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LACUNARY CONVERGENCE OF SERIES IN L_0 REVISITED

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Abstract

A simpler proof is given for the recent result of I. Labuda and the author that a series in the space $L_0(\lambda)$ is subseries convergent if each of its lacunary subseries converges.

Let (S, Σ, λ) be a measure space which we assume throughout to be *locally finite* (if $\lambda(A) > 0$ then there is a $B \subset A$ with $0 < \lambda(B) < \infty$) and such that the space $L_0(\lambda)$ is sequentially complete. Here $L_0(\lambda)$ denotes the topological vector space of all (equivalence classes) of scalar measurable functions on S equipped with the topology of convergence in measure λ on sets of finite measure. A subseries $\sum_k f_{n_k}$ of a given series $\sum_n f_n$ is said to be *lacunary* if $n_{k+1} - n_k \to \infty$. Recall that the subseries convergence and unconditional convergence of series coincide in sequentially complete topological vector spaces.

The purpose of this note is to present a simpler proof for the following result established in a recent joint paper of I. Labuda and the author [DL1]; see also [DL2] for its extension to the case of Bochner spaces $L_0(\lambda, E)$. The reader is referred to these two papers for historical information and relevant literature.

Theorem. *The space $L_0(\lambda)$ has the Lacunary Convergence Property. That is, a series in $L_0(\lambda)$ is subseries convergent provided each of its lacunary subseries converges.*

What we actually proved first was an analogue of the above result for *positive* series (see [DL2, Thm. 5.1]). That was relatively easy and

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elementary, but our attempts to reduce somehow the case of general series to that of positive series failed. It took us quite a time before a proof for the general case was found, and it turned out to be quite technical and not so elementary. In particular, it crucially depended on a result of Orlicz asserting that

if a series $\sum_n f_n$ *in* $L_0(\lambda)$ *is unconditionally convergent, then*

$$\sum_n |f_n(s)|^2 < \infty \ a.e.;$$

see [O1, Hilfsatz] and, for stronger forms, [O2, Thm. 8], and [MO, Thm. 1]. (The proof in [MO] is particularly simple and self-contained.)

The proof presented here is also based on Orlicz's theorem, but formulated in a somewhat sharper form (stated below) which allows us for arguments very much like those used in the case of positive series. Roughly speaking, we work here with positive series of the form $\sum_k |\sum_{i\in N_k} f_i|^2$.

The sharper form of Orlicz's theorem needed here can easily be deduced from the former using the following general fact: If a series $\sum_n x_n$ in a topological vector space is subseries convergent, then for every sequence (N_k) of disjoint subsets of $\mathbb{N}$ also the series $\sum_k(\sum_{i\in N_k} x_i)$ is subseries convergent.

Orlicz's Theorem. *For a series* $\sum_n f_n$ *in* $L_0(\lambda)$*, the following are equivalent.*

(a) $\sum_n f_n$ *is subseries convergent.*

(b) *For every disjoint sequence* (N_k) *of finite subsets of* $\mathbb{N}$,

$$\sum_{k=1}^{\infty} \Big| \sum_{i\in N_k} f_i(s) \Big|^2 < \infty \qquad \lambda\text{-a.e.}$$

(c) *For every disjoint sequence* (N_k) *of finite subsets of* $\mathbb{N}$,

$$\sum_{i\in N_k} f_i \to 0 \qquad \lambda\text{-a.e.}$$

Let K be a subset of $\mathbb{N}$. It is said to be *r-rare* for some $r \in \mathbb{N}$ if $|k - k'| \geq r$ for all distinct $k, k' \in K$; *lacunary* if for every $r \in \mathbb{N}$ there is n such that the set $\{k \in K : k \geq n\}$ is r-rare. Evidently, if $K = \{n_1 < n_2 < \ldots\}$ is infinite, then it is lacunary iff $n_{k+1} - n_k \to \infty$. Also, as easily seen, K is lacunary iff it is the union of a sequence (K_r) of finite sets such that each K_r is r-rare and $\max K_r + r \leq \min K_{r+1}$.

Proof of the Theorem. The basic idea of the proof is the same as in [DL1]: Assume that a series $\sum_n f_n$ in $L_0(\lambda)$ is not subseries convergent although all of its lacunary subseries converge. Then we construct a lacunary subseries $\sum_k f_{n_k}$ such that, for a suitable sequence (g_n) of its "blocks", we have $\sum_n |g_n(s)|^2 = \infty$ on a set of positive λ measure. However, since also the series $\sum_n g_n$ is subseries convergent, this would contradict Orlicz's theorem. A more detailed argument runs as follows.

