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# Extending algebraic actions.

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**Abstract**

There is a well-known procedure -induction- for extending an action of a subgroup $H$ of a Lie group $G$ on a topological space $X$ to an action of $G$ on an associated space. Induction can also extend a smooth action of a subgroup $H$ of a Lie group $G$ on a manifold $M$ to a smooth action of $G$ on an associated manifold. In this paper elementary methods are used to show that induction also works in the category of (nonsingular) real algebraic varieties and regular or entire maps if $G$ is a compact abelian Lie group.

## Introduction

If $H$ a closed subgroup of the Lie group $G$ and $f : X \to Y$ is an $H$-equivariant function then $f$ may be extended to a $G$-equivariant function $F : G \times_H X \to G \times_H Y$ by $F(g,x) = gf(x)$. Here $G \times_H X$ denotes the quotient space of $G \times X$ obtained by identifying $(gh, x)$ with $(g, hx)$, cf. [P], page 31 or [B], page 79. The set $G \times_H X$ is usually denoted by $ind_H^G(X)$ and the map $F$ is usually denoted by $ind_H^G(f)$. If is a smooth $H$-equivariant map between $H$-manifolds $X$ and $Y$ then $ind_H^G(f) : ind_H^G(X) \to ind_H^G(Y)$ is a smooth $G$-equivariant map of $G$-manifolds. If $X$ and $Y$ are (nonsingular) real algebraic $H$-varieties and $H$ has finite index in $G$ and $G$ is compact then it was shown in [DM] that $ind_H^G(X)$ and $ind_H^G(Y)$ can be given the structure of (nonsingular) algebraic varieties such that if $f$ is a regular (resp. entire rational) $H$-equivariant map then $ind_H^G(f) : ind_H^G(X) \to ind_H^G(Y)$ is a regular (resp. entire rational) $G$-equivariant map. In this note we show that

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the same result is true if $G$ is a compact abelian Lie group-the assumption that $H$ has finite index in $G$ is not necessary. The proof is entirely elementary.

A much stronger result has recently been obtained by G. Schwarz [S]; he shows, among other results, that if $X$ is a (nonsingular) real algebraic $H-$variety, $H$ is a closed subgroup of $G$ and $G$ is compact then $ind_H^G(X)$ can be given the structure of a (nonsingular) algebraic $G-$variety in a functorial way.

A good reference for definitions and background is [D].

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# 1 Definitions

Let $G$ be a compact Lie group and $V$ a representation space of $G$, that is, a real vector space on which $G$ acts via linear maps. Let $p_i : V \to R$, $i = 1, 2, \cdots, n$ be polynomial maps. Then the set $M = \{v \in V \mid p_i(v) = 0 \text{ for } i = 1, 2, \cdots, n\}$ is called a real algebraic $G-$ variety if it is invariant under the action of $G$. If the differentials $\{dp_i\}$ at $x \in M$ span a space of constant rank then $M$ is said to be a nonsingular real algebraic $G-$variety.

If $X \subset V$ and $Y \subset W$ are real algebraic varieties a map $f : X \to Y$ said to be regular if it is the restriction of a polynomial map $F : V \to W$. We say that $f$ is an entire rational map if there polynomial maps $h : V \to W, k : V \to R$, such that $k$ does not vanish on $X$ and $f = h/k$. If $X \subset V$ and $Y \subset W$ are real algebraic $G-$varieties and $f : X \to Y$ is equivariant as well as regular then $f$ is a regular equivariant map. Is is shown in [DMS] that if $f$ is a regular equivariant map then there is an **equivariant** polynomial map $F : V \to W$ such that $f$ is the restriction of $F$ to $X$; similarly, if $f$ is an equivariant entire rational map, there exist **equivariant** polynomial maps $h : V \to W$, $k : V \to R$, such that $k$ does not vanish on $X$ and $f = h/k$.

# 2

**Theorem.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic $H-$varieties and regular $H-$equivariant maps (resp. $H-$equivariant entire rational maps) to the category of (nonsingular) real algebraic $G-$varieties and regular $G-$equivariant maps (resp. $G-$equivariant entire rational maps).*

The proof will follow from a series of lemmas.

The following lemma allows us to reduce the proof to a special case.

