REVISTA MATEMÁTICA COMPLUTENSE
(2000) vol. XIII, num. 2, 341-347
ISSN 1139-1138

BAIRE-LIKE SPACES $C(X, E)$

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Abstract

We characterize Baire-like spaces $C_c(X, E)$ of continuous functions defined on a locally compact and Hewitt space X into a locally convex space E endowed with the compact-open topology.

1 Introduction and preliminary facts

Throughout this note "lcs" will stand for "Hausdorff locally convex topological vector space". The word "space" will mean "completely regular Hausdorff topological space". $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of the real and rational numbers, respectively. By $C_c(X, E)$ we denote the space of all continuous functions defined on X with values in a lcs E endowed with the compact-open topology. As usual $C_c(X)$ denotes the space $C_c(X, \mathbb{R})$. For a space X by βX and υX we denote the Stone-Čech and the Hewitt compactification of X, respectively. The space X is called a *Hewitt space* if $X = \upsilon X$, cf. [5].

In [18] Saxon defined a lcs E to be *Baire-like* if given an increasing sequence (A_n) of closed absolutely convex subsets of E covering E, there is an integer $n \in \mathbb{N}$ such that A_n is a neighbourhood of zero. When the sequence (A_n) is required to be *bornivorous*, E is said to be *b-Baire-like*, cf. [19]. Clearly Baire $\Rightarrow$ Baire-like $\Rightarrow$ barrelled. Every metrizable lcs is b-Baire-like, see also [16] and [4]. Recall that a lcs E is *barrelled* (*quasibarrelled*), if every closed absolutely convex and absorbing (bornivorous) subset of E is a neighbourhood of zero of E. Every metrizable barrelled space is Baire-like, see also [1], [6], [7], [9], [11], [21]. It is known that the spaces of Pettis or Bochner integrable

*This research was supported by Komitet Badań Naukowych (State Comittee for Scientific Research), Poland, grant no. 2P03A05115.

1991 Mathematics Subject Classification: 46A30, 54C35.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2000

functions are not Baire spaces but Baire-like, [2], [3]. In contrast to the Baire spaces, cf. [12], Baire-like spaces have "good" properties. For instance, Saxon showed [18] that Grothendieck's factorization theorem for closed linear maps from a locally convex Baire space into an (LF)-space remains true for closed linear maps from a Baire-like space into an (LB)-space. Recall that Baire-like spaces are also stable under arbitrary products, quotients, countably codimensional subspaces, etc., cf. [18], [16], [15].

Mendoza [13] realized that the space $c_0(E)$ of sequences in E converging to zero, endowed with the uniform topology, is essential for the study of $C_c(X,E)$. In this paper, applying rather known techniques, we prove the following

Theorem. *If X is pseudo-finite, i.e. every compact subset of X is finite, then $C_c(X,E)$ is Baire-like iff E and $C_c(X)$ are Baire-like. If X is locally compact and Hewitt and X is not pseudo-finite, then $C_c(X,E)$ is Baire-like iff $c_0(E)$ is Baire-like.*

The proof heavily depends on the following result that we established in [10].

(+) *If X is locally compact and Hewitt and (x_n) is a sequence in $\beta X\backslash X$, then there exists a continuous function $f:\beta X\to[0,1]$ which is positive on X and vanishes on a subsequence of (x_n).*

On the other hand, as we proved in [8],

(*) *the space $c_0(E)$ is Baire-like iff E is barrelled and the strong dual $E'_b=(E',\beta(E',E))$ is strong fundamentally ℓ_1-bounded.*

Some particular cases of the theorem were proved in [7], [8], [10], [11], [13], [14]. For instance, $C_c(X)$ is Baire-like provided X is locally compact and Hewitt. The assumption "X is locally compact" cannot be removed; the space $C_c(\mathbb{Q})$ is barrelled but not Baire-like. If X is first countable, then $C_c(X)$ is a bornological Baire-like space iff X is locally compact and Hewitt.

