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An approach to shape covering maps.

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Abstract

In this note we give an approach to shape covering maps which is comparable to that of $*$-fibrations [5]. The introduced notion conserves some important properties of usual covering maps.

1 Introduction

In [2] D.S. Coram and P.F. Duvall, Jr. introduced the notion of approximate fibration and showed that several important properties of Hurewicz fibrations carry over, with suitable modifications, to approximate fibrations. Coram and Duvall proved, for example, that the fibers are FANRs and that if the base space is path connected then all the fibers have the same shape.

In [5,6] S. Mardešić and T.R. Rushing introduced the notion of shape fibrations. Shape fibrations are defined in the spirit of the ANR-sequence approach to shape theory. It is shown that shape fibrations coincide with approximate fibrations whenever the base space and total space are ANR's.

Recently, A. Giraldo [3] and A. Giraldo and J.M.R. Sanjurjo [4] have given an intrinsic description of shape fibrations with the near lifting of near multivalued paths property.

The multifibrations introduced by these authors represent a formally stronger concept than that of shape fibration.

Because the theory of covering maps appears much richer in geometric content than that of Hurewicz fibrations we consider to be very

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appealing to have a comparable theory of shape covering maps.

In this paper we define a notion of shape covering map which is comparable to that of $*$–fibration.

At first we recall from [5,6] some notions and properties concerning $*$–fibrations and shape fibrations.

We consider inverse sequences (towers) $\underline{E} = (E_i, q_{ij})$, $\underline{B} = (B_i, r_{ij})$ of metric compacta (called for short compact sequences). If all E_i and B_i are compact ANR's we speak of ANR-sequences. A level preserving map of sequences (abbreviated as level map) $\underline{p} : \underline{E} \to \underline{B}$ is a sequence of maps $\underline{p} = (p_i)$, where $p_i : E_i \to B_i$, and for every i and every $j \geq i$ the following diagram commutes

(1)
$$\begin{tikzcd} E_i \arrow[d, "p_i"'] & E_j \arrow[l, "q_{ij}"'] \arrow[d, "p_j"] \\ B_i & B_j \arrow[l, "r_{ij}"] \end{tikzcd}$$

Definition 1. [5] *A level map $\underline{p} : \underline{E} \longrightarrow \underline{B}$ has the homotopy lifting property (HLP) with respect to a space X provided that each i admits a $j \geq i$ such that for any maps $h_j : X \longrightarrow E_j$, $H_j : X \times I \longrightarrow B_j$ with*

(2) $$p_j h_j = H_{j0},$$

there exists a homotopy $\widetilde{H}_i : X \times I \longrightarrow E_i$ with

(3) $$\widetilde{H}_{i0} = q_{ij} h_j \quad \text{and}$$

(4) $$p_i \widetilde{H}_i = r_{ij} H_j.$$

Every such j is called a lifting index for i.

Definition 2. [7] *Let $\underline{p} : \underline{E} \longrightarrow \underline{B}$ be a level map of compact sequences. Let $\varprojlim \underline{E} = (E, q_i)$ and $\varprojlim \underline{B} = (B, r_i)$, where $q_i : E \longrightarrow E_i$ and $r_i : B \longrightarrow B_i$ are the natural projections. The unique map $p : E \longrightarrow B$ such that for every i the following diagram commutes*

(5)
$$\begin{tikzcd} E_i \arrow[d, "p_i"'] & E \arrow[l, "q_i"'] \arrow[d, "p"] \\ B_i & B \arrow[l, "r_i"] \end{tikzcd}$$

is said to be induced by $\underline{p}$ or to be the limit of $\underline{p}$.

Definition 3. [5] *A map between metric compacta* $p : E \longrightarrow B$ *is called a* $*$*-fibration provided it is induced by a level map* $\underline{p} : \underline{E} \longrightarrow \underline{B}$ *between compact sequences satisfying the HLP with respect to any metric space* X *and the lifting index does not depend on* X.

If $\underline{E}$ and $\underline{B}$ are required to be compact ANR-sequences and if the HLP for $\underline{p}$ is replaced with the AHLP (approximate homotopy lifting property) which means that each i and each $\varepsilon > 0$ admit a $j \geq i$ such that (2) and (3) imply

$$d(p_i \widetilde{H}_i, r_{ij} H_j) < \varepsilon, \tag{6}$$

then $p : E \longrightarrow B$ is called a *shape fibration.*

In [5] it is proved that every shape fibration $p : E \longrightarrow B$ between metric compacta is a $*$–fibration but the converse is false.

