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Boundary singularities of solutions of sublinear elliptic equations.

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Abstract

Let Ω be a domain of $\mathbb{R}^N$, $N \geq 3$, such that $0 \in \partial\Omega$. In this paper we study the behavior near 0 of any nonnegative solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ of equation of the type $-\Delta u + a(x)u^q = 0$ where $0 < q < 1$ and function a behaves like a power of $|x|$.

1 Introduction

In this article we study the boundary behavior of the nonnegative solutions of sublinear elliptic equations of the type

$$-\Delta u + a(x)u^q = 0 \tag{1}$$

in a domain Ω of $\mathbb{R}^N$, $N \geq 3$, $q \in (0,1)$, with a possible isolated singularity at one point of the boundary. More precisely we assume that $0 \in \partial\Omega$ is the singular point and $a \in C^1(\Omega)$ with :

$$a(x) = |x|^\sigma(1 + o(1)) \tag{2}$$

$$|\nabla a(x)| = O(|x|^{\sigma-1}) \tag{3}$$

near 0, where σ is a given real.

Our first question is the following : let $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ be a nonnegative solution of (1) in Ω such that

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$$u = \phi \quad on \quad \partial\Omega \setminus \{0\} \tag{4}$$

where ϕ is a given continuous function on $\partial\Omega$; can we extend u as a continuous function defined in whole $\overline{\Omega}$? If not, the second point is to describe the precise behavior of u near 0.

This boundary singularity problem for sublinear elliptic equation is a new type of problem. In the superlinear case, the problem has been studied by analystic methods by Gmira and Véron [9] and Sheu [12] in the regular case, Fabbri and Véron [8] in the non regular case and by probabilistic methods by Le Gall [10] and Dynkin and Kuznetsov [7]. Recall that the singularity is removable only in the case $q > N+1/(N-1)$, when $\sigma = 0$.

When the singular point lies in Ω, equation (1) has been studied in the superlinear case $q > 1$ in [6], [11] and [13] and in the sublinear case $q < 1$ in [1] and [2].

In the present work, we consider the case where Ω is a ball, for example

$$\Omega = B(x_0, \frac{1}{2}) \quad with \quad x_0 = \frac{e_N}{2},$$

where $(e_1, ..., e_N)$ is the canonical basis of $\mathbb{R}^N$. Our results depend on the relative positions of q, N and σ. The principal point is to obtain a priori estimates near 0 for the solutions of (1). In that aim, we first use two change of variables which lead us to a problem in the half space

$$\mathbb{R}^{N+} = \{x \in \mathbb{R}^N / x_N > 0\}.$$

More precisely we introduce the following Kelvin transform:

$$u(x) = |y + e_N|^{N-2} v(y) \quad with \quad y + e_N = \frac{x}{|x|^2} \tag{5}$$

where $e_N = (0, ..., 0, 1) \in \mathbb{R}^N$. A straightforward computation implies that v satisfies

$$-\Delta v(y) + |y + e_N|^{(N-2)q-(N+2)} a\left(\frac{y + e_N}{|y + e_N|^2}\right) v^q(y) = 0 \tag{6}$$

for all $y \in \mathbb{R}^{N+}$. Remark that the singular point 0 is reduced to infinity by this transform. Now we use the classical Kelvin transform :

$$v(y) = |z|^{N-2} w(z) \quad with \quad z = \frac{y}{|y|^2}. \tag{7}$$

If v satisfies (6) in $\mathbb{R}^{N+}$, then w is solution of

$$-\Delta w + b(z)w^q = 0 \tag{8}$$

in $\mathbb{R}^{N+}$ with the singularity at 0 and where $b(z) = |z|^{(N-2)q-(N+2)}|y + e_N|^{(N-2)q\ -(N+2)}a\left(\frac{y+e_N}{|y+e_N|^2}\right)$. Because of (2), we have

$$b(z) = |z|^{\sigma}(1 + o(1)) \quad near \quad 0. \tag{9}$$

Once we are reduced to an equation in $\mathbb{R}^{N+}$, we make a new change of variables which leads us to an equation in the infinite cylinder $C = \mathbb{R} \times S_+^{N-1}$ where S_+^{N-1} is the hemisphere of S^{N-1} contained in $\mathbb{R}^{N+}$: defining

