g_1 + p_{j1}q_{12}L_1 * g_2$$

$$+ p_{j2}q_{22}L_2 * g_2||_{H^{s+1}} \leq C_0||g||_{H^{s-1} \times H^{s-1}},$$

where

$$C_0 = \max \left\{ \frac{1}{\min(b_1, \lambda_1)}, \frac{1}{\min(b_1, \lambda_2)} \right\} ||P|| \, ||P^{-1}||,$$

with $||P|| = \sum\limits_{j,\ell=1}^{2} |p_{j\ell}|$

■

Using Lemma 2.1 we can write (2.3) in the space $H^s \times H^s$ with $s \geq 2$ as

$$U_t = \mathcal{L}U, \quad U(\,\cdot\,,0) = \varphi, \tag{2.4}$$

where $\mathcal{L} = M^{-1}B \frac{\partial^2}{\partial x^2}$. The initial value problem (2.4) has a unique solution $U \in C^1([0,\infty); H^s \times H^s)$, which is given by $U(\,\cdot\,,t) = E(t)\varphi$, where $E(t)$ denotes the semigroup of linear operators on $H^s \times H^s$ generated by $\mathcal{L}$. Therefore, the components u, v of U are the unique functions in $C^1([0,\infty]; H^s)$ satisfying (2.1)–(2.2).

Next we will find explicit formulas for u and v which will allow us to derive some decay estimates.

Taking the Fourier transform of (2.1) and (2.2) we obtain

$$\widehat{U}_t(z,t) + A(z)\widehat{U}(z,t) = 0, \qquad \widehat{U}(z,0) = \hat{\varphi}(z)$$

where $\widehat{U} = \binom{\hat{u}}{\hat{v}}$, $\hat{\varphi} = \binom{\hat{\varphi}_1}{\hat{\varphi}_2}$ and

$$A(z) = \frac{z^2}{\sigma(z)} \begin{bmatrix} (b_1 + z^2)\varepsilon_1 & -a_3 z^2 \\ -b_2 a_3 \varepsilon_1 z^2 & (1 + z^2)\varepsilon_2 \end{bmatrix}$$

with $\sigma(z) = b_1 + (1 + b_1)z^2 + (1 - a_3^2 b_2)z^4 > 0$.
Therefore

$$\widehat{U}(z,t) = \exp\left(-tA(z)\right)\hat{\varphi}(z). \tag{2.5}$$

The eigenvalues of the matrix $A(z)$ are given by

$$\lambda_1 = \lambda_1(z) = \frac{z^2}{2\sigma(z)}\left((b_1 + z^2)\varepsilon_1 + (1 + z^2)\varepsilon_2 + \delta(z)\right)$$

$$\lambda_2 = \lambda_2(z) = \frac{z^2}{2\sigma(z)}\left((b_1 + z^2)\varepsilon_1 + (1 + z^2)\varepsilon_2 - \delta(z)\right),$$

where

$$\delta(z) = \left\{ \left[(b_1 + z^2)\varepsilon_1 - (1 + z^2)\varepsilon_2 \right]^2 + 4a_3^2 b_2 \varepsilon_1 \varepsilon_2 z^4 \right\}^{1/2}.$$

Computing the matrix $\exp(-t(A(z))$ and using (2.5) we find

$$\begin{aligned}
\hat{u}(z,t) &= a_{11}(z,t)\hat{\varphi}_1(z) + a_{12}(z,t)\hat{\varphi}_2(z) \\
\hat{v}(z,t) &= a_{21}(z,t)\hat{\varphi}_1(z) + a_{22}(z,t)\hat{\varphi}_2(z)
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
a_{11}(z,t) &= (\lambda_1 - \lambda_2)^{-1}\left[(\lambda_1 - d)\exp(-t\lambda_1) - (\lambda_2 - d)\exp(-t\lambda_2) \right] \\
a_{12}(z,t) &= (\lambda_1 - \lambda_2)^{-1}c^{-1}(\lambda_1 - d)(\lambda_2 - d)\left[\exp(-t\lambda_1) - \exp(-t\lambda_2) \right] \\
a_{21}(z,t) &= (\lambda_1 - \lambda_2)^{-1}c\left[\exp(-t\lambda_2) - \exp(-t\lambda_1) \right] \\
a_{22}(z,t) &= (\lambda_1 - \lambda_2)^{-1}\left[(\lambda_1 - d)\exp(-t\lambda_2) - (\lambda_2 - d)\exp(-t\lambda_1) \right]
\end{aligned} \tag{2.7}$$

and

$$c = -b_2 a_3 \varepsilon_1 \frac{z^4}{\sigma(z)}, \qquad d = (1 + z^2)\varepsilon_2 \frac{z^2}{\sigma(z)}.$$

Using the inverse of the Fourier transform in (2.6) we obtain

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a_{11}(z,t)\hat{\varphi}_1(z) + a_{12}(z,t)\hat{\varphi}_2(z)]\exp(ixz)dz \tag{2.8}$$

$$v(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a_{21}(z,t)\hat{\varphi}_1(z) + a_{22}(z,t)\hat{\varphi}_2(z)]\exp(ixz)dz \tag{2.9}$$