2 A characterization of $*$–fibrations

We give a characterization of $*$–fibrations by analogy with the case of Hurewicz fibrations.

For a level map $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ between metric compacta and for an index i, we consider the space $D_i = \{(e_i, \omega_i) \in E_i \times B_i^I \mid p_i(e_i) = \omega_i(0)\}$.

Then, for $j \geq i$, we can define a map $d_{ij} : D_j \longrightarrow D_i$ by $d_{ij}(e_j, \omega_j) = (q_{ij}(e_j), r_{ij}\omega_j)$ and then we obtain an inverse sequence $\underline{D} = (D_i, d_{ij})$. On the other hand, we can consider the inverse sequence $\underline{E}^I = (E_i^I, e_{ij})$, with $e_{ij} : E_j^I \longrightarrow E_i^I$ given by $e_{ij}(\theta_j) = q_{ij}\theta_j$ for $j \geq i$.

Definition 4. *A lifting morphism for a level map* $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ *is a morphism of inverse systems*

$$\underline{\lambda} = (\lambda_{ij}) : \underline{D} \longrightarrow \underline{E}^I, \quad \text{where} \tag{7}$$

$$\lambda_{ij} : D_j \longrightarrow E_i^I, \quad \text{for a } j \geq i, \text{ and satisfying} \tag{8}$$

$$\lambda_{ij}(e_j, \omega_j)(0) = q_{ij}(e_j), \quad \text{and} \tag{9}$$

$$p_i \lambda_{ij}(e_j, \omega_j) = r_{ij}\omega_j, \quad \text{for any } (e_j, \omega_j) \in D_j. \tag{10}$$

We say that $\underline{p}$ has an approximate lifting morphism if each i and each $\varepsilon > 0$ admit an index j and a map $\lambda_{ij} : D_j \longrightarrow E_i^I$ such that the function $i \longrightarrow j$ and the maps λ_{ij} define a morphism of inverse system from $\underline{D}$ to $\underline{E}^I$, satisfying the condition (9) and

$$(11) \qquad d(p_i\lambda_{ij}(e_j,\omega_j), r_{ij}\omega_j) < \varepsilon \ \text{ for any } (e_j,\omega_j) \in D_j.$$

Theorem 1. *Let $p : E \longrightarrow B$ be a map between metric compacta induced by a level map $\underline{p} : \underline{E} \longrightarrow \underline{B}$ between compact sequences. Then p is a $*$-fibration if and only if $\underline{p}$ has a lifting morphism.*

Proof. Suppose that $\underline{p}$ has a lifting morphism. For an index i, denote by j the index from Definition 4 corresponding to i. Let X be an arbitrary space and two maps $h_j : X \longrightarrow E_j$, $H_j : X \times I \longrightarrow B_j$ with $H_j(x,0) = p_jh_j(x)$, $(\forall)x \in X$. For $x \in X$, define $\omega_j(x) : I \longrightarrow B_j$ by $\omega_j(x)(t) = H_j(x,t)$. Then we have $\omega_j(x)(0) = H_j(x,0) = p_jh_j(x)$ and therefore $p_i(q_{ij}h_j(x)) = r_{ij}(p_jh_j(x)) = (r_{ij}H_j)(x,0)$. Hence $(q_{ij}h_j(x), r_{ij}(\omega_j(x))) \in D_i$ and we can define $\widetilde{H}_i : X \times I \longrightarrow E_i$ by $\widetilde{H}_i(x,t) = \lambda_{ij}(h_j(x),\omega_j(x))(t)$. For this we have: $\widetilde{H}_i(x,0) = q_{ij}h_j(x)$ and $p_i\widetilde{H}_i(x,t) = p_i\lambda_{ij}(h_j(x),\omega_j(x))(t) = p_i\lambda_{ij}(h_j(x),\omega_j(x))(t) = r_{ij}\omega_j(x) = r_{ij}H_j(x,t)$. Then $\underline{p} : \underline{E} \longrightarrow \underline{B}$ has the HLP with respect to X and therefore $p : E \longrightarrow B$ is a $*$-fibration.