Recall that a lcs E is *fundamentally ℓ_1-bounded* (or has property (B)), cf. [17] or [16], if for every bounded subset H of $\ell_1(E)$, there exists a closed disc B of E such that $\sum_{n=1}^{\infty}p_B(x_n)\leq 1$ for all $(x_n)\in H$, where p_B denotes the Minkowski functional of B. A lcs E is *strong fundamentally ℓ_1-bounded*, see [8], *if E is fundamentally ℓ_1-bounded and the space $\ell_1(E)$ satisfies property (s)* and a lcs E is said to satisfy *property* (s) if for every decreasing sequence (H_n) of absolutely convex subsets of E such that for any $p \in \mathcal{F}(E)$ there exists $m \in \mathbb{N}$ with $\sup_{x\in H_m} p(x) < \infty$, then there is $k \in \mathbb{N}$ such that $\sup_{x\in H_k} p(x) < \infty$ for every $p \in \mathcal{F}(E)$.

For a lcs E by $\mathcal{F}(E)$ and $\mathcal{U}(E)$ we denote the set of all continuous seminorms and absolutely convex neighbourhoods of zero on E, respectively. By E' we denote the topological dual of E. An increasing sequence (A_n) of absolutely convex and closed subsets of a lcs E is *absorbing* if it covers E. It is *bornivorous* if for every bounded subset B of E there exists $n \in \mathbb{N}$ such that $B \subset A_n$. Recall that in a barrelled space every absorbing sequence is bornivorous, cf. [16], 8.1.23.

2 Proof of Theorem

It turns out, cf. [13], [20], that

(**) *$C_c(X)$ is barrelled iff every bounding subset of X is relatively compact. If X is pseudo-finite, then $C_c(X,E)$ is barrelled iff E and $C_c(X)$ are barrelled. If X is not pseudo-finite, then $C_c(X,E)$ is barrelled iff E and $C_c(X)$ are barrelled and E'_b is fundamentally ℓ_1-bounded.*

We start with the following

Lemma 1. *Let X be infinite compact. Then $C_c(X,E)$ is Baire-like iff $c_0(E)$ is Baire-like.*

Proof. Assume that $C_c(X,E)$ is Baire-like. Since $C_c(X)\otimes_\epsilon E$ is a *large subspace* of $C_c(X,E)$, cf. [16], p. 414, it follows that $C_c(X)\otimes_\epsilon E$ is b-Baire-like. By 11.4.46 of [16] the space

$$Y = (c_0 \otimes_r (C_c(X))') \otimes_\pi (C_c(X) \otimes_\epsilon E)$$

has a quotient isomorphic to $Z = c_0 \otimes_\epsilon E$. On the other hand, applying the argument of [16], 11.2.4, one deduces that Y is b-Baire-like. Consequently Z is b-Baire-like, so $c_0(E)$ is b-Baire-like, since it contains a dense b-Baire-like space Z. Since a barrelled space is Baire-like iff it is b-Baire-like, (**) applies to deduce that the space $c_0(E)$ is Baire-like. For the converse assume that $c_0(E)$ is Baire-like. Since $U = c_0 \otimes_\epsilon E$ is a large subspace of $c_0(E)$, cf. proof of 11.5.9 of [16], U is b-Baire-like. On the other hand the space

$$Y_0 = (C_c(X)\otimes_\epsilon \ell_1) \otimes_\pi (c_0 \otimes_\epsilon E)$$

has a quotient isomorphic to the space $C_c(X) \otimes_\epsilon E$, cf. 11.4.46 of [16]. Proceeding as above one gets that Y_0 is b-Baire-like, so $C_c(X, E)$ is b-Baire-like. By (**) $C_c(X, E)$ is barrelled, so it is Baire-like.

From Lemma 1 it follows immediately that if $C_c(X, E)$ is Baire-like for some infinite compact X, then $C_c(Y, E)$ is Baire-like for any infinite compact Y.

