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The density condition in projective tensor products.

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Abstract

In this paper we modify a construction due to J. Taskinen to get a Fréchet space F which satisfies the density condition such that the complete injective tensor product $l_2\tilde{\otimes}_\epsilon F_b'$ does not satisfy the strong dual density condition of Bierstedt and Bonet. In this way a question that remained open in [14] is solved.

1 Introduction and notation

The density condition (DC) was introduced by S. Heinrich in the context of ultrapowers of locally convex spaces, see [12]. K. D. Bierstedt and J. Bonet investigated the (DC) and the strong dual density condition (SDDC) in [1] - [4]. The (DC) and the (SDDC) play an important role in the theory of Köthe echelon spaces [1] - [4], for extension of linear operators [10], and in the theory of unbounded operator *-algebras [13].

Many locally convex spaces are in fact topological tensor products. In [15], [16], and [7], A. Peris developed a method to define topological properties by operator. By this procedure he obtained good stability properties in injective tensor products. The author studied this method in the case of (DC), (SDDC), and (DF)-spaces in [14]. It was an open problem in [14] to find a Fréchet space F that satisfies (DC) and a Banach space X such that the complete injective tensor product $X\tilde{\otimes}_\epsilon F_b'$ does not satisfy the (SDDC). We will solve that problem by a modification of a counterexample to J. Taskinen, in [17] or [8], 35.9. For this we

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use duality properties between injective and projective tensor products.

The notation for locally convex spaces is standard. If E is a locally convex space, then $\mathcal{U}(E)$ stands for a basis of absolutely convex closed 0-neighborhoods and $\mathcal{B}(E)$ stands for the system of all absolutely convex bounded sets in E. By $\Gamma(M)$ we denote the absolutely convex hull of M. If E and F are locally convex spaces, then $E \otimes_\epsilon F$ and $E \otimes_\pi F$ stand for the injective tensor product and projective tensor product, respectively. We denote the completions by $E \tilde{\otimes}_\epsilon F$ and $E \tilde{\otimes}_\pi F$, respectively. $L_b(E, F)$ means the space of all continuous linear mappings from E into F endowed with the topology of uniform convergence on the bounded sets of E. For $K \subset E$, $L \subset F$, and M is a linear subspace of $L_b(E, F)$, then we write $W(K, L) := \{T \in M : T(K) \subset L\}$. If X is a Banach space, then B_X denotes its closed unit ball. We denote by FIN the class of all finite-dimensional Banach spaces.

2 The main result

We start with some definitions.

Definition 1. *(1) Let F denote a metrizable space and $(U_k)_{k=1}^\infty$ a countable basis of closed absolutely convex 0-neighborhoods in F. The space F is said to satisfy the* **density condition** *-briefly (DC)- if the following holds*

$$\forall (\lambda_k)_{k=1}^\infty \subset]0,\infty[\quad \forall n \in \mathbf{N} \quad \exists m \in \mathbf{N} \quad \exists M \in \mathcal{B}(F) :$$
$$\bigcap_{k=1}^{m} \lambda_k U_k \subset U_n + M .$$

(2) Let E denote a locally convex space with an increasing fundamental sequence $(M_k)_{k=1}^\infty$ of bounded sets. E is said to satisfy the **strong dual density condition** *-briefly (SDDC)- if the following holds*

$$\forall (\lambda_k)_{k=1}^\infty \subset]0,\infty[\quad \forall n \in \mathbf{N} \quad \exists m \in \mathbf{N} \quad \exists U \in \mathcal{U}(E) :$$
$$M_n \cap U \subset \Gamma\left(\bigcup_{k=1}^{m} \lambda_k M_k\right)$$

and the space E is said to satisfy the **strong dual density condition by operator** *-briefly (SDDCO)- if the following holds*

$$\forall (\lambda_k)_{k=1}^\infty \subset]0,\infty[\quad \forall n \in \mathbf{N} \quad \exists m \in \mathbf{N} \quad \exists U \in \mathcal{U}(E)$$

$$\exists (Q_k)_{k=1}^m \text{ lin. op. on } E:$$

$$\sum_{k=1}^m Q_k = I_E \quad \text{and} \quad Q_k(M_n \cap U) \subset \lambda_k M_k , \qquad k = 1, \ldots, m .$$

Quasinormable Fréchet spaces and Fréchet-Montel spaces are examples of spaces satisfying (DC), see [12]. By taking polars it follows that the Fréchet space F satisfies (DC) if and only if the strong dual F'_b satisfies (SDDC), see [1]. It is readily seen, that (SDDCO) implies (SDDC). The strong dual space of an (FBa)-space with (DC) or a (DF)-space satisfying the strict Mackey condition are examples for spaces satisfying (SDDCO), see [14].