Conversely, suppose that $p : E \longrightarrow B$ is a $*$-fibration. For an index i, denote by $j \geq i$ its corresponding index by Definition 1. We can consider the maps $h_j : D_j \longrightarrow E_j$ given by $h_j((e_j,\omega_j)) = e_j$ and $H_j : D_j \times I \longrightarrow B_j$ with $H_j((e_j,\omega_j),t) = \omega_j(t)$.

For these maps we have $H_j((e_j,\omega_j),0) = \omega_j(0) = p_j(e_j) = p_jh_j((e_j,\omega_j))$. Then, by hypothesis, there exists $\widetilde{H}_i : D_j \times I \longrightarrow E_i$ with $\widetilde{H}_i((e_j,\omega_j)),0) = q_{ij}h_j(e_j,\omega_j) = q_{ij}(e_j)$ and $p_i\widetilde{H}_i((e_j,\omega_j),t) = r_{ij}H_j((e_j,\omega_j),t) = r_{ij}\omega_j(t)$. Define $\lambda_{ij} : D_j \longrightarrow E_i^I$ by $\lambda_{ij}((e_j,\omega_j))(t) = \widetilde{H}_i((e_j,\omega_j),t)$. Then: $\lambda_{ij}((e_j,\omega_j))(0) = \widetilde{H}_i((e_j,\omega_j),0) = q_{ij}(e_j)$ and $p_i\lambda_{ij}(e_j,\omega_j)(t) = p_i\widetilde{H}_i((e_j,\omega_j),t) = r_{ij}\omega_j(t)$. This proves that the λ_{ij} define a lifting morphism for $\underline{p}$.

Analogously we can prove:

Theorem 2. *Let $p : E \longrightarrow B$ be a map between metric compacta induced by a level map $\underline{p} : \underline{E} \longrightarrow \underline{B}$ of* ANR*-sequences. Then p is a shape fibration if and only if $\underline{p}$ has an approximate lifting morphism.*

3 Shape covering maps

Definition 5. A map between metric compacta $p : E \longrightarrow B$ is called *a shape covering map* provided it is induced by a level map $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ between compact sequences satisfying the following condition:

For every space B_i there exists an open cover $\mathcal{U}_i$ such that $j \geq i$ implies $r_{ij}(\mathcal{U}_j) \subset \mathcal{U}_i$ (i.e. $r_{ij}(U_j) \in \mathcal{U}_i$ for any $U_j \in \mathcal{U}_j$)[1], for $U_i \in \mathcal{U}_i$ we have $p_i^{-1}(U_i) = \cup \widetilde{U}_i$ a disjoint union of open subset of E_i and each i admits a $j \geq i$ with the property that $p_j \mid q_{ij}^{-1}(\widetilde{U}_i) : q_{ij}^{-1}(\widetilde{U}_i) \longrightarrow r_{ij}^{-1}(U_i)$ is a homeomorphism.

Remark 1. Easily follows that a shape covering map is surjective.

Remark 2. Obviously, if $p_i : E_i \longrightarrow B_i$ are covering maps for any index i, then $\varprojlim p_i$ is a shape covering map.

Corollary 1. *If $p : E \longrightarrow B$ is a shape covering map, then any of its fibers $p^{-1}(b)$ is a discrete subspace of E.*

Proof. Choose an index i and let $j \geq i$ be as in Definition 5. We can prove that $p^{-1}(b) \cap q_j^{-1}(q_{ij}^{-1}(\widetilde{U}_i))$ consists of a single point. Indeed, if we suppose that $e_1, e_2 \in p^{-1}(b) \cap q_j^{-1}(q_{ij}^{-1}(\widetilde{U}_i))$ then we have $p(e_1) = p(e_2) = b$ and $q_j(e_1), q_j(e_2) \in q_{ij}^{-1}(\widetilde{U}_i)$. But $p_j q_j(e_1) = r_j p(e_1) = r_j p(e_2) = p_j q_j(e_2)$ and this implies $q_j(e_1) = q_j(e_2)$.

Then $q_i(e_1) = q_{ij} q_j(e_1) = q_{ij} q_j(e_2) = q_i(e_2)$. Because this equality holds for each index i, we have $e_1 = e_2$.