Recall that if D is an absolutely convex subset of $C_c(X, E)$, a *hold* K of D is a compact subset of βX such that $f \in C_c(X, E)$ belongs to D if its continuous extension f^β of βX into βE is identically zero on a neighbourhood of K. The intersection $k(D)$ of all holds of an absolutely convex set D in $C_c(X, E)$ is again a hold, [20], II.1.2, and it is called a *support* of D. If moreover D is bornivorous, then $k(D)$ is contained in υX, [20], II.2.4, II. 1.2, II.1.4.

Lemma 2. *Let X be locally compact and Hewitt. Let (D_n) be a bornivorous sequence in $C_c(X, E)$ covering $C_c(X, E)$. Then there exists $m \in \mathbb{N}$ such that $k(D_m) \subset X$.*

Proof. If this fails, for every $n \in \mathbb{N}$ there exists $x_n \in k(D_n) \setminus X$. Let f be a function as in (+). Since (D_n) is increasing we may assume that $f(x_n) = 0$, $n \in \mathbb{N}$. The sets $A_m = \{y \in \beta X : f(y) > m^{-1}\}$ are open in βX and form an increasing sequence which covers X. Since $x_n \notin \overline{A_n}$ for $n \in \mathbb{N}$,

$$k(D_n) \not\subset \overline{A_n}$$

for every $n \in \mathbb{N}$, where the closure is taken in βX. This implies that $\overline{A_n}$ is not a hold of D_n for any $n \in \mathbb{N}$. Hence there exists a sequence $f_n \in C_c(X, E) \setminus D_n$ such that its extension $f_n^\beta = 0$ on some neighbourhood of $\overline{A_n}$. Since (f_n) converges to zero in $C_c(X, E)$, there exists $p \in \mathbb{N}$ such that $f_n \in D_p$ for all $n \in \mathbb{N}$, a contradiction.

Proof of Theorem. Assume X is locally compact and Hewitt but not pseudo-finite. If $C_c(X, E)$ is Baire-like, then E is Baire-like and $C_c(K, E)$ is Baire-like for any infinite compact K in X. Indeed, the restriction $f \to f|K$ defines a linear map of $C_c(X, E) \to C_c(K, E)$ which is open and has a dense range. Next, Lemma 1 applies to conclude that $c_0(E)$ is Baire-like (or equivalently E is barrelled and E'_b is strong fundamentally ℓ_1-bounded by (*)). Conversely, if $c_0(E)$ is

Baire-like, then the space $C_c(\beta X, E)$ is Baire-like (by Lemma 1). Finally we prove that $C_c(X,E)$ is Baire-like. By (*) and (**) the space $C_c(X,E)$ is barrelled. Let (D_n) be an absorbing sequence in $C_c(X,E)$. Since $C_c(\beta X, E)$ is Baire-like, we get $m \in \mathbb{N}$ and $h \in \mathcal{F}(E)$ such that $\{\varphi \in C_c(\beta X, E) : \sup_{x\in X} h(\varphi(x)) \leq 1\} \subset D_m \cap C_c(\beta X, E)$. Since $C_c(X,E)$ is barrelled we apply [16], 8.1.23, to deduce that (D_n) is bornivorous. By Lemma 2 there exists $n \geq m$ such that $k(D_n) \subset X$. Finally, using the local compactness of X and following the argument of IV.4.3, [20], one gets $\{\varphi \in C_c(X,E) : \sup_{x\in k(D_n)} h(\varphi(x)) \leq 1\} \subset D_n$. Hence $D_n \in \mathcal{U}(C_c(X,E))$ and consequently $C_c(X,E)$ is Baire-like.