We also obtain from (2.7) the following inequalities

$$|a_{ij}(z,t)| \le \alpha \exp\left(-\frac{z^2}{1 + z^2}\beta t \right), \quad i,j = 1,2 \tag{2.10}$$

where

$$\alpha = \max\left\{ 1, \left(\frac{\varepsilon_2}{b_2 \varepsilon_1} \right)^{\frac{1}{2}}, \frac{1}{2}\left(\frac{b_2 \varepsilon_1}{\varepsilon_2} \right)^{\frac{1}{2}} \right\}$$

and

$$\beta = \varepsilon_1 \varepsilon_2 \{ \max[b_1 \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + 2|a_3|(b_2 \varepsilon_1 \varepsilon_2)^{1/2}]\}^{-1}.$$

Remark. In the above calculation we assumed that $a_3 \ne 0$. In the case $a_3 = 0$ we have

$$a_{11}(z,t) = \exp\left(-\frac{z^2}{1 + z^2}\varepsilon_1 t \right), \ a_{22}(z,t) = \exp\left(-\frac{z^2}{1 + z^2}\varepsilon_2 t \right),$$

$$a_{12}(z,t) = a_{21}(z,t) = 0,$$

and we can replace the values of α and β in (2.10) by $\alpha = 1$ and $\beta = \min\left(\varepsilon_1, \dfrac{\varepsilon_2}{\min(1, b_1)}\right)$ without changes on the final results stated in the next theorem.

Theorem 2.3. *Let $\varphi \in H^s \times H^s$ for $s \geq 2$ and $a_3^2 b_2 < 1$. Then we have the following estimate for the solution of (2.1)–(2.2):*

$$\|(u, v)\|_{L^\infty \times L^\infty} \leq C(1 + t)^{-1/4} \|\varphi\|_{H^1 \times H^1}. \qquad (2.11)$$

Moreover, if also $\varphi \in L^1 \times L^1$, then

$$\|(u, v)\|_{L^\infty \times L^\infty} \leq C(1 + t)^{-1/2} \left(\|\varphi\|_{H^1 \times H^1} + \|\varphi\|_{L^1 \times L^1}\right) \qquad (2.12)$$

and

$$\|(u, v)\|_{L^2 \times L^2} \leq C(1 + t)^{-1/4} \left(\|\varphi\|_{L^2 \times L^2} + \|\varphi\|_{L^1 \times L^1}\right). \qquad (2.13)$$

Proof. Taking the norm L^∞ in (2.8) and using (2.10) we obtain

$$\|u(\,\cdot\,, t)\|_{L^\infty} \leq \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |a_{11}(z, t)| \, |\hat{\varphi}_1(z)| \, dz$$
$$+ \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |a_{12}(z, t)| \, |\hat{\varphi}_2(z)| \, dz$$

$$\leq \tfrac{\alpha}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{1}{1 + z^2} \exp\left(-\frac{2z^2}{1 + z^2} \beta t \right) \right]^{1/2} (\|\varphi_1\|_{H^1} + \|\varphi_2\|_{H^1}).$$
$$(2.14)$$

Observe that

$$\int_{|z| \leq 1} \frac{1}{1 + z^2} \exp\left(-\frac{2z^2}{1 + z^2} \beta t \right) dz \leq 2e^\beta \int_0^1 e^{-\beta(1+t)z^2} \, dz$$
$$\leq e^\beta \, \beta^{-\frac{1}{2}} \, \Gamma(\tfrac{1}{2})(1 + t)^{-\frac{1}{2}} \qquad (2.15)$$

and

$$\int_{|z| \geq 1} \frac{1}{1 + z^2} \exp\left(-\frac{2z^2}{1 + z^2} \beta t \right) dz \leq 2e^{-\beta t} \int_0^\infty \frac{dz}{1 + z^2} \leq C(1 + t)^{-1/2},$$
$$(2.16)$$

for all $t \geq 0$, where in (2.15) $\Gamma(\,\cdot\,)$ denotes the classical Gamma function.

Substitution of (2.15) and (2.16) in (2.14) give us the estimate

$$||u(\,\cdot\,,t)||_{L^\infty} \le C(1+t)^{-1/4}\,||\varphi||_{H^1\times H^1}\,, \quad t\ge 0\,. \tag{2.17}$$

Similarly, using (2.9) and (2.10) we prove that (2.17) is also true for $v(\,\cdot\,,t)$. This proves (2.11).

Now, observe that

$$\frac{1}{\sqrt{2\pi}} \int_{|z|\le 1} |a_{1j}(z,t)|\,|\hat{\varphi}_j(z)|\,dz$$

$$\le \frac{\alpha}{\sqrt{2\pi}} \int_{|z|\le 1} \exp\left(-\frac{z^2}{1+z^2}\,\beta t\right)|\hat{\varphi}_j(z)|\,dz$$

$$\le \frac{\alpha}{\pi} e^{\frac{1}{2}\beta}\left(\int_0^1 e^{-\frac{1}{2}\beta(1+t)z^2}\,dz\right)||\varphi_j||_{L^1}$$

$$\le \frac{\alpha}{\pi\sqrt{2}} e^{\frac{1}{2}\beta}\,\beta^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2})(1+t)^{-\frac{1}{2}}\,||\varphi_j||_{L^1}\,,$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{|z|\ge 1} |a_{1j}(z,t)|\,|\hat{\varphi}_j(z)|\,dz \le \frac{2\alpha}{\sqrt{2\pi}}\left(\int_1^\infty \frac{e^{-2\beta t}}{1+z^2}\,dz\right)^{\frac{1}{2}}||\varphi_j||_{H^1}$$

$$\le C\,e^{-\beta t}\,||\varphi_j||_{H^1} \le C(1+t)^{-\frac{1}{2}}\,||\varphi_j||_{H^1}\,, \quad j=1,2$$

for all $t\ge 0$.