Theorem 3. *Let $p : E \longrightarrow B$ be a shape covering map and $f, g : X \longrightarrow E$ two maps which are liftings of the same map* (i.e. $pf = pg$). *Then if the space X is connected and if the maps f and g coincide in a point of X, it follows the equality $f = g$.*

Proof. Consider an arbitrary fixed index i and let $j \geq i$ be the index correspoding to i by Definition 5.

Let $X_1 = \{x \in X \mid q_j f(x) = q_j g(x)\}$. We prove that X_1 is open in X. If $x \in X_1$, let $U_i \in \mathcal{U}_i$ containing $r_i(pf(x)) = r_i(pg(x))$ or $p_i(q_i f(x)) = p_i(q_i g(x)) \in U_i$ and therefore $q_i(f(x)) \in p_i^{-1}(U_i)$. Then, there exists $\widetilde{U}_i$ such that $q_i(f(x)) \in \widetilde{U}_i$. This implies that $q_{ij} q_j(f(x)) \in \widetilde{U}_i$ or $q_j(f(x)) \in q_{ij}^{-1}(\widetilde{U}_i)$. Then $(q_j \circ f)^{-1}(q_{ij}^{-1}(\widetilde{U}_i)) \cap (q_j \circ g)^{-1}(q_{ij}^{-1}(\widetilde{U}_i))$ is an open set of X.

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[1] As the referee observes, this condition can be replaced by the condition: $r_{ij}(U_j)$ is contained in some open set of $\mathcal{U}_i$ because later only this condition is used.

This intersection contains x because $(q_j \circ f)(x) = q_j(f(x)) \in q_{ij}^{-1}(\widetilde{U}_i)$ and since $x \in X_1$, it follows $(q_j \circ g)(x) = (q_j f)(x) \in q_{ij}^{-1}(\widetilde{U}_i)$. Moreover, this intersection is contained in X_1. Indeed, if x_1 belongs to this intersection, then $(q_j f)(x_1) \in q_{ij}^{-1}(\widetilde{U}_i)$, $(q_j g)(x_1) \in q_{ij}^{-1}(\widetilde{U}_i)$ and $pf(x_1) = pg(x_1)$ implies $r_j pf(x_1) = r_j pg(x_1)$ which in turn implies that $p_j(q_j f(x_1)) = p_j(q_j g(x_1))$ and because $p_j \mid q_{ij}^{-1}(\widetilde{U}_i)$ is a homeomorphism, it follows that $(q_j f)(x_1) = (q_j g)(x_1)$ and thus $x_1 \in X_1$. In this way we proved that X_1 is an open set in X. But on the other hand, since E_j is a Hausdorff space, it follows that $X_2 = \{x \in X \mid q_j f(x) \neq q_j g(x)\}$ is also an open set of X. Then, since $X_1 \neq \emptyset$ and X is a connected space, it follows that $X_2 = \emptyset$. Hence $q_j f = q_j g$. Then we deduce: $q_i f = q_{ij} q_j f = q_{ij} q_j g = q_i g$ and this, by the definition of the inverse limit, implies the equality $f = g$.

Corollary 2. *Every shape covering map has the unique lifting property.*

Proof. Let $p : E \longrightarrow B$ be a shape covering map. The uniqueness of the lift of a path $\omega : I \longrightarrow B$ follows from Theorem 3. Then if $\omega : I \longrightarrow B$ is a path with $\omega(0) = b$ and $e \in p^{-1}(b)$, we deduce from Definition 5 that there exists for each i a path $\widetilde{\omega}_i : I \longrightarrow E_i$ such that $p_i \widetilde{\omega}_i = r_i \omega$ and $\widetilde{\omega}_i(0) = q_i(e)$. But the homeomorphism from Definition 5 implies $q_{ij} \widetilde{\omega}_j = \widetilde{\omega}_i$ for $j \geq i$. It follows that there exists $\widetilde{\omega} : I \longrightarrow E$ satisfying $p_i \widetilde{\omega} = \widetilde{\omega}_i$. For this we have $\widetilde{\omega}(0) = e$ and $p\widetilde{\omega} = \omega$.

Theorem 4. *Every shape covering map is a $*$-fibration.*

Proof. Let $p : E \to B$ be a shape covering map. We will prove that p is a $*$-fibration by constructing for p a lifting morphism and using Theorem 1.