$$V(t,\theta) = |z|^{N-1}w(z) = r^{N-1}w(r,\theta) \tag{10}$$

where (r,θ) are the spherical coordinate of z and $t = -lnr$. Because of (8), V satisfies :

$$\begin{cases} V_{tt} + NV_t + (N-1)V + \Delta_{S^{N-1}}V = g(t,\theta)V^q & in \quad C \\ V(t,.) = \Psi(t,.) \quad on \quad \partial S_+^{N-1} = S^{N-2} \end{cases} \tag{11}$$

where g is some nonnegative function in C and $\Psi(t,.)$ is some nonnegative function on S^{N-2} with $\max_{S^{N-2}} \Psi(t,.) = O(e^{-(N-1)t})$ when t tends to $+\infty$.

In the first time, we give a priori estimate result. For this we introduce the first eigenfunction Φ_1 of the Laplacian in $W_0^{1,2}(S_+^{N-1})$ where S_+^{N-1} is the hemisphere of S^{N-1} contained in $\mathbb{R}^{N+}$. The function Φ_1 is normalized by $\|\Phi_1\|_\infty = 1$ and satisfies

$$\begin{cases} -\Delta_{S^{N-1}}\Phi_1 = (N-1)\Phi_1 & in \quad S_+^{N-1} \\ \Phi_1 = 0 \quad on \quad S^{N-2}. \end{cases} \tag{12}$$

Now the main point is to prove that any estimate of the mean value

$$\overline{V}(t) = \int_{S_+^{N-1}} V(t,\theta)\Phi_1(\theta)d\theta \tag{13}$$

implies an analogous estimates on V. Then we are reduced to give estimates on $\overline{V}$, wich reduces the problem to the resolution of ordinary differential inequalities. In that way we get a priori estimates for all nonnegative solution of (1) satisfying (4) for all continuous function ϕ. Our main result concerning the priori estimates is the following.

Theorem 1. *Assume ϕ is a continuous function on ∂B. Let $u \in C^2(B) \cap C^0(\overline{B} \setminus \{0\})$ be any nonnegative solution of (1) satisfying (4). Then we have :*

(i) If $q < \min\left(1, \frac{N+1+\sigma}{N-1}\right)$, then

$$u(x) = O(|x|^{1-N}) \quad near \quad 0. \tag{14}$$

(ii) If $q > \frac{N+1+\sigma}{N-1}$, then

$$u(x) = O(|x|^{\frac{2+\sigma}{1-q}}) \quad near \quad 0. \tag{15}$$

(iii) If $q = \frac{N+1+\sigma}{N-1}$, then

$$u(x) = O(|x|^{1-N}|ln|x||^{\frac{1}{1-q}}) \quad near \quad 0. \tag{16}$$

Our results show that two effects one fighting each other, the nonlinear and the linear one, as it was the case in the interior problem [1], [2]. The nonlinear effect is governed by the possible existence of particular solutions of (8) when $b(z) = |z|^\sigma$, given by :

$$w^*(z) = C(N, q, \sigma)|z|^\gamma \quad where \quad \gamma = \frac{2+\sigma}{1-q}. \tag{17}$$

The linear effect is governed by the solution of Poisson equation :

$$\begin{cases} -\Delta P = 0 & \text{in} \quad I\!R^{N+} \\ P(0) = \delta_0 \end{cases} \tag{18}$$

where δ_0 is the Dirac mass at the origin. Recall that P is given by $P(z)=P(r,\theta) = C_N r^{1-N}\Phi_1(\theta)$.

In a second part we prove more precise convergence results by using some techniques adapted to equations in an infinite cylinder, still used in [11], [4], [3], [1].

Our main result is then the following :

Theorem 2. *Assume ϕ is a nonnegative continuous function on ∂B, identically equal to 0 in a neighborhood of 0 in ∂B. Let $u \in C^2(B) \cap C^0(\overline{B} \setminus \{0\})$ be any nonnegative solution of (1) satisfying (4).*

(i) Assume $q < 1 < \frac{N+1+\sigma}{N-1}$ (hence $2+\sigma > 0$). Then, using Kelvin transforms (5) and (7), there exist $l \geq 0$ such that :

$$\lim_{|r| \to 0} |r|^{N-1} w(r, \theta) = l \Phi_1(\theta) \quad uniformly \quad on \quad S_+^{N-1}. \tag{19}$$

with $(r, \theta) \in \mathbb{R}_+^* \times S_+^{N-1}$ is the spherical coordinates of z in $\mathbb{R}^{N+}$.