Substitution of these inequalities in (2.14) give us that

$$||u(\,\cdot\,,t)||_{L^\infty} \le C(1+t)^{-\frac{1}{2}}\,(||\varphi||_{H^1\times H^1}+||\varphi||_{L^1\times L^1}),\quad t\ge 0\,. \tag{2.18}$$

A similar estimate can be proved for $v(\,\cdot\,,t)$. This proves (2.12).

Finally, using Plancherel's theorem and (2.6) we obtain

$$||u(\,\cdot\,,t)||_{L^2}^2 = ||\hat{u}(\,\cdot\,,t)||_{L^2}^2 \le 2\int_{-\infty}^\infty |a_{11}(z,t)|^2|\hat{\varphi}_1(z)|^2\,dz$$

$$\tag{2.19}$$

$$+2\int_{-\infty}^\infty |a_{12}(z,t)|^2|\hat{\varphi}_2(z)|^2\,dz.$$

Estimating each integral in (2.19) as before we have

$$2\int_{-\infty}^\infty |a_{1j}(z,t)|^2|\hat{\varphi}_j(z)|^2 dz \le \frac{\alpha^2}{\pi} e^{\beta}\beta^{-\frac{1}{2}}\,\Gamma(\frac{1}{2})(1+t)^{-\frac{1}{2}}\,||\varphi_j||_{L^1}^2$$

$$+4\alpha^2\,e^{-\beta t}\,||\varphi_j||_{L^2}^2 \le C(1+t)^{-\frac{1}{2}}\,(||\varphi_j||_{L^1}^2+||\varphi_j||_{L^2}^2),\quad t\ge 0\,,$$

Thus

$$||u(\,\cdot\,,t)||_{L^2} \leq C(1+t)^{-\frac{1}{4}} \left(||\varphi||_{L^2 \times L^2} + ||\varphi||_{L^1 \times L^1}\right), \quad t \geq 0.$$

A similar estimate is also true for $v(\,\cdot\,,t)$. This proves (2.13). ∎

Remark. Under the hypothesis of Theorem 2.3, using interpolation we immediately obtain

$$||(u,v)||_{L^q \times L^q} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})} \left(||\varphi||_{H^1 \times H^1} + ||\varphi||_{L^1 \times L^1}\right)$$

for all $2 \leq q \leq \infty$. We also easily obtain from (2.8)-(2.9) and (2.10) the following estimate

$$||E(t)\varphi||_{H^s \times H^s} \leq \sqrt{2\alpha} ||\varphi||_{H^s \times H^s}, \quad t > 0. \tag{2.20}$$

3 Existence of solutions of the nonlinear system

In this section we shall prove that the initial value problem (1.4)-(1.5) has a unique global solution $U(\,\cdot\,,t)$ in the class $C^1([0,\infty); H^s \times H^s)$, provided that $s \geq 2$, $a_3^2 b_2 < 1$ and $\varphi \in H^s \times H^s$ and that this solution depends continuously on the initial data.

We begin observing that for all U, V in $H^s \times H^s$ the following inequalities are true

$$\begin{aligned} &||F(U) - F(V)||_{H^s \times H^s} \\ &\leq C_1(||U||_{H^s \times H^s} + ||V||_{H^s \times H^s})^p \, ||U-V||_{H^s \times H^s}, \quad s \geq 1 \end{aligned} \tag{3.1}$$

$$\begin{aligned} &||M^{-1}\partial_x[F(U) - F(V)]||_{H^s \times H^s} \\ &\leq C_0 C_1(||U||_{H^{s-1} \times H^{s-1}} \\ &+ ||V||_{H^{s-1} \times H^{s-1}})^p \, ||U-V||_{H^{s-1} \times H^{s-1}}, \quad s \geq 2 \end{aligned} \tag{3.2}$$

where F is defined in (1.7) and C_0, C_1 are positive constants with C_0 as in the Corollary 2.2.

The proof of (3.1) follows easily from the fact that when $s > \frac{1}{2}$, $H^s(\mathbb{R})$ is a Banach algebra and (3.2) is an immediate consequence of Corollary 2.2 and (3.1).

Theorem 3.1. (Local existence). *Let $s \geq 2$, $a_3^2 b_2 < 1$ and $\varphi \in H^s \times H^s$. Then there exists $T > 0$ and a unique solution $U \in C^1([0,T]; H^s \times H^s)$ of (1.4)-(1.5).*

Proof. Using Lemma 2.1 we can write the integral equation associated with (1.4)–(1.5):

$$U(\,\cdot\,,t) = E(t)\varphi - \int_0^t E(t-\sigma)M^{-1}\partial_x\,F(U(\,\cdot\,,\sigma))\,d\sigma \qquad (3.3)$$

where $E(t)$ denotes the semigroup generated by $\mathcal{L} = M^{-1}B\,\frac{\partial^2}{\partial x^2}$. We consider the space of functions

$$Y_T = \{U \in C([0,T]; H^s \times H^s) \text{ such that}$$

$$\sup_{0\leq t\leq T} ||U(\,\cdot\,,t) - E(t)\varphi||_{H^s\times H^s} \leq 1 \text{ and } U(\,\cdot\,,0) = \varphi\}$$