The following propositions characterize the (SDDCO) for F'_b, where F is a Fréchet space, by properties in projective and injective tensor products.

Proposition 2. *Let F be a Fréchet space and let X be a Banach space. The following assertions are equivalent:*

1. *$X\tilde{\otimes}_\pi F$ ($X \otimes_\pi F$, resp.) satisfies the condition*
   $\forall (\lambda_k)_{k=1}^\infty \subset]0,\infty[\quad \forall n \in \mathbf{N} \quad \exists m \in \mathbf{N} \quad \exists M \in \mathcal{B}(F) \ :$
   $$\bigcap_{k=1}^m \lambda_k \bar{\Gamma}(B_X \otimes U_k) \subset \bar{\Gamma}(B_X \otimes U_n) \ + \ \bar{\Gamma}(B_X \otimes M) \ .$$
2. *$L_b(X, F'_b)$ satisfies the strong dual density condition (SDDC).*

Proof. We are going to prove the result for the complete projective tensor product $X\tilde{\otimes}_\pi F$. It is not hard to see that $X\tilde{\otimes}_\pi F$ satisfies condition *1* if and only if $X \otimes_\pi F$ does.

The proposition follows by polarity in the pairing $\langle X\tilde{\otimes}_\pi F, L(X, F'_b)\rangle$, using the fact $(\bar{\Gamma}(B_X \otimes U_k))^\circ = W(B_X, U_k^\circ)$. Remark that $W(B_X, U_k^\circ)$ is w*-compact and hence $\Gamma(\bigcup_{k=1}^m \lambda_k W(B_X, U_k^\circ))$ is closed, see [1], the proof of Proposition 1.2. ◇

Proposition 3. *Let F be a Fréchet space. The following assertions are equivalent:*

1. *$X\tilde{\otimes}_\pi F$ ($X \otimes_\pi F$, resp.) satisfies the condition*
$\forall(\lambda_k)_{k=1}^{\infty} \subset]0,\infty[\quad \forall n \in \mathbf{N} \quad \exists m \in \mathbf{N} \quad \exists M \in \mathcal{B}(F)$:

$$\bigcap_{k=1}^{m} \lambda_k \bar{\Gamma}(B_X \otimes U_k) \subset \bar{\Gamma}(B_X \otimes U_n) \; + \; \bar{\Gamma}(B_X \otimes M)$$

for each Banach space X.

2. *F_b' satisfies the strong dual density condition by operator (SD-DCO).*

3. *$X \otimes_\epsilon F_b'$ ($L_b(X, F_b')$, resp.) satisfies the strong dual density condition (SDDC) for each Banach space X.*

Proof. By Theorem 1.6 in [14] the strong dual F_b' satisfies (SDDCO) if and only if $X\tilde{\otimes}_\epsilon F_b'$ ($X \otimes_\epsilon F_b'$ and $L_b(X, F')$, resp.) has the strong dual density condition (SDDC) for each Banach space X. The rest follows by Proposition 2. $\diamond$

Example 4. Let λ be an (FM)-sequence space in the class Ω, for the definition of this class see [11]. By the Remark (b) after Proposition 3.2 in [11] the space $X\tilde{\otimes}_\pi\lambda$ satisfies (DC) for each Banach space X. Let $L_{ind}(X, \lambda_b')$ be the bornological space associated to $L_b(X, \lambda_b')$. Then, we omit the proof, $L_{ind}(X, \lambda_b')$ has (SDDC). By Proposition 3.2 in [11] the topologies on $L_{ind}(X, \lambda_b')$ and $L_b(X, \lambda_b')$ induce the same topology on a bounded subset. Thus $L_b(X, \lambda_b')$ has (SDDC) for each Banach space X and λ_b' satisfies (SDDCO).

We give another argument that λ_b' satisfies (SDDCO). By [5], 3. Prop. (a) λ is a decomposable T-space. It follows by [5], 5. Theorem that λ is even an (FBa)-space. We get $L_b(X, \lambda_b') \cong (X\tilde{\otimes}_\pi\lambda)_b'$. Since $X\tilde{\otimes}_\pi\lambda$ has (DC), see [4], 1.7. Corollary, it follows the (SDDC) for $L_b(X, \lambda_b')$.