If $l = 0$, then u can be extended to a continuous function in $\overline{B}$. In that case • if $\sigma + 1 + q \leq 0$, then

$$u(x) = O(|x|^\gamma) \quad near \quad 0 \tag{20}$$

with $\gamma = \frac{2+\sigma}{1-q}$. Using Kelvin transforms (5) and (7), the limit set in $C^2(S_+^{N-1})$ of $r^{-\gamma} w(r, .)$ as r goes to 0 is contained in the set of nonnegative solutions of

$$\begin{cases} \Delta_{S^{N-1}} \omega + \gamma(\gamma - 2 + N)\omega - \omega^q = 0 & in \quad S_+^{N-1} \\ \omega = 0 \quad on \quad S^{N-2}. \end{cases} \tag{21}$$

- *If $\sigma + 1 + q > 0$, then there exists $k \geq 0$ such that*

$$\lim_{|r| \to 0} |r|^{-1} w(r, \theta) = k \Phi_1(\theta) \quad uniformly \quad on \quad S_+^{N-1}. \tag{22}$$

Moreover, if $k = 0$, then (20) holds and we have the same property as above.

(ii) Assume $q < \frac{N+\sigma+1}{N-1} \leq 1$ (hence $2+\sigma \leq 0$). Then (19) holds and if $l = 0$, then $u \equiv 0$ near the origin.

(iii) Assume $\frac{N+\sigma+1}{N-1} < q < 1$. Then as in (i) we have (20) and the inclusion property. Moreover, if $a(x) = |x|^\sigma$ and $\lim_{n \to +\infty} r_n^{-\gamma} w(r_n, .) = 0$ for some sequence $r_n \to 0$, then u is identically equal to 0 near the origin.

Our paper is organized as follows :

1. Introduction

2. Preliminary results
3. A priori estimates
4. Convergence results.

2 Preliminary results

Let Cl the infinity cylinder defined by $Cl = [1, +\infty) \times S_+^{N-1}$. For all function V defined on Cl, we denote $\overline{V}$ the average of V defined on $[1, +\infty)$ as in (13).

We start this section with some result which allows us to claim that a nonnegative solution V of some elliptic equation in Cl is bounded as soon as its average $\overline{V}$ in $[1, +\infty[$ is.

Proposition 1. *Let $(a_1, a_2, b_1, b_2, c_1) \in I\!R \times I\!R^* \times I\!R^3$. Assume that g is a nonnegative bounded function on Cl. Let $V \in C^2(Cl) \cap C(\overline{Cl})$ be any nonnegative solution of*

$$V_{tt} + \left(\frac{a_1}{t} + a_2\right) V_t + \frac{1}{t}\left(\frac{b_1}{t} + b_2\right) V + c_1 V + \Delta_{S^{N-1}} V = g(t, \theta) V^q \quad in \quad Cl \tag{23}$$

satisfying

$$V = \Psi \quad on \quad [1, +\infty) \times S^{N-2} \tag{24}$$

with $\Psi \in C([1, +\infty) \times S^{N-2})$ be a nonnegative function and $\max_{S^{N-2}} \Psi(t, .) = O(e^{-\beta t})$ for some $\beta > 0$.

If $\overline{V}$ is bounded on $[1, +\infty[$, then V belongs to $L^\infty(Cl)$.

This proposition ensues from the two following lemmas. They are an adaptation of some result of [5] for a problem with the other sign in the cylinder, of the type

$$\begin{cases} W_{tt} + a_0 W_t - lW + \Delta_{S^{N-1}} W + W^Q = 0 \quad in \quad Cl \\ W = 0 \quad on \quad [1, +\infty) \times S^{N-2} \end{cases}$$

where a_0, l are constants, with $l > 0$, in the superlinear case $Q > 1$.