**Lemma 1.** *Let $G$ be a compact abelian Lie group, let $H$ be a closed subgroup and let $n = \text{dimension}\, G - \text{dimension}\, H$; $n \geq 0$. Then there is a sequence of groups $H = G_0 \subset G_1 \subset \ldots G_n \subset G$ such that $G_{j+1}/G_j \approx S^1$ for $j = 0, 1\, n-1$ and $G_n$ has finite index in $G$.*

**Proof of lemma.** If $n = 0$ then $H$ and $G$ have the same components of the identity, $H_e = G_{e'}$ so $G/H$ is a quotient of $G/G_e$ which is finite so $G_0$ has finite index in $G$. If $n > 0$ we proceed by induction on $n$.

There is a one to one correspondence between (closed) subgroups of $G$ containing $H$ and (closed) subgroups of $G/H$. If $n > 0$, let $K$ be a circle subgroup of $G/H$, let $\pi : G \to G/H$ be the projection and let $G_1 = \pi^{-1}(K)$. Now dimension $G-$dimension $G_1 = n-1$ so we are done.

∎

**Remark 1.** Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Choose a circle subgroup $S^1 \subset G$ such that $S^1$ is not contained in $H$; then $G$ is the quotient of $S^1 \times H$ by a cyclic group $Z_m = S^1 \cap H$. (If $S^1 \cap H = e$ then $m = 1$). We will henceforth assume that such a circle group has been chosen once and for all. We let $\zeta = e^{2\pi i/m}$; $\zeta$ is a generator for the group $Z_m = S^1 \cap H$.

We wish to define the induction functor $ind_H^G()$ from the category of (nonsingular) algebraic $H-$varieties and regular $H-$equivariant maps (resp. $H-$equivariant entire rational maps) to (nonsingular) algebraic $G-$varieties and regular $G-$equivariant maps (resp. $G-$equivariant entire rational maps). In view of Lemma 1 it suffices to define $ind_H^G()$ as the composition of induction maps from $G_j$ to $G_{j+1}$ for $j = 0, 1, n-1$ and from $G_n$ to $G$. Since $G_n$ has finite index in $G$, the functor $ind_{G_n}^{G}()$ exists by the result of [DM1]. Thus, we need only consider the special case $G_j$ to $G_{j+1}$, that is, we have proved:

**Lemma 2.** *It suffices to prove the theorem in the special case in which dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$.*

We next show that without loos of generality we may assume that our $H-$varieties are imbedded in complex representations on which $G$ acts.

**Lemma 3.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Let $X$ be a (nonsingular) $H-$variety in the real $H-$representation space $V$. Then we may find a complex $G-$representation space $V' = V \otimes_R C$ such that $X$ is an $H-$invariant (nonsingular) real algebraic variety in $V'$. Moreover, this procedure is functorial, that is, if $f : X \to Y$ is an $H-$equivariant regular (resp. entire rational) map between algebraic $H-$varieties $X \subset V$ and $Y \subset W$ where $V$ and $W$ are real $H-$representation spaces then there is an $H-$equivariant polynomial map $F' : V' \to W'$ between the complex $G-$representation spaces $V'$ and $W'$ such that $F' \mid X = f$ (resp. $f = h'/k'$ where $h' : V' \to W', k' : V' \to R$ are polynomial maps such that $k'$ does not vanish on $X$).*

**Proof.** Let $p_i : V \to R, i = 1, 2, \ldots, n$, be polynomial maps such that $X = \{v \in V \mid p_i(v) = 0 \text{ for } i = 1, 2, \ldots, n\}$. (If $X$ is nonsingular we assume that the differentials $\{dp_i\}$ at $x \in X$ span a space of constant rank). Then we have that $X = \{v \in V \otimes_R C \mid v = \bar{v}, p'_i(v) = 0 \text{ for } i = 1, 2, \ldots, n\}$ where $p'_i(v)$ denotes the same polynomials but with complex arguments rather than real arguments. We note that the equations $v = \bar{v}, p'_i(v) = 0$ may be regarded as real polynomial equations on the underlying real vector space of $V'$. Moreover, if $f : X \to Y$ is an $H-$equivariant regular map then the $H-$equivariant polynomial map $F' : V' \to W'$ between the complex $G-$representation spaces $V'$ and $W'$ given by extending the definition of $F$ to complex arguments satisfies $F' \mid_X = f$. Similar remarks apply if $f$ is an entire rational $H-$equivariant map.