By Orlicz's Theorem, there is a disjoint sequence (N_k) of finite subsets of $\mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \left| \sum_{i \in N_k} f_i(s) \right|^2 = \infty \qquad \text{on a set } D \text{ with } 0 < \lambda(D) < \infty.$$

For $r = 1, 2, \ldots$ let

$$A_{rj} = \{j + (k-1)r : k = 1, 2, \ldots\}, \qquad j = 1, \ldots, r.$$

Obviously, for each k the sets $M_{rjk} = A_{rj} \cap N_k$ $(j = 1, \ldots, r)$ form a partition of N_k. Furthermore, by the Cauchy-Schwartz inequality, we have

$$\left(\sum_{k=1}^{\infty} \left| \sum_{i \in N_k} f_i(s) \right|^2 \right)^{1/2} \leq \sum_{j=1}^{r} \left(\sum_{k=1}^{\infty} \left| \sum_{i \in M_{rjk}} f_i(s) \right|^2 \right)^{1/2}.$$

Therefore, if

$$D_{rj} = \left\{ s \in D : \sum_{k=1}^{\infty} \left| \sum_{i \in M_{rjk}} f_i(s) \right|^2 = \infty \right\},$$

then $D = D_{r1} \cup \ldots \cup D_{rr}$. Before proceeding note the following easy consequence of Egoroff's theorem:

If a positive series $\sum_k h_k$ in $L_0(\lambda)$ is such that $\sum_k h_k(s) = \infty$ on a set A of finite λ measure, then for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is an interval I in $\mathbb{N}$ such that $m \leq \min I$ and $\sum_{k\in I} h_k(s) \geq 1$ on a measurable set $B \subset A$ with $\lambda(A\backslash B) < \varepsilon$.

Applying this inductively with respect to $r = 1, 2, \ldots$ and, for each r, with respect to $j = 1, \ldots, r$ (taking $\varepsilon = 2^{-r}/r$ and $E = D_{rj}$) to the series

$$\sum_{k=1}^{\infty} \Big| \sum_{i\in M_{rjk}} f_i(s) \Big|^2 \quad \text{on the set} \quad D_{rj} \quad (j = 1, \ldots, r),$$

we find for each r intervals $I_{r1}, \ldots, I_{rr}$ in $\mathbb{N}$ and measurable sets $E_{rj} \subset D_{rj}$ so that if

$$K_{rj} = \bigcup_{k\in I_{rj}} M_{rjk} \qquad \text{for } j = 1, \ldots, r,$$

$$K_r = K_{r1} \cup \ldots \cup K_{rr}, \quad E_r = E_{r1} \cup \ldots \cup E_{rr},$$

then

$$\max K_{rj} + r < \min K_{r,j+1} \quad \text{for } j = 1, \ldots, r-1,$$
$$\max K_r + r < \min K_{r+1}, \qquad \lambda(D\backslash E_r) < 2^{-r},$$

and

$$\sum_{k\in I_{rj}} \Big| \sum_{i\in M_{rjk}} f_i(s) \Big|^2 \geq 1 \qquad \text{on } E_{rj} \quad \text{for } j = 1, \ldots, r.$$

In consequence,

$$\sum_{j=1}^{r} \sum_{k\in I_{rj}} \Big| \sum_{i\in M_{rjk}} f_i(s) \Big|^2 \geq 1 \qquad \text{on } E_r.$$

Arrange all the sets M_{rjk} in a single sequence (M_n) so that for each r,

$$\{M_n : p_r \leq n < p_{r+1}\} = \{M_{rjk} : j = 1, \ldots, r,\ k \in I_{rj}\},$$

where $1 = p_1 < p_2 < \ldots$; note that $\{M_n : p_r \leq n < p_{r+1}\}$ is a partition of K_r. Then

$$\sum_{p_r \leq n < p_{r+1}} |g_n(s)|^2 \geq 1 \quad \text{on } E_r, \text{ where } \quad g_n = \sum_{i\in M_n} f_i(s).$$

It follows that $\sum_{n=1}^{\infty} |g_n(s)|^2 = \infty$ a.e. on D. Now, by the conditions above, the union K of the sets K_r is lacunary. Hence, by assumption, the series $\sum_{i \in K} f_i$ is subseries convergent, and so is the series $\sum_n g_n$. We have thus got a contradiction with Orlicz's theorem.

■

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