Now assume that X is pseudo-finite and $C_c(X)$ and E are Baire-like. Then, by (**), $C_c(X,E)$ is barrelled. Clearly $C_c(X,E)$ is dense in E^X. Let (A_n) be an increasing sequence of closed absolutely convex subsets of $C_c(X,E)$ covering it. Then

$$E^X = \bigcup_n \overline{A_n},$$

the closure is taken in E^X, cf. [16], 8.2.27. Since E^X is Baire-like, [16], 9.2.6, we deduce that some A_n is a neighbourhood of zero in $C_c(X,E)$. Clearly $C_c(X)$ and E are Baire-like provided $C_c(X,E)$ is Baire-like.

The author wishes to thank the referee for valuable comments.

References

[1] N. Berscheid, *Baire properties of locally convex spaces,* Note di Matem. **16** (1996), 227-265.

[2] L. Drewnowski, M. Florencio, P. J. Paúl, *Some new classes of Banach-Mackey spaces,* Manuscr. Math. **76** (1992), 341-351.

[3] L. Drewnowski, M. Florencio, P. J. Paúl,*The space of Pettis integrable functions is barrelled,* Proc. Amer. Math. Soc. **34** (1992), 687-694.

[4] J. C. Ferrando, M. López Pellicer, L. M. Sánchez-Ruiz, *Metrizable barrelled spaces,* Pitman Research Notes in Math., Longman, 1995.

[5] L. Gillman, M. Jerison, *Rings of continuous functions,* van Norstrand Reinhold Comp., New York, 1987.

[6] J. Kąkol, *Sequential closure conditions and Baire-like spaces,* Arch. Marh. **60** (1993), 277-282.

[7] J. Kąkol, *Strongly Lindelöf spaces, Baire-type property and sequential closure conditions for inductive limits of metrizable spaces, Proceedings of the first Inter. Workshop on Funct. Analysis, Trier 1994,* Walter de Gruyter, 1996, pp. 227-239.

[8] J. Kąkol, T. Gilsdorf, L. Sánchez-Ruiz, *Baire-likeness of spaces $\ell_\infty(E)$ and $c_0(E)$,* Periodica Math. Hung. (to appear).

[9] J. Kąkol, L. M. Sánchez-Ruiz, *A note on Baire and ultrabornological property of spaces $C_p(X, E)$,* Arch. Math. **67** (1996), 493-499.

[10] J. Kąkol, W. Śliwa, *Strongly Hewitt spaces,* to appear in Topology and Appl.

[11] W. Lehner, *Über die Bedeutung gewisser Varianten des Bairéschen Kategorienbegriffs für die Funktionenräume $C_c(T)$,* Dissertation, München (1978).

[12] R. A. McCoy, I. Ntantu, *Topological Properties of Spaces of Continuous Functions,* Lecture Notes in Math., New York, 1988.

[13] J. Mendoza, *Necessary and sufficient condition for $C(X, E)$ to be barrelled or infrabarrelled,* Simon Stevin **57** (1983), 103-123.

[14] J. Mendoza, *A barrelled criteria for $c_0(E)$,* Arch. Math. **40** (1983), 156-158.

[15] P. P. Narayanaswami, S. S. A. Saxon, *(LF)-spaces, Quasi-Baire spaces and the strongest locally convex topology,* Math. Ann. **274** (1986), 627-641.

[16] P. Pérez Carreras, J. Bonet, *Barrelled locally convex spaces,* Math. Studies, North-Holland, Amsterdam, 1987.

[17] A. Pietsch, *Nuclear locally convex spaces,* Springer-Verlag, Heidelberg-New York, 1972.

[18] S. A. Saxon, *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology,* Math. Ann., **197** (1972), 87-106.

[19] S. A. Saxon, A. R. Todd, *A property of locally convex Baire spaces,* Math. Ann. **206** (1973), 23-34.

[20] J. Schmets *Spaces of vector-valued continuous functions,* Lecture Notes in Math., New York, 1983.

[21] A. R. Todd, *$C_k(X)$ and a Property of (db)-Spaces,* Ann. Math. Pura. Appl. (IV) **128** (1980), 317-323.

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Recibido: 29 de Septiembre de 1999
Revisado: 3 de Mayo de 2000