We need only verify that the $H$ action on $V$ can be extended to a $G$ action on $V'$. Since $H$ is abelian every irreducible complex representation of $H$ is one dimensional; in particular, we may write $V'$ as a sum of one (complex) dimensional $H$–irreducible spaces, $V' = \sum_{s=1}^{a} V_s$. Thus, to prove that we may extend the action of $H$ on $V'$ to $G$ it is sufficient to show that we may extend the action of $H$ on each $V_s$ to $G$. In view of remark 1 above, that means defining an action of the circle subgroup $S^1$ on each $V_s$ that extends the action of $Z_m$ on each $V_s$. The element $\zeta$ in $H$ acts on $V_s$ via complex multiplication by $\zeta^r$ for some $r$ that depends on $s$. We let $t \in S^1$ act on $V_s$ via complex multiplication by $t^r$. That extends the action to $G$. (Complex multiplication by $t^{r}$ for $r' \equiv r$ mod $m$ would also work).

■

**Definition.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Let $\alpha : G \to S^1 = U(1)$ be the homomorphism with kernel $H$ given by $\alpha(t) = t^m$ for $t \in S^1$, $\alpha(h) = 1$ for $h \in H$. We denote by $\bar{C}$ the complex numbers with the $G$ action given by $\alpha$.*

**Notation.** We denote the **action** of the complex number $t$ on the vector $v$ by $t * v$; $tv$ will denote *multiplication* of the vector $v$ by the complex number $t$.

The proof of the following lemma can be found in [P], page 31. It is given here for the sake of completeness. The assertion is that $V' \times 1$ is a slice in $V' \times S^1$ at $(0, 1)$.

**Lemma 4.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Let $X$ be a (nonsingular) $H$–variety in the complex $G$–representation space $V'$ and let $i : X \to V'$ be the inclusion. Let $j : G \times_H X \to V' \oplus \bar{C}$ be given by $j(g, x) = (gi(x), \alpha(g))$. Then $j$ is a $G$–equivariant imbedding of $G \times_H X \to V' \oplus \bar{C}$.*

**Proof.** To see that $j$ is $G$–equivariant we note that $j(g'(g, x)) = j(g'g, x) = (g'gi(x), \alpha(g'g)) = g'(gi(x), \alpha(g)) = g'j(g, x)$.

If $j(g, x) = j(g', y)$ then $\alpha(g) = \alpha(g')$ so $g' = hg$ for some $h$ in $H$. Then $j(g, x) = (gi(x), \alpha(g))$ and $j(g', y) = j(gh, y) = (ghi(y), \alpha(gh)) = (g\,i(hy), \alpha(gh))$ so $i(x) = i(hy)$ and thus, since $i$ is an imbedding,

$x = hy$. Hence $j$ is one to one. The statement that $j$ is an imbedding follows from the fact that $i$ is a closed map and $G$ is compact.

■

We will now show that $j(G \times_H X)$ is a (nonsingular) algebraic $G$–variety in $V' \oplus \bar{C}$. Since every point in $j(G \times_H X)$ is of the form $g(x, 1)$ for some $x \in X$, and $X$ is the zero set of some polynomials $p_i'$ we must try to write $(v, w)$ as $g(x, 1)$ for some $g$ and some $x$ and then verify that $p_i'(x) = 0$ for $i = 1, 2, \ldots, n$. If $w\bar{w} = 1$ then $w \in S^1$ and that is necessary and sufficient for $w$ to equal $\alpha(g)$ for some $g \in G$ since $\alpha$ is onto. By remark 1, any $g$ in $G$ is of the form $t\,h$ for some $t \in S^1$ and some $h \in H$ so if $w = \alpha(g)$ then $w = t^m$ and $\bar{w} = t^{-m}$. (Note that $t$ is only determined up to an $m^{th}$ root of unity). Then $(v, w)$ can be written as $t * (\bar{t} * v, 1)$. Thus, we only need the equations $w\bar{w} = 1, p_i'(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n$. (Since $g$ is not unique, the equations do not appear to be well defined no less polynomial in $v$ and $w$ and their conjugates).

We interpreted the equations as follows: $V' = \sum_{s=1}^{a} V_s$; the action of $t \in S^1 \subset G$ on $V_j \subset V'$ is given by $t * v_j = t^{r_j} v_j$ for some integer $r_j$. Now we extend the action of $S^1$ on $V'$ to an "action" of $C$ on $V'$ by $q * v_j = q^{r_j} v_j$ for $q$ in $C$. Finally, define $p_i'(\bar{w}^{1/m} * v)$ by $p_i'(\bar{w}^{1/m} * v) = p_i'(\bar{q} * v)$ where $q$ is any $m^{th}$ root of $w$, the $^-$ indicates complex conjugate, and the $*$ indicates the "action" of $C$ on $V'$. To put it more concretely, one computes $p_i'(v, w)$ by first choosing $q$ an $m^{th}$ root of $w$ (recall $Z_m = S^1 \cap H$) and then replacing every occurrence of the variable $v_j$ in the polynomial $F$ by $\bar{q}^{r_j} v_j$.