with the norm $||U||_{Y_T} = \sup\limits_{0\leq t\leq T} ||U(\,\cdot\,,t)||_{H^s\times H^s}$, where $s \geq 2$.
Let us denote by $\mathcal{P}(U)(\,\cdot\,,t)$ the right-hand side of (3.3). Using inequality (3.1) we easily see that $\mathcal{P}\colon Y_T \to Y_T$ is a contraction if T is chosen sufficiently small. (In fact it sufficient to take T as $T = [2^p C_1\sqrt{2\alpha}(1 + \sqrt{2\alpha}||\varphi||_{H^s\times H^s})^{p+1}]^{-1}$, where C_1 and α are defined in (3.1) and (2.20) respectively). By Banach Fixed Point Theorem it follows that there exists a unique $U \in Y_T$ satisfying (3.3).
To calculate U_t let $V(t) = \int_0^t E(t-\sigma)M^{-1}\partial_x\,F(U(\,\cdot\,,\sigma))\,d\sigma,\ 0 \leq t \leq T$.
A direct calculation shows that

$$V_t(\,\cdot\,,t) = \mathcal{L}\left[\int_0^t E(t-\sigma)M^{-1}\partial_x\,F(U(\,\cdot\,,\sigma))\,d\sigma\right] + M^{-1}\partial_x\,F(U(\,\cdot\,,t)).$$

Since $U(\,\cdot\,,t) = E(t)\varphi - V(t)$, then we conclude that

$$U_t(\,\cdot\,,t) = \mathcal{L}\,U(\,\cdot\,,t) - M^{-1}\partial_x\,F(U(\,\cdot\,,t))$$
$$= M^{-1}B\,U_{xx}(\,\cdot\,,t) - M^{-1}\partial_x\,F(U(\,\cdot\,,t)),$$

for all $0 \leq t \leq T$. This proves, in view of Corollary 2.2 and inequality (3.1), that $U_t \in C([0,T]; H^s \times H^s)$ and that $U(\,\cdot\,,t)$ satisfies (1.4). Obviously $U(\,\cdot\,,0) = \varphi$.

To prove uniqueness observe that if U, V are any solutions of (1.4)–(1.5) in the class $C^1([0,T]; H^s \times H^s)$ with $s \geq 2$, then $W = U - V$ is also in the same class and satisfies

$$W_t = \mathcal{L}W - M^{-1}\partial_x[F(U) - F(V)], \quad W(\,\cdot\,,0) = 0. \qquad (3.4)$$

Using (3.4) we have that

$$\frac{d}{d\sigma}E(t-\sigma)W(\cdot,\sigma) = -E(t-\sigma)M^{-1}\partial_x[F(U(\cdot,\sigma)) - F(V(\cdot,\sigma))].$$

Integrating this equation from 0 to t we obtain

$$W(\cdot,t) = -\int_0^t E(t-\sigma)M^{-1}\partial_x[F(U(\cdot\,\sigma)) - F(V(\cdot,\sigma))]\,d\sigma.$$

This equation with (2.20) and (3.2) imply that

$$\|W(\cdot,t)\|_{H^s \times H^s}$$

$$\leq \sqrt{2\alpha}C_0 C_1 \sup_{0 \leq t \leq T}(\|U(\cdot,t)\|_{H^s \times H^s} +$$

$$\|V(\cdot,t)\|_{H^s \times H^s})^p \int_0^t \|W(\cdot,\sigma)\|_{H^s \times H^s}\,d\sigma.$$

By Gronwall's inequality we obtain $W = 0$.

∎

Using well known techniques (see [13] for example) we obtain the following

Corollary 3.2. *Under the hypothesis of Theorem 3.1 there exists a unique solution $U \in C^1([0,T^*);H^s \times H^s)$ of (1.4)–(1.5) with the property that either $T^* = \infty$ or $T^* < \infty$ and $\lim\limits_{t\uparrow T^*}\|U(\cdot,t)\|_{H^s \times H^s} = \infty$.*

Lemma 3.3. *Let $s \geq 2$, $a_3^2 b_2 < 1$ and $\varphi = (\varphi_1,\varphi_2) \in H^s \times H^s$. If $U \in C^1([0,T);H^s \times H^s)$ is a solution of (1.4)–(1.5), then there exists a positive constant $C_2 = C_2(b_1,b_2,a_3)$ such that*

$$\|U(\cdot,t)\|_{H^1 \times H^1} \leq C_2\|\varphi\|_{H^1 \times H^1}, \quad 0 \leq t < T. \tag{3.5}$$

Proof. Let u, v be the components of U. Multiply the first equation in (1.1) by $b_2 u$ and the second equation by v, integrate in the whole space and add to obtain

$$\frac{1}{2}\frac{d}{dt}\{b_2\|u\|_{L^2}^2 + b_2\|u_x\|_{L^2}^2 + b_1\|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + 2a_3 b_2(u_x,v_x)_{L^2}\}$$

$$+b_2\varepsilon_1\|u_x\|_{L^2}^2 + \varepsilon_2\|v_x\|_{L^2}^2 = 0. \tag{3.6}$$

Integration of (3.6) in time give us

$$b_2||u||_{L^2}^2 + b_2||u_x||_{L^2}^2 + b_1||v||_{L^2}^2 + ||v_x||_{L^2}^2 + 2a_3 b_2 (u_x, v_x)_{L^2}$$

$$+2\int_0^t (b_2\varepsilon_1||u_x(\,\cdot\,,\sigma)||_{L^2}^2 + \varepsilon_2||v_x(\,\cdot\,,\sigma)||_{L^2}^2)\,d\sigma = b_2||\varphi_1||_{L^2}^2 \qquad (3.7)$$

$$+b_2||\partial_x\varphi_1||_{L^2}^2 + b_1||\varphi_2||_{L^2}^2 + ||\partial_x\varphi_2||_{L^2}^2 + 2a_3 b_2 (\partial_x\varphi_1, \partial_x\varphi_2)_{L^2}\,.$$