For an index i, denote by $j \geq i$, the index correspoding to i by Definition 5. Let $(e_j, \omega_j) \in D_j$. Then $\{\omega_j^{-1}\mathcal{U}_j)\}$ is an open cover of the compact metric space $I = [0, 1]$. By Lebesgue's Theorem, there exists a division $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\omega_j([t_{k-1}, t_k]) \subset U_{kj} \in \mathcal{U}_j$. Hence $\omega_j([t_0, t_1]) \subset U_{1j}$ and since $p_j(e_j) = \omega_j(0) \in U_{1j} \Longrightarrow . p_i(q_{ij}(e_j)) = (r_{ij}\omega_j)(0)$. If $r_{ij}(\omega_j(0)) \in U_i \in \mathcal{U}_i$, then $q_{ij}(e_j) \in p_i^{-1}(U_i)$ and suppose that $q_{ij}(e_j) \in \widetilde{U}_{1i}$ with $p_j | q_{ij}^{-1}(\widetilde{U}_{1i}) : q_{ij}^{-1}(\widetilde{U}_{1i}) \longrightarrow r_{ij}^{-1}(U_i)$ a homeomorphism. Denote by $\omega_j | [t_{k-1}, t_k]$ the path defined by $(\omega_j | [t_{k-1}, t_k])(t) = \omega_j[(1-t)t_{k-1} + t t_k]$ and let $\widetilde{\omega}_{1j} = q_{ij} \circ (p_j \mid q_{ij}^{-1}(\widetilde{U}_{1i}))^{-1} \circ \omega_j | [0, t_1]$. We have $\omega_j | [t_1, t_2](I) \subset U_{2j}$, suppose that $\widetilde{\omega}_{1j}(1) \in \widetilde{U}_{2j}$ and define

$\widetilde{\omega}_{2j} = q_{ij} \circ (p_j | q_{ij}^{-1}(\widetilde{U}_{2i}))^{-1} \circ \omega_j | [t_1, t_2]$. We continue until $\widetilde{\omega}_{m-1\,j}$ and then we define $\lambda_{ij}(e_j, \omega_j) = \widetilde{\omega}_{1j} * \widetilde{\omega}_{2j} * \cdots * \widetilde{\omega}_{m-1\,j}$. Easily follows that $\lambda_{ij} : D_j \longrightarrow E_i^I$ is a continuous map and then it is immediate that these maps define a lifting morphism for $\underline{p}$.

By Theorem 4 and by Theorem 3 from [5], we obtain:

Corollary 3. *If* $p : E \longrightarrow B$ *is a shape covering map, and* $x, y \in B$ *can be joined by a path in* B*, then the fibers* $p^{-1}(x)$ *and* $p^{-1}(y)$ *have the same shape.*

Remark 3. By Corollary 1 it follows that in the conditions of Corollary 3, the fibers $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

Corollary 4. *If* $p : E \longrightarrow B$ *is a shape covering map as limit of a level map* $\underline{p} : \underline{E} \longrightarrow \underline{B}$ *between compact* ANR*-sequences then* p *is a shape fibration.*

Using now Theorem 3 from [6], Corollary 1 and Corollary 5 from [7, p.117] we deduce:

Corollary 5. *If* $p : E \longrightarrow B$ *is a shape covering map as limit of a level map* $\underline{p} : \underline{E} \longrightarrow \underline{B}$ *between compact* ANR*-sequences and if* $e \in E$*,* $b = p(e)$*, then the following sequence of* pro*-groups*

$$0 \longrightarrow \text{pro} - \pi_n(E, e) \xrightarrow{p_*} \text{pro} - \pi_n(B, b) \longrightarrow 0 \tag{12}$$

is exact for $n \geq 2$ *and* p_* *is a monomorphism of* pro*-groups for* $n \geq 1$.

Corollary 6. *If in Corollary* 5 E *and* B *are compact* ANR*'s, then*

$$p_* : \pi_n(E, e) \longrightarrow \pi_n(B, b) \tag{13}$$

is an isomorphism for $n \geq 2$ *and a monomorphism for* $n = 1$.