Then we have:

**Lemma 5.** *Under the assumptions of Lemma 4 let $X$ be the $H$–variety in the complex $G$–representation space $V'$ given by $\{v \in V' \mid p_i'(v) = 0$ for $i = 1, 2, \ldots, n\}$; then $j(G \times_H X) = \{(v, w) \in V' \oplus \bar{C} \mid w\bar{w} = 1, p_i'(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n\}$. Moreover, $j(G \times_H X)$ si a (nonsingular) $G$–variety in $V' \oplus \bar{C}$.*

**Proof.** We will show that

i) the zeroes of the $p_i''s$ do not depend on which $m^{th}$ root $q$ is chosen to compute the value of $p_i$,

ii) the "equations" vanish exactly on $j(Gx_H X)$,

iii) the equations are equivalent to (have the same zero set as) polynomial equations in the variables, $v$, $w$ and their conjugates, and

iv) if $X$ is nonsingular then $j(Gx_H X)$ is nonsingular.

i) If $p_i'(\bar{w}^{1/m} * v) = 0$, that is, if $p_i'(\bar{q} * v) = 0$ for some choice of $q$ then for any choice of $q$, $p_i'(\bar{q} * v) = 0$ because the $p_i'$ are $H-$invariant, and two $q's$ differ by an $m^{th}$ root of unity any $m^{th}$ root of unity is a power of $\zeta \in H$.

ii) We note that a point of the form $(x, 1)$ satisfies $w\bar{w} = 1$ and $p_i'(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n$, if and only if $x$ in $X$ since we may take $q = 1$. Since the equations are invariant under $G$ they vanish precisely on $j(G \times_H X)$.

iii) As interpreted, $p_i'(v, w)$ is a polynomial in $v$ and $q$; we must show that it is equivalent to polynomials in $v$ and $w$.

Fix $i$ and write

$\diamondsuit$ $p_i'(v, w) = P_0(v, w) + P_1(v, w)q + P_2(v, w)q^2 \ldots + P_{m-1}(v, w)q^{m-1}$ by grouping together monomials that have the same power of $q$ mod $m$ and replacing each occurrence of $q^m$ by $w$. Thus $P_0(v, w)$ consists of all those monomials in which $q$ appears to a power divisible by $m$. Each $P_i$ is a polynomial in $v$ and $w$.

Now let $(v, w)$ be a point in $j(Gx_H X)$, that is, $p_i'(v, w) = 0$. Consider the system of $m$ linear equations obtained from $\diamondsuit$) by substituting $q\zeta^j$ for $q$. By i) the result in each cases is $p_i'(v, w) = 0$ so we get:

$$
\begin{aligned}
0 &= P_0(v, w) + P_1(v, w)q + P_2(v, w)q^2 \ldots + P_{m-1}(v, w)q^{m-1} \\
0 &= P_0(v, w) + P_1(v, w)q\zeta + P_2(v, w)q^2\zeta^2 \\
&\qquad \ldots + P_{m-1}(v, w)q^{m-1}\zeta^{m-1} \\
&\cdot \\
&\cdot \\
0 &= P_0(v, w) + P_1(v, w)q\zeta^{m-1} + P_2(v, w)q^2\zeta^{2(m-1)} \\
&\qquad \ldots + P_{m-1}(v, w)q^{m-1}\zeta^{(m-1)(m-1)}
\end{aligned}
$$

These equations have the form:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(m-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & \zeta^{(m-1)^2} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 q \\ P_2 q^2 \\ \cdots \\ P_{m-1} q^{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

Since the coefficient matrix is a Vandermonde matrix with nonzero determinant $\left( \prod_{0<i<j<m} (\zeta^j - \zeta^i) \right)$ there is a unique solution to the system of equations which is $P_j(v, w)q^j = 0$ for all $j$. Thus, $p'_i(v\,w) = 0$ if and only if the polynomials $P_j(v\,w) = 0$ for all $j = 1, 2, \ldots m-1$. (There is a different set of $P'_j s$ for each $i$; we write them as $P^i_j$ if we need to be precise). Hence, the set $j(G \times_H X)$ is the zero set of a family of polynomials, i.e., $j(G \times_H X)$ is a real algebraic $G-$variety.