Observe that if $a_3^2 b_2 < \xi < 1$, then $|2a_3^2 b_2(u_x, v_x)_{L^2}| \le b_2\xi||u_x||_{L^2}^2 + \frac{a_3^2 b_2}{\xi}||v_x||_{L^2}^2$. Using this fact in (3.7) we obtain

$$b_2||u||_{L^2}^2 + b_2(1-\xi)||u_x||_{L^2}^2 + b_1||v||_{L^2}^2 + \left(1 - \frac{a_3^2 b_2}{\xi}\right)||v_x||_{L^2}^2$$

$$+2\int_0^t \left(b_2\varepsilon_1||u_x(\,\cdot\,,\sigma)||_{L^2}^2 + \varepsilon_2||v_x(\,\cdot\,,\sigma)||_{L^2}^2\right)d\sigma \qquad (3.8)$$

$$\le b_2||\varphi_1||_{L^2}^2 + b_2(1+\xi)||\partial_x\varphi_1||_{L^2}^2 + b_1||\varphi_2||_{L^2}^2 + \left(1 + \frac{a_3^2 b_2}{\xi}\right)||\partial_x\varphi_2||_{L^2}^2\,.$$

Now, choosing $\xi = \frac{1}{2}(a_3^2 b_2 + 1)$ in (3.8) we obtain (3.5), with C_2 depending only on b_1, b_2 and a_3.

■

Theorem 3.4. (Global existence). *Let* $s \ge 2$, $a_3^2 b_2 < 1$ *and* $\varphi \in H^s \times H^s$. *Then the initial value problem (1.4)-(1.5) has a unique solution* $U \in C^1([0,\infty); H^s \times H^s)$.

Proof. By Theorem 3.1 and Corollary 3.2 it sufficient to prove that if U is any solution of (1.4)-(1.5) in the class $C^1([0,T); H^s \times H^s)$, $s \ge 2$, then there exists a positive constant $C = C(s, T, ||\varphi||_{H^s \times H^s})$ such that $\sup_{0 \le t < T} ||U(\,\cdot\,,t)||_{H^s \times H^s} \le C$. Since U also satisfies the integral equation (3.3) for $0 \le t < T$, then using (2.20) and (3.2) we have that

$$||U(\,\cdot\,,t)||_{H^s \times H^s}$$
$$\le \sqrt{2\alpha}||\varphi||_{H^s \times H^s} + \sqrt{2\alpha}\, C_0 C_1 \int_0^t ||U(\,\cdot\,,\sigma)||_{H^{s-1} \times H^{s-1}}^{p+1}\,d\sigma. \qquad (3.9)$$

From (3.9) and (3.5) it follows that

$$||U(\,\cdot\,,t)||_{H^2 \times H^2} \le \sqrt{2\alpha}\, \max\,(1, C_2^{p+1})(1 + C_0 C_1\, T||\varphi||_{H^2 \times H^2}^p)||\varphi||_{H^2 \times H^2}.$$

By induction we obtain

$$
\sup_{0 \leq t < T} \|U(\,\cdot\,,t)\|_{H^m \times H^m}
$$

$$
\leq \eta^{(p+2)^{m-2}} \left(1 + C_0 C_1 T \|\varphi\|_{H^m \times H^m}\right)^{(p+2)^{m-2}} \|\varphi\|_{H^m \times H^m} \tag{3.10}
$$

where $m = 2, 3, \ldots$ and $\eta = \sqrt{2\alpha} \max(1, C_2^{p+1})$.

Now, for any $s > 2$ we choose an integer m such that $m < s \leq m+1$ and use (3.9), (3.10) and the Sobolev imbeddings $H^s \hookrightarrow H^m \hookrightarrow H^{s-1}$ to conclude that

$$
\begin{aligned}
\|U(\,\cdot\,,t)\|_{H^s \times H^s} &\leq \sqrt{2\alpha} \|\varphi\|_{H^s \times H^s} + \sqrt{2\alpha}\, C_0 C_1 \int_0^t \|U(\,\cdot\,,\sigma)\|_{H^m \times H^m}^{p+1} \, d\sigma \\
&\leq \eta^{(p+2)^{m-1}} \left(1 + C_0 C_1 T \|\varphi\|_{H^s \times H^s}^p\right)^{(p+2)^{m-1}} \|\varphi\|_{H^s \times H^s} \\
&\leq \eta^{(p+2)^{s-1}} \left(1 + C_0 C_1 T \|\varphi\|_{H^s \times H^s}^p\right)^{(p+2)^{s-1}} \|\varphi\|_{H^s \times H^s}
\end{aligned}
$$

for all $0 \leq t < T$. This proves the theorem.

■

The solution of (1.4)-(1.5) obtained in Theorem 3.4 depends continuously on the initial data in the following sense:

Theorem 3.5. (Continuous dependence). *Let $s \geq 2$, $a_3^2 b_2 < 1$ and U_n, U the solutions of (1.4)-(1.5) in $C^1([0,\infty); H^s \times H^s)$ corresponding to the initial data φ_n, φ in $H^s \times H^s$, respectively. If $\varphi_n \to \varphi$ in $H^s \times H^s$ then*

$$
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left(\|U_n(\,\cdot\,,t) - U(\,\cdot\,,t)\|_{H^s \times H^s} \right.
$$
$$
\left. + \|\partial_t U_n(\,\cdot\,,t) - \partial_t U(\,\cdot\,,t)\|_{H^s \times H^s}\right) = 0 \,,
$$

for all fixed $T > 0$.