For more examples of Fréchet spaces such that the strong dual spaces satisfy (SDDCO), see [14].

Now we present a slight modification of a counterexample for a (FBa)-space due to J. Taskinen, see [17] or [8], 35.9. We will get the first example for a (DF)-space E such that E satisfies (SDDC) and E does not satisfy (SDDCO).

We fix a Banach space X that is not an $\mathcal{L}_1$-space (for example $X := l_2$). Then there are $G_{m,l} \in FIN$ with subspaces $M_{m,l} \subset G_{m,l}$ for $m, l \in \mathbf{N}$ such that there exist $z_{m,l} \in X \otimes M_{m,l}$ with

$$\begin{aligned} \pi(z_{m,l}; X, G_{m,l}) &\leq (m+l)^{-(m+l)^2} \quad \text{and} \\ \pi(z_{m,l}; X, M_{m,l}) &= (m+l)^{-(m+l)} , \end{aligned}$$

see [8], 23.5. Let $P_{m,l} : G_{m,l} \longrightarrow G_{m,l}$ be a continuous projection onto $M_{m,l}$ and we define

$$\|y\|_{m,l,k} := \|P_{m,l}(y)\|_{m,l} + k\|y - P_{m,l}(y)\|_{m,l}$$

for $m, l, k \in \mathbf{N}$ and $y \in G_{m,l}$. It follows that $\|\cdot\|_{m,l} \leq \|\cdot\|_{m,l,k}$ and $\|y\|_{m,l} = \|y\|_{m,l,k}$ for $y \in M_{m,l}$. Now, we define the spaces:

$$\begin{aligned} G^{\circ}_{m,l,k} &:= (G_{m,l}, \|\cdot\|_{m,l,k}) \quad ; \qquad G_{m,l,k} := (G_{m,l}, \|\cdot\|_{m,l}) \\ H^{\circ} &:= l_2((G^{\circ}_{m,l,k})^{\infty}_{m,l,k=1}) \quad ; \qquad H := l_2((G_{m,l,k})^{\infty}_{m,l,k=1}) . \end{aligned}$$

Then we get $H^{\circ} \subset H$ and $\|\cdot\|_H \leq \|\cdot\|_{H^{\circ}}$. If $J_{m,l,k} : G_{m,l,k} \hookrightarrow H$ is the canonical injection, then $z_{m,l,k} := (I_X \otimes J_{m,l,k})(z_{m,l})$ satisfies

$$\begin{aligned} \pi(z_{m,l,k}; X, H) &\leq (m+l)^{-(m+l)^2} \quad \text{and} \\ \pi(z_{m,l,k}; X, H^{\circ}) &\leq (m+l)^{-(m+l)} . \end{aligned} \tag{1}$$

We assume that

$$\begin{aligned} &\exists m, l, t \in \mathbf{N} \quad \forall k \in \mathbf{N} : \\ &z_{m,l,k} \in \Gamma\left(B_X \otimes \left(\frac{1}{2}(m+l)^{-(m+l)} B_H \cap t B_{H^{\circ}}\right)\right) . \end{aligned}$$

Then by projection on the (m, l, k)-th component of H there is a representation of $z_{m,l}$ by

$$z_{m,l} = \sum_{i=1}^{N} \lambda_i x_i \otimes y_i = \sum_{i=1}^{N} \lambda_i x_i \otimes P_{m,l}(y_i) \in X \otimes G_{m,l}$$

with

$$\sum_{i=1}^{N} |\lambda_i| \leq 1 , \quad \|x_i\|_X \leq 1 , \quad \|y_i\|_{m,l} \leq \frac{1}{2}(m+l)^{-(m+l)} , \quad \|y_i\|_{m,l,k} \leq t .$$

Since $P_{m,l}(y_i) \in M_{m,l}$, it follows that

$$\begin{aligned}(m+l)^{-(m+l)} &= \pi(z_{m,l}; X, M_{m,l}) \le \sum_{i=1}^{N} |\lambda_i| \|x_i\|_X \|P_{m,l}(y_i)\|_{m,l} \\ &\le \sum_{i=1}^{N} |\lambda_i| (\|y_i\|_{m,l} + \|y_i - P_{m,l}(y_i)\|_{m,l}) \\ &\le \frac{1}{2}(m+l)^{-(m+l)} + \frac{t}{k} .\end{aligned}$$