iv) If $X$ is a nonsingular variety then the differentials $\{dp'_i\}$ at $x \in X$ span a space of constant rank = dimension $V'-$dimension $X$. To see that the differentials $dP^i_j$ and $d(w\bar{w})$ span a space of constant rank = dimension $V' \oplus \bar{C}-$dimension $j(Gx_HX) = 2+$dimension $V' - (1+$ dimension $X) = 1+$ dimension $V'-$dimension $X$ at points $(x, 1)$ in dimension $j(Gx_HX)$ we note that the $dp'_i$ are linear combinations of the $dP^i_j$ because the $p'_i$ are linear combinations of the $P^i_j$ and that $w\bar{w}$ does not involve the $v_j$ coordinates. Hence, the dimension of the space spanned is at least 1+ dimension $V'-$dimension $X$; it cannot be more since dimension $j(Gx_HX) = 1+$ dimension $X$. To check at other points $(v, w)$ in $j(Gx_HX)$ one translates the problem to $V' \times 1$ by multiplying by a suitable element of $G$.

■

In view of lemma 5 we may make the following definition.

**Definition.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Let $X$ be a (nonsingular) $H-$variety in the complex $G-$representation space $V'$ given by $\{v \in V' \mid p'_i(v) = 0$ for $i = 1, 2, \cdots, n\}$. Then $ind^G_H(X)$ is* *the (nonsingular) $G-$variety in $V' \oplus \bar{C}$ given by*

$$\{(v, w) \in V' \oplus \bar{C} \mid w\bar{w} = 1, P_j^i(v, w) = 0$$

*for*

$$i = 1, 2, \cdots, n, j = 0, 1, 2, \cdots, m-1\}.$$

**Lemma 6.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. Let $F : V' \to W'$ be an $H-$equivariant polynomial map between the complex $G-$representation spaces $V'$ and $W'$; then there is a $G-$equivariant polynomial map $F' : V' \oplus \bar{C} \to W' \oplus \bar{C}$ between the complex $G-$representation spaces $V' \oplus \bar{C}$ and $W' \oplus \bar{C}$ such that $F'(v, 1) = (F(v), 1)$.*

**Proof.** Since $G$ is abelian we have that $W' = \sum_{j=1}^{b} W_j$ where the $W_j$ are one dimensional $G-$invariant subspaces. Since the map $F'$ will have the desired properties if and only if the projections of $F'$ onto each of the factors has the desired properties it suffices to prove the lemma in the case that $W'$ is one dimensional. That is, we consider a polynomial $F(v_1, v_2, \cdots, v_s)$ from $V'$ to $W'$ where the action of $t \in S^1 \subset G$ on $V'$ is given by $t * v_j = t^{r_j} v_j$ for some integer $r_j$ and where the action of $t \in S^1 \subset G$ on $W'$ is given by $t * w = t^d w$ for some integer $d$. As in lemma 5 we extend the action of $S^1$ on $V'$ and $W'$ to an "action" of $C$ on $V'$ and $W'$ by $q * v_j = q^{r_j} v_j$ and $q * w = q^d w$ for $q$ in $C$. Finally, define $F'(v, z) = (q * F(\bar{q} * v), z)$ where $q$ is any $m^{th}$ root of $z$, the $^-$ indicates complex conjugate, and the $*$ indicates the "action" of $C$ on $V'$ and $W'$. To put it more concretely, one computes $F'(v, z)$ by first choosing $q$, an $m^{th}$ root of $z$ (recall $Z_m = S^1 \cap H$) and then replacing every occurrence of the variable $v_j$ in the polynomial $F$ by $\bar{q}^{r_j} v_j$ and then multiplying the result by $q^d$.

We will now verify that.

i) $F'$ is well defined, that is, that $F'(v, z)$ is independent on the choice of $q$.

ii) $F'$ is $G-$equivariant.

iii) $F'(v, 1) = (F(v), 1)$ and

iv) $F'$ is a polynomial in $v$ and $z$.

i) Note that $q$ is only determined up to an $m^{th}$ root of unity and any $m^{th}$ root of unity is a power of $\zeta \in H$. Then $F'(v,z) = (q * F(\bar{q} * v), z) = (q\zeta\bar{\zeta} * F(\bar{q} * v), z)$, which, by the $H-$equivariance of $F$ is equal to $(\zeta q * F(\bar{\zeta}\bar{q} * v), z)$ so $F'$ is well defined.