Proof. Fixed $T > 0$, using the integral equation (3.3) with φ_n, φ and (2.20),(3.2) we obtain the estimate

$$
\|U_n(\,\cdot\,,t) - U(\,\cdot\,,t)\|_{H^s \times H^s} \leq \sqrt{2\alpha} \|\varphi_n - \varphi\|_{H^s \times H^s} +
$$
$$
\sqrt{2\alpha} C_0 C_1 \sup_{0 \leq t \leq T} \left(\|U_n(\,\cdot\,,t)\|_{H^s \times H^s} + \right.
$$
$$
\left. \|U(\,\cdot\,,t)\|_{H^s \times H^s}\right)^p \int_0^t \|U_n(\,\cdot\,,\sigma) - U(\,\cdot\,,\sigma)\|_{H^s \times H^s} d\sigma
$$

for all $0 \le t \le T$. Therefore, Gronwall's inequality implies that

$$\sup_{0 \le t \le T} ||U_n(\,\cdot\,,t) - U(\,\cdot\,,t)||_{H^s \times H^s} \le \sqrt{2\alpha} ||\varphi_n - \varphi||_{H^s \times H^s}\, e^{\omega_n T} \quad (3.11)$$

where

$$\omega_n = \sqrt{2\alpha} C_0 C_1 \sup_{0 \le t \le T} (||U_n(\,\cdot\,,t)||_{H^s \times H^s} + ||U(\,\cdot\,,t)||_{H^s \times H^s})^p, \; n = 1, 2, \dots$$

Moreover, using (1.4) with U_n and U and (3.11) we have

$$||\partial_t U_n(\,\cdot\,,t) - \partial_t U(\,\cdot\,,t)||_{H^s \times H^s} \quad (3.12)$$
$$\le ||\mathcal{L}U_n(\,\cdot\,,t) - \mathcal{L}U(\,\cdot\,,t)||_{H^s \times H^s}$$
$$+ ||M^{-1}\partial_x[F(U_n(\,\cdot\,,t) - F(U(\,\cdot\,,t))]||_{H^s \times H^s}$$
$$\le C_0 \max(\varepsilon_1 b_1, b_2) ||U_n(\,\cdot\,,t) - U(\,\cdot\,,t)||_{H^s \times H^s}$$
$$+ \omega_n ||U_n(\,\cdot\,,t) - U(\,\cdot\,,t)||_{H^s \times H^s}$$
$$\le \sqrt{2\alpha}[C_0 \max(\varepsilon_1 b_1, b_2) + \omega_n]\, e^{\omega_n T} ||\varphi_n - \varphi||_{H^s \times H^s}, \quad 0 \le t \le T.$$

Since $\varphi_n \to \varphi$ in $H^s \times H^s$, then using the same arguments presented in the proof of Theorem 3.4 we show that (ω_n) is a bounded sequence. Thus, the conclusion of Theorem 3.5 follows from (3.11) and (3.12).

■

Remarks. (1) The same results given by Theorems 3.1, 3.4 and 3.5 above can be proved for (1.3) in the case of $\varepsilon_1 = \varepsilon_2 = 0$. In fact, for this case we can replace the hypothesis $s \ge 2$ by $s \ge 1$. These results improve those one presented in [5].

(2) It is also important to observe that in some particular cases, for example, $b_1 = b_2 = 1, a_1 = a_2, |a_3| < 1$ and $\varepsilon_1 = \varepsilon_2 = 0$, system (1.1) has solitary wave solutions of the form $u(x,t) = v(x,t) = \psi(x - ct)$, where $c > 0$ and

$$\psi(x) = [\frac{1}{2}(p+1)(p+2)\frac{c}{1 + a_1(p+1)} sech^2 \frac{p}{2}(\frac{1}{1+a_3})^{1/2}x]^{1/p}$$

Thus, an argument similar to that of page 579 of [9] shows that (in this case) the continuous dependence in Theorem 3.5 can't be uniform for all $t > 0$.

4   Decay estimates to the nonlinear system

Let us denote by u, v the components of the global solution U of (1.4)–(1.5) obtained in the last section. From Theorem 3.4 and Lemma 3.3 we deduce that

$$\int_0^\infty (||u_x(\,\cdot\,,\sigma)||_{L^2}^2 + ||v_x(\,\cdot\,,\sigma)||_{L^2}^2)\,d\sigma < +\infty. \tag{4.1}$$

Theorem 4.1. *Under the conditions of Theorem 3.4 and $p \geq 3$ we have the following decay estimate*

$$U(\,\cdot\,,t)||_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{4}}\,||\varphi||_{H^1 \times H^1}, \quad t \geq 0. \tag{4.2}$$

Moreover, if also $\varphi \in L^1 \times L^1$, then

$$||U(\,\cdot\,,t)||_{L^q \times L^q} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})}\,(||\varphi||_{H^1 \times H^1} + ||\varphi||_{L^1 \times L^1}), \quad t \geq 0 \tag{4.3}$$

for all $2 \leq q \leq \infty$.