Lemma 1. *Under the assumptions of proposition 1, for all $\gamma \in]1, \frac{1}{1-q}[$, there exists $K = K(\gamma, N, q) > 0$ such that for all $t \geq 2$:*

$$\int_t^{t+1} \int_{S_+^{N-1}} 1_{V \neq 0} \frac{|DV|^2}{V^\beta} \Phi_1 d\theta ds \leq K \tag{25}$$

where $\beta = 2 - \frac{1}{\gamma}$, $|DV|^2 = (V_t)^2 + |\nabla_{S^{N-1}} V|^2$ *and* $1_{V\neq 0}$ *denotes the caracteristic function of the set* $\{(t,\theta) \in Cl / V(t,\theta) \neq 0\}$.

Proof. Since V can vanish, we consider the function $U = V + \varepsilon$ for $\varepsilon \in (0,1)$. Because of (23), U satisfies

$$\begin{aligned} U_{tt} \;\; + \;\; & \left(\frac{a_1}{t} + a_2\right) U_t + c_1 U + \Delta_{S^{N-1}} U + \frac{1}{t}\left(\frac{b_1}{t} + b_2\right) U \qquad (26) \\ & \leq g(t,\theta) U^q + c_1 \varepsilon + \frac{\varepsilon}{t}\left(\frac{b_1}{t} + b_2\right) \end{aligned}$$

in Cl. Now set $U = W^\gamma$, then $W \geq \varepsilon^{\frac{1}{\gamma}}$ in Cl and from (26), W satisfies in Cl

$$\begin{aligned} W_{tt} \;\; + \;\; & \left(\frac{a_1}{t} + a_2\right) W_t + \Delta_{S^{N-1}} W + \frac{c_1}{\gamma} W \\ + \;\; & \frac{1}{t\gamma}\left(\frac{b_1}{t} + b_2\right) W + \frac{\gamma - 1}{W}(W_t^2 + |\nabla_{S^{N-1}} W|^2) \\ & \leq \frac{C_1}{\gamma} W^{\gamma(q-1)+1} + \frac{c_1}{\gamma}\varepsilon^{\frac{1}{\gamma}} + \frac{\varepsilon^{\frac{1}{\gamma}}}{\gamma t}\left(\frac{b_1}{t} + b_2\right) \qquad (27) \end{aligned}$$

where C_1 is a positive constant independant on t and θ. Multiplying (27) by Φ_1 and integrating on S_+^{N-1}, the function $\overline{W}$ introduced in (13) satisfies

$$\begin{aligned} \overline{W}_{tt} \;\; + \;\; & \left(\frac{a_1}{t} + a_2\right) \overline{W}_t + \left(\frac{c_1}{\gamma} - (N-1)\right) \overline{W} \\ + \;\; & \frac{1}{\gamma t}\left(\frac{b_1}{t} + b_2\right) \overline{W} + \int_{S_+^{N-1}} A(t,\theta) d\theta \\ - \int_{S^{N-2}} \Psi \frac{\partial \Phi_1}{\partial \nu} d\theta & \leq \frac{C_1}{\gamma} \int_{S_+^{N-1}} (W\Phi_1)^j d\theta + C_2 \varepsilon^{\frac{1}{\gamma}} \qquad (28) \end{aligned}$$

in $[1, +\infty)$, where $A(t,\theta) = \frac{\gamma-1}{W}(W_t^2 + |\nabla_{S^{N-1}} W|^2)\Phi_1(\theta)$, $j = \gamma(q-1) + 1 \in (0,1)$ and $C_2 = (\int_{S_+^{N-1}} \Phi_1 d\theta)[C_1 \gamma^{-1} + \gamma^{-1} \max(0, \max_{t>1}(b_1 t^{-2} + b_2 t^{-1}))]$. Then from Jensen inequality and observing that $-\partial\Phi_1/\partial\nu \geq 0$ on S^{N-2}, we get :

$$\overline{W}_{tt} + \left(\frac{a_1}{t} + a_2\right) \overline{W}_t + \int_{S_+^{N-1}} A(t,\theta) d\theta \leq \frac{C_1}{\gamma} \overline{W}^j + C_2 \varepsilon^{\frac{1}{\gamma}}$$