This is a contradiction for large k and it follows that

$$\begin{aligned}&\forall m, l, t \in \mathbf{N} \quad \exists k \in \mathbf{N} : \\ &z_{m,l,k} \not\in \Gamma\left(B_X \otimes \left(\frac{1}{2}(m+l)^{-(m+l)} B_H \cap t B_{H^\circ}\right)\right) .\end{aligned} \tag{2}$$

Now set for brevity

$$f_{n,m,l}(x) := (n+1)^{(m+l)} \|x\|_{H^\circ} \vee (n+1)^{(m+l)^2} \|x\|_H \quad \text{for} \quad x \in H^\circ$$

and

$$g_{n,m,l}(x) := (n+1)^{(m+l)^2} \|x\|_H \quad \text{for} \quad x \in H .$$

Then the Fréchet space F_0 is defined as double sequence space

$$\Big\{ (x_{ij})_{i,j=1}^{\infty} \subset H^\circ : \Big((f_{n,m,l}(x_{ij}))_{i=1,\ldots,\infty; j<n} , (g_{n,m,l}(x_{ij}))_{i=1,\ldots,\infty; j \ge n} \Big) \in l_\infty(\mathbf{N} \times \mathbf{N}) \quad \forall n \in \mathbf{N} \Big\} ,$$

and a decreasing basis of 0-neighborhoods is given by $(U_n)_{n=1}^{\infty}$, where

$$\begin{aligned}U_n : &= \big((C_{n,i,1} \cap D_{n,i,1})_i , \ldots , (C_{n,i,n-1} \cap D_{n,i,n-1})_i , \\ &\quad (D_{n,i,n})_i , \ldots , (D_{n,i,j})_i , \ldots \big) , \\ &\qquad\quad \uparrow n \qquad\qquad\quad \uparrow j\end{aligned} \tag{3}$$

$$C_{n,i,j} := \frac{B_{H^\circ}}{(n+1)^{(i+j)}} , \quad D_{n,i,j} := \frac{B_H}{(n+1)^{(i+j)^2}} \tag{4}$$

and $(\ldots)_i$ stands for a column vector.

Lemma 5. *The Fréchet space F_0 satisfies the density condition (DC).*

Proof. Given any positive sequence $(\lambda_k)_{k=1}^{\infty}$ and $n \in \mathbb{N}$. Choose $m > n$ with $\frac{\lambda_{n+1}}{(n+2)^m} \leq \frac{1}{(n+1)^m}$. This implies $\frac{\lambda_{n+1}}{(n+2)^{(i+j)}} \leq \frac{1}{(n+1)^{(i+j)}}$ and $\frac{\lambda_{n+1}}{(n+2)^{(i+j)^2}} \leq \frac{1}{(n+1)^{(i+j)^2}}$ for all $i + j \geq m$ and the following relations

$$\lambda_{n+1} C_{n+1,i,j} = \frac{\lambda_{n+1}}{(n+2)^{(i+j)}} B_{H^\circ} \subset \frac{1}{(n+1)^{(i+j)}} B_{H^\circ} = C_{n,i,j} \quad \text{and}$$
$$\lambda_{n+1} D_{n+1,i,j} \subset D_{n,i,j}$$

for $i + j \geq m$. By a short calculation we get

$$\lambda_{n+1} U_{n+1} \cap \lambda_m U_m \subset$$

$$\begin{pmatrix} \lambda_m C_{m,1,1}, & \lambda_m C_{m,1,2}, & \ldots, & \lambda_m C_{m,1,m-1}, & 0, & \ldots \\ \lambda_m C_{m,2,1}, & \lambda_m C_{m,2,2}, & \ldots, & \lambda_m C_{m,2,m-1}, & 0, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ \lambda_m C_{m,m-1,1}, & \lambda_m C_{m,m-1,2}, & \ldots, & \lambda_m C_{m,m-1,m-1}, & 0, & \ldots \\ 0, & 0, & \ldots, & 0, & 0, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \end{pmatrix}$$

$$+ \begin{pmatrix} 0, & \ldots, & 0, & D_{n,1,m}, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ 0, & \ldots, & 0, & D_{n,m-1,m}, & \ldots \\ C_{n,m,1} \cap D_{n,m,1}, & \ldots, & D_{n,m,m-1}, & D_{n,m,m}, & \ldots \\ C_{n,m+1,1} \cap D_{n,m+1,1}, & \ldots, & D_{n,m+1,m-1}, & D_{n,m+1,m}, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots \end{pmatrix}$$

$$\subset M + U_n ,$$

where M is a bounded subset of F_0. Hence

$$\bigcap_{k=1}^{m} \lambda_k U_k \subset \lambda_{n+1} U_{n+1} \cap \lambda_m U_m \subset M + U_n$$

and the lemma is proved. $\diamond$

Theorem 6. *Let X be a Banach space that is not an $\mathcal{L}_1$-space and let F_0 be constructed as above. Then $(F_0)'_b$ satisfies the strong dual density condition (SDDC) and $L_b(X,(F_0)'_b)$ does not satisfy the (SDDC). The space $(F_0)'_b$ does not satisfy the strong dual density condition by operator (SDDCO).*