Proof. Since $U(\,\cdot\,,t)$ also satisfies the integral equation (3.3) for any $t \geq 0$, then we can take the norm $L^\infty \times L^\infty$ in (3.3) and use (2.11) to obtain

$$||U(\,\cdot\,,t)||_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{4}}\,||\varphi||_{H^1 \times H^1}$$

$$\tag{4.4}$$

$$+C\int_0^t (1+t-\sigma)^{-\frac{1}{4}}\,||M^{-1}\partial_x\,F(U(\,\cdot\,,\sigma))||_{H^1 \times H^1}\,d\sigma.$$

To estimate $||M^{-1}\partial_x\,F(U)||_{H^1 \times H^1}$, let $\partial_x\,F(U) = g = (g_1, g_2)$, where

$$g_1 = b_1 u^p u_x + b_1 a_1 v^p v_x + b_1 a_2 (u^p v)_x$$
$$g_2 = v^p v_x + b_2 a_2\,u^p u_x + b_2 a_1 (u v^p)_x\,.$$

From Corollary 2.2 it follows that

$$||M^{-1}\partial_x\,F(U)||_{H^1 \times H^1} \leq C_o ||g||_{L^2 \times L^2} \tag{4.5}$$

Using the inequality $||w||_{L^\infty} \leq ||w||_{L^2}^{1/2}\,||w||_{L^2}^{1/2}$ valid for any $w \in H^1(\mathbb{R})$, the imbedding $H^1 \hookrightarrow L^\infty$, and lemma 3.3 we estimate $||g_1||_{L^2}$ as follows

$$||g_1||_{L^2} \leq b_1 ||u||_{L^\infty}^{p-2}\,||u_x||_{L^2}^2\,||u||_{L^2} + b_1 |a_1|\,||v||_{L^\infty}^{p-2}\,||v||_{L^2} ||v_x||_{L^2}^2$$

$$+ b_1 p |a_2| \, \|u\|_{L^\infty}^{p-3} \|u_x\|_{L^2}^2 \|u\|_{L^2} \|v\|_{L^\infty}$$

$$+ b_1 |a_2| \, \|u\|_{L^\infty}^{p-2} \|u_x\|_{L^2} \|u\|_{L^2} \|v_x\|_{L^2}$$

$$\leq C(\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)\|U\|_{L^\infty \times L^\infty} \tag{4.6}$$

A similar estimate is also true for $\|g_2\|_{L^2}$. From these estimates and (4.5) we conclude that

$$\|M^{-1}\partial_x \, F(U)\|_{H^1 \times H^1} \leq C(\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)\|U\|_{L^\infty \times L^\infty} \, . \tag{4.7}$$

Substitution of (4.7) in (4.4) give us

$$\|U(\, \cdot \, t)\|_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{4}} \|\varphi\|_{H^1 \times H^1}$$

$$+ C \int_0^t (1+t-\sigma)^{-\frac{1}{4}} \, H(\sigma)\|U(\, \cdot \, , \sigma)\|_{L^\infty \times L^\infty} \, d\sigma,$$

where $H(t) = \|u_x(\, \cdot \, , t)\|_{L^2}^2 + \|v_x(\, \cdot \, , t)\|_{L^2}^2$ is an integrable function on $[0, \infty)$ by (4.1) .

By a Gronwall's lemma (see Lemma 1 of [11]) it follows (4.2).

Now taking the norms $L^\infty \times L^\infty$ and $L^2 \times L^2$ in (3.3) and using (2.12)-(2.13) we obtain respectively

$$\|U(\, \cdot \, , t)\|_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{2}} \left(\|\varphi\|_{H^1 \times H^1} + \|\varphi\|_{L^1 \times L^1} \right)$$

$$+ C \int_0^t (1+t-\sigma)^{-\frac{1}{2}} \left(\|M^{-1}\partial_x \, F(U(\, \cdot \, , \sigma))\|_{H^1 \times H^1} \right.$$

$$\left. + \|M^{-1}\partial_x \, F(U(\, \cdot \, , \sigma))\|_{L^1 \times L^1} \right) d\sigma \tag{4.8}$$

and

$$\|U(\, \cdot \, , t)\|_{L^2 \times L^2} \leq C(1+t)^{-\frac{1}{4}} \left(\|\varphi\|_{L^2 \times L^2} + \|\varphi\|_{L^1 \times L^1} \right)$$

$$+ C \int_0^t (1+t-\sigma)^{-\frac{1}{4}} \left(\|M^{-1}\partial_x \, F(U(\, \cdot \, , \sigma))\|_{L^2 \times L^2} \right.$$

$$\left. + \|M^{-1}\partial_x \, F(U(\, \cdot \, , \sigma))\|_{L^1 \times L^1} \right) d\sigma \tag{4.9}$$

Let us work with (4.9) first. Using the same notation as before and lemma 2.1 we have

$$\|M^{-1}\partial_x \, F(U)\|_{L^2 \times L^2} \leq \sum_{j=1}^2 (\|k_{j1} * g_1\|_{L^2} + \|k_{j2} * g_2\|_{L^2})$$

$$\leq \sum_{j=1}^2 (\|k_{j1}\|_{L^1}\|g_1\|_{L^2} + \|k_{j2}\|_{L^1}\|g_2\|_{L^2})$$

and

$$||M^{-1}\partial_x F(U)||_{L^1 \times L^1} \leq \sum_{j=1}^{2} (||k_{j1} * g_1||_{L^1} + ||k_{j2} * g_2||_{L^1})$$

$$\text{(4.10)}$$

$$\leq \sum_{j=1}^{2} (||k_{j1}||_{L^1}||g_1||_{L^1} + ||k_{j2}||_{L^1}||g_2||_{L^1})$$

Estimating $||g_1||_{L^2}$ as in (4.6) we find

$$||g_1||_{L^2} \leq C(||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U||_{L^2 \times L^2} .$$