$$-\left(\frac{c_1}{\gamma}-(N-1)\right)\overline{W}-\frac{1}{t\gamma}\left(\frac{b_1}{t}+b_2\right)\overline{W} \tag{29}$$

in $[1,+\infty)$. On the other hand, Jensen inequality, the fact that $\Phi_1^\gamma \leq \Phi_1 \leq 1$ and that $\overline{V}$ is bounded imply that there exists $D>0$ such that for all $t \geq 1$: $(\overline{W}(t))^\gamma \leq \overline{U}(t) \leq D(1+\varepsilon)$. Therefore $\overline{W}$ is bounded on $[1,+\infty)$. From (29) we deduce that there exists $C_3>0$ such that

$$0 \leq \int_{S_+^{N-1}} A(t,\theta)d\theta \leq C_3 - \overline{W}_{tt} - \left(\frac{a_1}{t}+a_2\right)\overline{W}_t \tag{30}$$

for all $t \geq 1$. Integrating twice (30) we obtain for all $t \geq 1$:

$$0 \leq \int_t^{t+1}\left(\int_s^{s+1}\left(\int_{S_+^{N-1}} A(\tau,\theta)d\theta\right)d\tau\right)ds \leq C_4 \tag{31}$$

where $C_4>0$ does not depend on t. Remark that for all nonnegative integrable function f, we have :

$$\int_t^{t+1}\left(\int_s^{s+1} f(\tau)\right)ds \geq \int_{t+\frac{1}{2}}^{t+1}\left(\int_{t+1}^{s+1} f(\tau)\right)ds \geq \frac{1}{2}\int_{t+1}^{t+\frac{3}{2}} f(\tau)d\tau.$$

Hence we deduce from (31) :

$$0 \leq \frac{1}{2}\int_{t+1}^{t+\frac{3}{2}}\int_{S_+^{N-1}} A(s,\theta)d\theta ds \leq C_4. \tag{32}$$

Since $V=W^\gamma-\varepsilon$, (32) implies for all $t \geq 2$:

$$0 \leq \int_t^{t+1}\int_{S_+^{N-1}} 1_{V\neq 0}\frac{|DV|^2}{(V+\varepsilon)^\beta}\Phi_1 d\theta ds \leq C_5 \tag{33}$$

where $\beta = 2-\gamma^{-1}$ and $C_5>0$ does not depend on t. Letting ε tend to 0 in (33) we obtain (25) using Fatou lemma.

Lemma 2. *Under the assumptions of proposition 1, for any $\varepsilon>0$ small enough there exists a positive constants K_1 such that for all $t \geq 2$:*

$$\int_t^{t+1}\int_{S_+^{N-1}} (V(s,\theta))^{\frac{N}{N-1}-\varepsilon} d\theta ds \leq K_1. \tag{34}$$

Proof. Here we follow the ideas of the proof of [5] theorem 4.1. Let $\tau \in (0,1)$ be fixed. From [5] lemma 4.1, there exists a unique solution ξ of problem

$$\begin{cases} -\Delta_{S^{N-1}}\xi = \Phi_1^{-\tau} & in \quad S_+^{N-1} \\ \xi = 0 & on \quad S^{N-2} \end{cases} \tag{35}$$

and there exists $K \geq 0$ such that $K^{-1}\Phi_1 \leq \xi \leq K\Phi_1$ on S_+^{N-1}. Defining $Z(t) = \int_{S_+^{N-1}} V(t,\theta)\xi(\theta)d\theta$, we deduce from (23) that

$$Z_{tt} + \left(\frac{a_1}{t} + a_2\right) Z_t + \frac{1}{t}\left(\frac{b_1}{t} + b_2\right) Z + c_1 Z - \int_{S^{N-2}} \Psi \frac{\partial \xi}{\partial \nu} d\theta =$$

$$\int_{S_+^{N-1}} V\Phi_1^{-\tau} d\theta + \int_{S_+^{N-1}} gV^q \xi d\theta$$

hence from (24) there exists $A \geq 0$ such that

$$\int_{S_+^{N-1}} V\Phi_1^{-\tau} d\theta \leq Z_{tt} + \left(\frac{a_1}{t} + a_2\right) Z_t + \frac{1}{t}\left(\frac{b_1}{t} + b_2\right) Z + c_1 Z + Ae^{-\beta t}. \tag{36}$$