Corollary 7. *Let* $p : E \longrightarrow B$ *be a shape covering map, with* E *a path connected space. If* $e_1, e_2 \in E$*, then there exists a path* ω *in* E *from* $p(e_2)$ *to* $p(e_1)$ *for which*

$$p_*\pi_1(E, e_1) = h_{[\omega]} p_*\pi_1(E, e_2). \tag{14}$$

Conversely, for any path ω *in* B *from* $p(e_1)$ *to* $b_2 \in B$ *there exists a point* $e_2 \in p^{-1}(b_2)$ *for which the above relation holds.*

Proof. Let $\widetilde{\omega}$ be a path in E from e_1 to e_2. Then $\pi_1(E, e_1) = h_{[\widetilde{\omega}]}\pi_1(E, e_2)$ and therefore $p_*\pi_1(E, e_1) = h_{[\widetilde{\omega}]}p_*\pi_1(E, e_2)$.

Conversely, for ω there exists $\widetilde{\omega} : I \longrightarrow E$ (Corollary 2) with $\widetilde{\omega}(0) = e_1$ and $p\widetilde{\omega} = \omega$. If $e_2 = \widetilde{\omega}(1)$, then the required relation holds.

The proof of the following theorem is a logical adaptation of the proof of the corresponding result for covering maps [8, Theorem 5, §2, Ch.4].

Theorem 5. *Let $p : E \longrightarrow B$ be a shape covering map, $e \in E$ and $b = p(e)$. Let X be a connected and locally path connected space. If $f : (X, x) \longrightarrow (B, b)$ admits a lift $\widetilde{f} : (X, x) \longrightarrow (E, e)$ then*

$$f_*\pi_1(X, x) \subset p_*\pi_1(E, e). \tag{15}$$

Conversely, if p is induced by the level map $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ between compact sequences such that

$$f_{i*}\pi_1(X, x) \subset p_{i*}\pi_1(E_i, q_i(e)) \tag{16}$$

where $f_i = r_i f$, then f admits a lift with respect to $p : (E, e) \longrightarrow (B, b)$.

4 A construction of shape covering maps

Consider a pointed metric compact space (B, b) and suppose that this is the limit of a pointed compact ANR-sequence $\underline{B} = ((B_i, b_i), r_{ij})$. Let $\mathcal{U}_i$ be an open cover of B_i. Then we can consider the group $\pi_1(\mathcal{U}_i, b_i)$. We recall that $\pi_1(\mathcal{U}_i, b_i)$ is the subgroup of the group $\pi_1(B_i, b_i)$ generated by the classes of the paths $(\omega_i * \omega_i') * \omega_i^{-1}$, where ω_i' is a loop situated in a term of $\mathcal{U}_i$ and ω_i is a path from b_i to $\omega_i'(0)$.

If we suppose that $r_{ij}(\mathcal{U}_j) \subset \mathcal{U}_i$ for $j \geq i$, then the relation $[(\omega_j * \omega_j') * \omega_j^{-1}] \in \pi_1(\mathcal{U}_j, b_j)$ implies $[(r_{ij}\omega_j * r_{ij}\omega_j') * (r_{ij}\omega_j)^{-1}] \in \pi_1(\mathcal{U}_i, b_i)$ and in this way we obtain an inverse sequence of groups $(\pi_1(\mathcal{U}_i, b_i), (r_{ij})_*)$ and we can consider the limit $\underleftarrow{\lim}\, \pi_1(\mathcal{U}_i, b_i)$. The inclusion morphisms $\pi_1(\mathcal{U}_i, b_i) \hookrightarrow \pi_1(B_i, b_i)$ induce the inclusion $\underleftarrow{\lim}\, \pi_1(\mathcal{U}_i, b_i) \hookrightarrow \underleftarrow{\lim}\, \pi_1(B_i, b_i) = \check{\pi}_1(B, b)$, because $\underleftarrow{\lim}$ is an exact functor.

Let $p : E \longrightarrow B$ be a shape covering map induced by a level map $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ of compact ANR-sequences. Let $\mathcal{U}_i$ an open cover of B_i satisfying the conditions of Definition 5.