Proof. Denote by $J_{i,j}$ the canonical injection of the (i,j)-th component into F_0 and by $Q_{i,j}$ the corresponding projection. Further let $q_n(\cdot)$ denote the n-th seminorm in $X \otimes_\pi F_0$. By (2), (4) and (4) we get

$$q_n((I_X \otimes J_{l,m})(z_{m,l,k})) \;\leq\; \frac{(n+1)^{(m+l)^2}}{(m+l)^{(m+l)^2}} \vee \frac{(n+1)^{(m+l)}}{(m+l)^{(m+l)}} \;<\; \varrho_n$$

for $\quad m < n \quad$ and

$$q_n((I_X \otimes J_{l,m})(z_{m,l,k})) \;\leq\; \frac{(n+1)^{(m+l)^2}}{(m+l)^{(m+l)^2}} \leq 1 \quad \text{for} \quad m \geq n\,.$$

It follows that the set $A := \{(I_X \otimes J_{l,m})(z_{m,l,k}) \;:\; m,l,k \in \mathbf{N}\}$ is bounded. There exists positive $(\lambda_j)_{j=1}^\infty$ such that $A \subset \bigcap_{j=1}^\infty \lambda_j \Gamma(B_X \otimes U_j)$. We assume $X \otimes_\pi F_0$ satisfies the condition *1* in Proposition 2. Following for each $n \in \mathbf{N}$ exist $p_n \in \mathbf{N}$ and $M_n \subset F_0$ bounded such that

$$\begin{aligned} A \;\subset\; \bigcap_{j=1}^{p_n} \lambda_j \Gamma(B_X \otimes U_j) \;&\subset\; \Gamma(B_X \otimes U_{n+1}) \;+\; \bar{\Gamma}(B_X \otimes M_n) \\ &\subset\; 2\Gamma(B_X \otimes U_{n+1}) \;+\; \Gamma(B_X \otimes M_n)\,. \end{aligned}$$

There are $\rho_i^n \geq 1$ with $M_n \subset \bigcap_{i=1}^\infty \rho_i^n U_i$ and we get

$$A \;\subset\; 3\Gamma(B_X \otimes (\rho_1^n U_1 \cap \rho_{n+1}^n U_{n+1}))\,.$$

Thus

$$\begin{aligned} z_{n,l,k} &= (I_X \otimes Q_{l,n})(I_X \otimes J_{l,n})(z_{n,l,k}) \\ &\in (I_X \otimes Q_{l,n})\Gamma(B_X \otimes (3\rho_1^n U_1 \cap 3\rho_{n+1}^n U_{n+1})) \\ &\subset \Gamma\left(B_X \otimes \left(\frac{3\rho_1^n}{2^{(n+l)^2}} B_H \cap \frac{3\rho_{n+1}^n}{(n+2)^{(n+l)}} B_{H^\circ}\right)\right)\,. \end{aligned}$$

Now, for arbitrary $n \in \mathbf{N}$ choose $l \in \mathbf{N}$ with $\frac{3\rho_l^n}{2^{(n+l)^2}} \leq \frac{1}{2(n+l)^{(n+l)}}$ and set $t := \frac{3\rho_{n+1}^n}{(n+2)^{(n+l)}}$. Thus

$$z_{n,l,k} \in \Gamma\left(B_X \otimes \left(\frac{1}{2(n+l)^{(n+l)}} B_H \cap t B_{H^\circ}\right)\right)$$

for all $k \in \mathbf{N}$ that is a contradiction to equation (3). It follows that $X \otimes_\pi F_0$ does not satisfy condition *1* in Proposition 2 and $L_b(X,(F_0)'_b)$ does not satisfy (SDDC). Clearly, the space $(F_0)'_b$ has (SDDC) by Lemma 5 and does not satisfy (SDDCO) by Proposition 3. ◇

Corollary 7. *If F_0 is the Fréchet space constructed above, the space $L_b(l_2,(F_0)'_b)$ does not satisfy the strong dual density condition (SDDC).*

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