On the other hand, since [5], there exist some constant μ and $\nu > 0$ such that the function $\eta = \Phi_1(\mu - \nu\Phi_1^{1-\tau})$ is a supersolution of (35). Since $\overline{V}$ is bounded:

$$0 \leq Z(t) \leq \int_{S_+^{N-1}} V\Phi_1(\mu - \nu\Phi_1^{1-\tau}) d\theta < \infty \tag{37}$$

Now integrating twice (36) between t and $t+1$ for all $t \geq 2$ and using (37) we obtain after integrate by part the term $(a_1/t + a_2)Z_t$:

$$\int_t^{t+1} \left(\int_s^{s+1} \left(\int_{S_+^{N-1}} V\Phi_1^{-\tau} d\theta \right) d\tau \right) ds \leq D$$

where $D > 0$ does not depend on t. Then as in lemma 1, we prove that there exists $K_\tau > 0$ such that for any $t \geq 2$:

$$\int_t^{t+1} \int_{S_+^{N-1}} u\Phi_1^{-\tau} d\theta d\tau \leq K_\tau. \tag{38}$$

Then from estimates (25) and (38) and using Holder and Sobolev inequalities, we deduce (34) as in [5], lemma 4.1.

We now give the proof of proposition 1 where the condition $q < 1$ highly occurs.

Proof of proposition 1. In this proof, for $l \in \mathbb{N}^*$, C_l denotes a positive constant independant on t. Set $f(t,\theta) = g(t,\theta)V^q - c_1 V - \frac{1}{t}\left(\frac{b_1}{t} + b_2\right)V$. We know that g is bounded on Cl and because of (34), Young inequality implies that for all $t \geq 2$:

$$\|f\|_{L^{\frac{N}{N-1}-\varepsilon}([t-1,t+1]\times\overline{S_+^{N-1}})} \leq C_1. \tag{39}$$

For all $j \geq 1$, define $K_t^{(j)} = [t - \frac{1}{j}, t + \frac{1}{j}] \times \overline{S_+^{N-1}}$. Because V satisfies (23), Calderon-Zygmund theory ensures that for all $t \geq 2$:

$$\|V\|_{W^{2,\frac{N}{N-1}-\varepsilon}(K_t^{(2)})} \leq C_2. \tag{40}$$

Then, since $\frac{N}{N-1} - \varepsilon < \frac{N}{2}$, Sobolev imbeddings imply :

$$\|V\|_{L^{p_1}(K_t^{(2)})} \leq C_2' \tag{41}$$

with $\frac{1}{p_1} = \frac{N-1}{N-\varepsilon(N-1)} - \frac{2}{N}$. Using Calderon-Zygmund theory with some $p_1 > \frac{N}{N-1} - \varepsilon$, we prove (40) with p_1 and $K_t^{(3)}$ respectively replacing by $\frac{N}{N-1} - \varepsilon$ and $K_t^{(2)}$.
Therefore Sobolev imbeddings imply :

$$If \quad p_1 > \frac{N}{2}, \quad then \quad \|V\|_{L^\infty(K_t^{(3)})} \leq C_3. \tag{42}$$

$$If \quad p_1 = \frac{N}{2}, \quad then \quad \|V\|_{L^p(K_t^{(3)})} \leq C_4 \quad \forall p \geq p_1. \tag{43}$$

Applying another time Calderon-Zygmund theory with some $p > p_1$, we obtain (40) with p and $K_t^{(4)}$ respectively replacing $\frac{N}{N-1} - \varepsilon$ and $K_t^{(2)}$ and we can use (42).

$$If \quad p_1 < \frac{N}{2}, \quad then \quad \|V\|_{L^{p_2}(K_t^{(2)})} \leq C_5 \tag{44}$$

with p_2 such that $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2}{N}$. Either $p_2 > \frac{N}{2}$ and we are under the condition of 42), or we use (39) with $p_2 > p_1$. We construct in that way a nondecreasing sequence (p_n) such that $\frac{1}{p_n} = \frac{1}{p_{n-1}} - \frac{2}{N}$. Thus there exists

$p_{n_0} > \frac{N}{2}$ and finally obtain the existence of some $j_0 \geq 1$ and $C > 0$ such that $\|V\|_{L^\infty(K_t^{(j_0)})} \leq C$. This achieves the proof.