ii) If $h \in H$ we have by definition that $F(h * (v, z)) = F'(hv, z) = (q * F(\bar{q}h * v), z) = (qh * F(\bar{q} * v), z) = hF(v, z)$ since $H$ acts trivially on $\bar{C}$. If $t \in S^1$ we have

$$\begin{aligned} F(t * (v, z)) &= F'(tv, t^m z) = (qt * F(\bar{q}\bar{t}t * v), t^m z) \\ &= t * (F(\bar{q} * v), z) = t * F(v, z). \end{aligned}$$

iii) $F'(v, 1) = (q * F(\bar{q} * v), 1)$ and the result follows by taking $q = 1$.

iv) $F'(v, z)$ is a polynomial in $v$ and $q$; we must show that it is a polynomial in $v$ and $z$.

Write

$\Diamond\Diamond) F'(v, z) = P_0(v, z) + P_1(v, z)q + P_2(v, z)q^2 \ldots + P_{m-1}(v, z)q^{m-1}$

by grouping together monomials that have the same power of $q$ mod $m$ and replacing each occurrence of $q^m$ by $z$. Thus $P_0(v, z)$ consists of all those monomials in which $q$ appears to a power divisible by $m$. Each $P_i$ is a polynomial in $v$ and $z$.

Now consider the system of $m$ linear equations obtained from $\Diamond\Diamond)$ by substituting $q\zeta^j$ for $q$. By i) the result is $F'(v, z)$ in each case.

$$F'(v, z) = P_0(v, z) + P_1(v, z)q + P_2(v, z) \ldots + P_{m-1}(v, z)q^{m-1}$$
$$F'(v, z) = P_0(v, z) + P_1(v, z)q\zeta + P_2(v, z)q^2\zeta^2 \ldots + P_{m-1}(v, z)q^{m-1}\zeta^{m-1}$$

$\cdot$

$\cdot$

$\cdot$

$$F'(v, z) = P_0(v, z) + P_1(v, z)q\zeta^{m-1} + P_2(v, z)q^2\zeta^{2(m-1)} \ldots + P_{m-1}(v, z)q^{m-1}\zeta^{(m-1)(m-1)}$$

These equations have the form

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(m-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & \zeta^{(m-1)^2} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 q \\ P_2 q^2 \\ \cdots \\ P_{m-1} q^{m-1} \end{bmatrix} = \begin{bmatrix} F'(v,z) \\ F'(v;z) \\ F'(v,z) \\ \cdots \\ F'(v,z) \end{bmatrix}$$

Since the coefficient matrix is a Vandermonde matrix with nonzero determinant there is a unique solution to the system of equations which is $P_0(v, z) = F'(v, z)$, $P_i = 0$ for $i > 0$. Thus, $F'(v, z)$ is a polynomial in $v$ and $z$. ∎

An immediate consequence of lemma 6 is .

**Lemma 7.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H$ and $G/H \approx S^1$. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic $H-$varieties and regular $H-$equivariant maps to the category of (nonsingular) real algebraic $G-$varieties and regular $G-$equivariant maps.*

**Remark 2.** Lemma 6 also applies to the case in which $F$ is an $H-$invariant real or complex function. Hence we have another corollary of lemma 6.

**Lemma 8.** *Let $H$ be a closed subgroup of the compact abelian Lie group $G$ with dimension $G = 1+$ dimension $H, G/H \approx S^1$. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic $H-$varieties and $H-$equivariant entire rational maps to the category of (nonsingular) real algebraic $G-$varieties and $G-$equivariant entire rational maps.*

**Remark 3.** Choices were made in defining $ind_H^G(X)$; there was a choice as to the extension to $G$ of a linear action of $H$ on the vector space $V'$ in lemma 3 and there was a choice made in the "composition series" $H = G_0 \subset G_1 \subset \ldots G_n \subset G$ in lemma 1. For any $H-$variety $X$ the identity map $i : X \to X$ is an $H-$equivariant regular map and if $ind_H^G(X)$ and $ind_H^G(X)'$ are the varieties that result from two different choices of extension in lemma 3 there are regular $G-$equivariant maps $ind_H^G(i) : ind_H^G(X) \to ind_H^G(X)', ind_H^G(i)' : ind_H^G(X)' \to ind_H^G(X)$ by lemma 6 with $ind_H^G(i)'^0\, ind_H^G(i)$ and $ind_H^G(i)^0\, ind_H^G(i)'$ the identity so that the result is, in fact, unique. It is a priori possible that different choices in lemma 1 might lead to different functors.

The proof of the theorems is now complete.

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