We estimate $||g_1||_{L^1}$ as follows

$$\begin{aligned}
||g_1||_{L^1} \leq\ & b_1||u||_{L^\infty}^{p-3}||u_x||_{L^2}^2||u||_{L^2}^2 \\
& + b_1|a_1|\,||v||_{L^\infty}^{p-3}||v_x||_{L^2}||u_x||_{L^2}||u||_{L^2}||v||_{L^2} \\
& + b_1 p|a_2|\,||u||_{L^\infty}^{p-3}||u_x||_{L^2}||v_x||_{L^2}||u||_{L^2}||v||_{L^2} \\
& + b_1|a_2|||u||_{L^\infty}^{p-3}||u_x||_{L^2}||v_x||_{L^2}||u||_{L^2}^2 \\
\leq\ & C(||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U||_{L^2 \times L^2} .
\end{aligned}$$

Similar estimates are also true for $||g_2||_{L^2}$ and $||g_2||_{L^1}$.
Thus

$$||M^{-1}\partial_x F(U)||_{L^2 \times L^2} \leq C(||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U||_{L^2 \times L^2}$$

$$\text{(4.11)}$$

$$||M^{-1}\partial_x F(U)||_{L^1 \times L^1} \leq C(||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U||_{L^2 \times L^2}$$

Combining (4.11) with (4.9) we obtain

$$\begin{aligned}
||U(\,\cdot\,,t)||_{L^2 \times L^2} \leq\ & C(1+t)^{-\frac{1}{4}}(||\varphi||_{L^2 \times L^2} + ||\varphi||_{L^1 \times L^1}) + \\
& C\int_0^t (1+t-\sigma)^{-\frac{1}{4}} H(\sigma)||U(\,\cdot\,,\sigma)||_{L^2 \times L^2}\,d\sigma
\end{aligned}$$

with $H(t)$ as before. Applying lemma 1 of [11] we conclude that

$$||U||_{L^2 \times L^2} \leq C(1+t)^{-\frac{1}{4}}(||\varphi||_{L^2 \times L^2} + ||\varphi||_{L^1 \times L^1}), \quad t \geq 0 \qquad \text{(4.12)}$$

Now, let us estimate $||M^{-1}\partial_x F(U)||_{L^1 \times L^1}$ in terms of $||U||_{L^\infty \times L^\infty}$. Using similar arguments as before, the Young inequality and (4.12) we have

$$||u^p u_x||_{L^1} \leq ||u||_{L^\infty}^{p-1}||u||_{L^2}||u_x||_{L^2} \leq ||u||_{L^2}^{p/2}||u_x||_{L^2}^{3/2}||u||_{L^\infty}$$

$$\leq (\frac{1}{4}||u||_{L^2}^{2p} + \frac{3}{4}||u_x||_{L^2}^2)||u||_{L^\infty} \leq C((1+t)^{-\frac{p}{2}} + ||u_x||_{L^2}^2)||u||_{L^\infty}$$

$$||u^{p-1}u_x v||_{L^1} \leq C((1+t)^{-\frac{p}{2}} + ||u_x||_{L^2}^2)||u||_{L^\infty}$$

$$||v^p v_x||_{L^1} \leq C((1+t)^{-\frac{p}{2}} + ||v_x||_{L^2}^2)||v||_{L^\infty}$$

We also have

$$||u^p v_x||_{L^1} \leq ||u||_{L^\infty}^{p-2}||u||_{L^2}||v_x||_{L^2}||u||_{L^\infty}$$

$$\leq ||u||_{L^2}^{p/2}||u_x||_{L^2}^{1/2}||v_x||_{L^2}||u||_{L^\infty}$$

$$\leq (||u||_{L^2}^{2p} + ||u_x||_{L^2}^2 + 2||v_x||_{L^2}^2)||u||_{L^\infty}$$

$$\leq C((1+t)^{-\frac{p}{2}} + ||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||u||_{L^\infty}$$

Consequently

$$||g_1||_{L^1} \leq C((1+t)^{-\frac{p}{2}} + ||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U||_{L^\infty \times L^\infty} \qquad (4.13)$$

A similar estimate is also true for $||g_2||_{L^1}$. This fact with (4.13) and (4.10) imply that

$$||M^{-1}\partial_x F(U)||_{L^1 \times L^1} \leq C((1+t)^{-\frac{p}{2}} + ||u_x||_{L^2}^2 + ||v_x||_{L^2}^2)||U|_{L^\infty \times L^\infty}$$
$$(4.14)$$

Combining (4.14)-(4.7) with (4.8) we obtain

$$||U(\cdot,t)||_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{2}}(||\varphi||_{H^1 \times H^1} + ||\varphi||_{L^1 \times L^1})$$
$$+C\int_0^t (1+t-\sigma)^{-\frac{1}{2}}\tilde{H}(\sigma)||U(\cdot,\sigma)||_{L^2 \times L^2}\,d\sigma$$

where

$$\tilde{H}(t) = (1+t)^{-\frac{p}{2}} + ||u_x(\cdot,t)||_{L^2}^2 + ||v_x(\cdot,t)||_{L^2}^2$$

is an integrable functions on $[0,\infty)$ by (4.1) and because $p > 2$. By lemma 1 of [11] it follows that

$$||U(\cdot,t)||_{L^\infty \times L^\infty} \leq C(1+t)^{-\frac{1}{2}}(||\varphi||_{H^1 \times H^1} + ||\varphi||_{L^1 \times L^1}), \quad t \geq 0 \quad (4.15)$$

Finally, from (4.12)-(4.15) and interpolation we obtain (4.3). This completes the proof of Theorem 4.1.   ■

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