We end this section with the following convergence lemmas which will allow us to prove theorem 2.

Lemma 3. *Let $(A, \alpha) \in \mathbb{R}^* \times \mathbb{R}^*_+$. Consider a nonnegative Holder function f in Cl satisfying :*

$$f(t,.) = O(e^{-\alpha t}) \quad uniformly \quad in \quad S_+^{N-1} \tag{45}$$

for large t. Let $Y \in C^2(\overline{Cl})$ be any nonnegative bounded solution of equation

$$Y_{tt} + AY_t + (N-1)Y + \Delta_{S^{N-1}} Y = f(t,\theta) Y^q \tag{46}$$

in Cl and satisfying

$$Y(t,.) = 0 \quad on \quad \partial S_+^{N-1} \tag{47}$$

for all t. Then Y_t and Y_{tt} tends to 0 in $L^2(S_+^{N-1})$ when t tends to infinity and there exists $l \geq 0$ such that

$$\lim_{t \to +\infty} Y(t,.) = l\Phi_1 \quad uniformly \quad on \quad S_+^{N-1}. \tag{48}$$

Proof. Since Y is bounded on $\overline{Cl}$, Calderon-Zygmund theory, Sobolev imbedding and Schauder theory imply that there exists a constant $C > 0$ such that

$$\|Y\|_{C^{2,\beta q}(\overline{Cl})} \leq C \tag{49}$$

with $\beta \in]0,1[$. Now define on the one hand the limit set

$$\Gamma(Y) = \bigcap_{t \geq 1} \overline{\bigcup_{\tau \geq t} Y(\tau,.)}^{C^2(S_+^{N-1})}. \tag{50}$$

As in [1], both Y_t and Y_{tt} tend to 0 in $L^2(S_+^{N-1})$ when t tends to infinity. Then $\Gamma(Y)$ is a connected compact subset of the set $E = \{\omega \in C^2(S_+^{N-1}) / -\Delta_{S^{N-1}}\omega = (N-1)\omega \quad in \quad S_+^{N-1}, \omega \geq 0 \quad and \quad \omega = 0 \quad on \quad \partial S_+^{N-1}\} = \{l\Phi_1 / l \in \mathbb{R}^+\}$.

On the other hand multiplying (46) by Φ_1, integrating on S_+^{N-1} and using (45) and (49), we obtain

$$0 \leq \overline{Y}_{tt} + A\overline{Y}_t \leq De^{-\alpha t} \tag{51}$$

for all $t \geq 1$ with $D > 0$ and $\overline{Y}$ defined in (13). Because of (51), the function $G : t \mapsto \overline{Y}_t + A\overline{Y} + \frac{D}{\alpha}e^{-\alpha t}$ is nonincreasing and lowerbounded on $[1, +\infty($. Therefore there exists $\tilde{l} \in \mathbb{R}$ such that $\tilde{l} = \lim_{t\to+\infty} G(t) = \lim_{t\to+\infty} A\overline{Y}(t)$ because $\overline{Y}_t$ tends to 0 in $L^2(S_+^{N-1})$.

Finally, because of (49) and the fact that $\Gamma(Y)$ is included in E, there exists $l \in \mathbb{R}^+$ and a sequence (t_n) converging to infinity such that $Y(t_n, .)$ tends to $l\Phi_1$ in $C^2(S_+^{N-1})$ as n tends to infinity. Thus we obtain $\tilde{l} = Al \int_{S_+^{N-1}} \phi_1^2(\theta)d\theta$. It would be the same for an other sequence and (48) holds.