Then if $([\omega_i * \omega'_i) * \omega_i^{-1}])_i \in \underleftarrow{\lim}\, \pi_1(\mathcal{U}_i, r_i p(e))$, for $e \in E$, denote by $\widetilde{\omega}_i, \widetilde{\omega}'_i : I \longrightarrow E_i$ the lifts of $\omega_j, \omega'_j : I \longrightarrow B_j$ as in Definition 1. By Definition 5, if ω'_i is a loop situated in a term of $\mathcal{U}_i$ then $\widetilde{\omega}'_j$ is a loop in E_j and this implies that $(\widetilde{\omega}_j * \widetilde{\omega}_j) * \widetilde{\omega}_j'^{-1}$ is a loop in E_j. It follows that we have

$$\underleftarrow{\lim}\, \pi_1(\mathcal{U}_i, (p(e)) \subset \check{p}_*(\check{\pi}_1(E, e)), \tag{17}$$

where $\check{p}_*$ is induced by the morphism of compact ANR-sequences

$$\underline{p} = (p_i) : ((E_i, q_i(e)), q_{ij}) \longrightarrow ((B_i, p_i q_i(e), r_{ij}).$$

The proof of the following theorem is very technical and long but it is a logical adaptation of the proof of Theorem 12 from [8, Ch.2,§5].

Theorem 6. *Let $p : E \longrightarrow B$ be a map between metric compacta which is induced by a level map $\underline{p} : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ of compact* ANR*-sequences. Suppose that p is a $*$-fibration (shape fibration) with the unique lifting property and that the spaces E, B are path connected. Then p is a shape covering map if and only if there exists a point $e \in E$ and each i admits an open cover $\mathcal{U}_i$ of B_i such that for $j \geq i$, $r_{ij}(\mathcal{U}_j) \subset \mathcal{U}_i$ and the inclusion* (17) *is verified.*

Theorem 7. *Let B be a compact metric space so that $B = \underleftarrow{\lim}(B_i, r_{ij})$, where $\underline{B} = (B_i, r_{ij})$ is a compact* ANR*-sequence with B_i connected and locally path connected space and let b be a point of B.*

Let H be a subgroup of the group $\check{\pi}_1(B, b)$ and suppose that each i admits an open cover $\mathcal{U}_i$ such that $r_{ij}(\mathcal{U}_j) \subset \mathcal{U}_i$ and $\underleftarrow{\lim}\, \pi_1(\mathcal{U}_i, r_i(b)) \subset H$. Then there exists a shape covering map $p : (E, e) \longrightarrow (B, b)$, induced by a level map $\underline{p} = (p_i) : \underline{E} = ((E_i, e_i)), q_{ij}) \longrightarrow ((B_i, r_i(b)), r_{ij})$ of pointed compact ANR*-sequences, such that $\check{p}_* \check{\pi}_1(E, e) = H$.*

Proof. $\check{\pi}_1(B, b) = \underleftarrow{\lim}\, \pi_1(B_i, r_i(b))$, for $r_i : B \longrightarrow B_i$ the inverse canonical projections. Then for each i, $(r_i)_*(H)$ is a subgroup of the group $\pi_1(B_i, r_i(b))$ and, by hypothesis, each i admits $j \geq i$ such that

$$(r_{ij})_* \pi_1(\mathcal{U}_j, r_j(b)) \subset (r_i)_*(H). \tag{18}$$

If ω_i, ω'_i are two paths in B_i with $\omega_i(0) = \omega'_i(0) = r_i(b)$, we put $\omega_i \sim \omega'_i$ if $\omega_i(1) = \omega'_i(1)$ and $[\omega_i * \omega_1'^{-1}] \in (r_i)_*(H)$. This is an equivalence relation.

The equivalence class of a path ω_i is denoted by $\langle\omega_i\rangle$ and we consider the set E_i of all these classes. A topology on E_i can be considered in the following way: if D_i is an arbitrary open set in B_i and if ω_i is a path from $r_i(b)$ at a point of D, then we consider the following subset of E_i,

$$\langle\omega_i, D_i\rangle = \{\langle\omega_i * \omega_i'\rangle \mid \omega_i'(0) = \omega_i(1), \omega'(I) \subset D_i\}.$$

Then $\{\langle\omega_i, Di\rangle \mid D_i$ an open set of B_i, ω_i a path in B_i, $\omega_i(0) = r_i(b)$ and $\omega_i(1) \in D_i\}$ is a base for a topology on the set E_i. With respect to this topology E_i is a compact ANR. This fact follows using Theorem 5.1 from [1, Ch.IV, p.88]. For $j \geq i$, we define $q_{ij} : E_j \longrightarrow E_i$ by $q_{ij}(\langle\omega_j\rangle) = \langle r_{ij}\omega_j\rangle$ and in this way we obtain a compact ANR–sequence $\underline{E} = (E_i, q_{ij})$. Also we can define the maps $p_i : E_i \longrightarrow B_i$ by $p_i(\langle\omega_i\rangle) = \omega_i(1)$ which are continuous. Indeed, if D_i is an open set of B_i, then $p_i^{-1}(D_i) \supset \langle\omega_i, D_i\rangle$, where $\langle\omega_i\rangle \in p_i^{-1}(D_i)$. The map p_i is also open because $p_i(\langle\omega_i, D_i\rangle)$ is a path component of the set D_i which contains $\omega_i(1)$ and since B_i is a locally path connected space, this component is an open set. The sequence $\underline{p} = (p_i) : \underline{E} = (E_i, q_{ij}) \longrightarrow \underline{B} = (B_i, r_{ij})$ is a level map of compact ANR–sequences and we consider $p : E \longrightarrow B$ the map induced by $\underline{p}$. Now it is immediate that $p : E \longrightarrow B$ is a shape covering map. Indeed, if $\mathcal{U}_i' = \{V_i \mid V_i$ a path–component of an $U_i \in \mathcal{U}_i\}$. We have $p_i^{-1}(V_i) = \cup_{\langle\omega_i\rangle \in p_i^{-1}(V_i)} \langle\omega_i, V_i\rangle$ and then $\mathcal{U}_i'$ are open covers of B_i satisfying with respect to $\underline{p} : \underline{E} \longrightarrow \underline{B}$ the conditions of Definition 5. This proves that $p : E \longrightarrow B$ is a shape covering map.

By Definition 5 and by analogy with the calculations made in the proof of Theorem 13 from [8, Ch.2, §5] we deduce the equality $(p_i)_*(q_{ij})_*\pi_1(E_j, q_j(e)) = (r_i)_*(H)$ if j corresponds to i by Definition 5. By this and because $\check{\pi}_1(E, e) = \varprojlim(\pi_1(E_j, q_j(e)), (q_{ij})_*)$, we obtain the required equality $\check{p}_*(\check{\pi}_1(E, e)) = H$.

Example. Let B be a compact metric space which is the limit $B = \varprojlim \underline{B}$ for a compact ANR–sequence $\underline{B} = (B_i, r_{ij})$ and suppose that the projections $r_i : B \longrightarrow B_i$ are open maps (see for example the Hawaiian earring or the Overton–Segal star construction [8, p.184–185]). If for a point $b \in B$ and an open cover $\mathcal{U}$ of B we have $\pi_1(\mathcal{U}, b) \subset H \subset \check{\pi}_1(B, b)$, then the open covers $r_i(\mathcal{U})$ satisfy the conditions from Theorem 7 and the shape covering map $p : E \longrightarrow B$ is induced by a sequence of covering maps $p_i : E_i \longrightarrow B_i$, with $(p_i)_*\pi_1(E_i, q_i(e)) = (r_i)_*(H)$, for a point $e \in p^{-1}(b)$.

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References

[1] Borsuk, K., *Theory of Retracts*, Warszawa 1967.

[2] Coram, D.S. and Duvall, Jr., P.F., *Approximate fibrations*, Rocky Mountain J. Math. 7(1977), 275–288.

[3] Giraldo, A., *Shape fibrations, multivalued maps and shape groups*, Preprint.

[4] Giraldo, A. and Sanjurjo, J.M.R., *Multifunctions. A class of shape fibrations with path lifting property*, Preprint.

[5] Mardešić, S. and Rushing, T.B., *Shape fibrations* I, General Topology and Its Applications 9(1978), 193–215.

[6] Mardešić, S. and Rushing, T.B., *Shape fibrations* II, Rocky Mountain J. Math. 9(1979), 283–298.

[7] Mardešić, S. and Segal, J., *Shape Theory*, North–Holland Publ. Co., Amsterdam, 1982.

[8] Spanier, E.H., *Algebraic Topology*, McGraw–Hill Book Co., New York, 1966.

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