In the same way, we can prove the analogous lemma:

Lemma 4. *Let $(A, B, \alpha) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}_+^*$. Consider the Holder non-negative function f in Cl satisfying :*

$$|f(t, .) - 1| = O(e^{-\alpha t}) \quad \textit{uniformly} \quad \textit{on} \quad S_+^{N-1}. \tag{52}$$

Let $Y \in C^2(\overline{Cl})$ be any nonnegative bounded solution of equation

$$Y_{tt} + AY_t + BY + \Delta_{S^{N-1}}Y = f(t, \theta)Y^q \tag{53}$$

in Cl and satisfying (47) for all t. Then the limit set $\Gamma(Y) = \bigcap_{t\geq 1} \overline{\bigcup_{\tau\geq t} Y(\tau, .)}^{C^2(S_+^{N-1})}$ is a connected compact subset of the set $\{\omega \in C^2(S_+^{N-1}) / \Delta_{S^{N-1}}\omega + B\omega - \omega^q = 0$ on $S_+^{N-1}, \omega \geq 0$ and $\omega = 0$ on $\partial S_+^{N-1}\}$.

3 A priori estimates

In this section, we consider a nonnegative solution of equation (1) and give an a priori estimate near 0 of this solution.

Proof of theorem 1. Considering both changes of variables (5), (7) and (10), the function V satisfies the equation (11) in the cylinder $Cl =$

$[1, +\infty) \times S_+^{N-1}$, where g is a nonnegative function in Cl, Holderian because of (3), satisfying because of (2) :

$$g(t,.) = O(e^{-\alpha t}) \quad uniformly \quad on \quad S_+^{N-1} \tag{54}$$

with $\alpha = N + 1 + \sigma - q(N-1)$. And

$$V = \Psi \geq 0$$

with $\Psi \in C(Cl)$ and satisfies for all $t \geq 1$:

$$\Psi(t,.) = O(e^{(1-N)t}) \quad uniformly \quad on \quad \partial S_+^{N-1}. \tag{55}$$

Now consider the function $\overline{V}$ defined in (13). Multiplying (11) by Φ_1 and integrating on S_+^{N-1}, we obtain for all $t \geq 1$:

$$\overline{V}_{tt} + N\overline{V}_t - \int_{\partial S_+^{N-1}} V(t,\tau)\frac{\partial \Phi_1}{\partial \nu}(\tau)d\tau = \int_{S_+^{N-1}} g(t,\sigma)V^q(t,\sigma)\Phi_1(\sigma)d\sigma. \tag{56}$$

Since $\frac{\partial \Phi_1}{\partial \nu}$ is nonpositive on ∂S_+^{N-1}, (54), (56) and Jensen inequality imply that there exists $C > 0$ such that for all $t \geq 1$

$$\overline{V}_{tt} + N\overline{V}_t \leq Ce^{-\alpha t}\overline{V}^q. \tag{57}$$

We now distinguish three cases :

(i)<u>$\alpha > 0$:</u>

If $\overline{V}$ is not bounded, then it is nondecreasing on an interval $[T, +\infty)$ with $T > 1$. Actually if $\overline{V}$ is not nondecreasing, there exists a sequence (t_n) of strict maxima of $\overline{V}$ such that $t_n \to +\infty$ and $\overline{V}(t_n) \to +\infty$. Let s_n be a real such that $\overline{V}(s_n) = \max_{[T,t_n]} \overline{V}(t)$, then we have $\overline{V}(t) \leq \overline{V}(s_n)$ for all $t \in [T, s_n]$. Integrate (57) on $[T, s_n]$, we obtain

$$\begin{aligned} -\overline{V}_t(T) + N\overline{V}(s_n) - N\overline{V}(T) &\leq C\overline{V}^q(s_n)\int_T^{s_n} e^{-\alpha t}dt \\ &\leq \frac{C}{\alpha}\overline{V}^q(s_n)e^{-\alpha T}. \end{aligned} \tag{58}$$

As $\overline{V}(t_n) \leq \overline{V}(s_n)$, $\overline{V}(t_n) \to +\infty$ and $q \in (0,1)$, we have a contradiction when n tends to infinity in (58).

Now we claim that $\overline{V}$ is bounded. Actually, if $\overline{V}$ is not bounded, $\overline{V}$is nondecreasing on $[T,+\infty[$ and then $\lim_{t\to+\infty}\overline{V}(t)=+\infty$. On the other hand, because of (57), the function $G : t \mapsto \overline{V}_t(t)+N\overline{V}(t)-C\int_2^t e^{-\alpha s}\overline{V}^q(s)ds$ is nonincreasing on $[T,+\infty)$. Therefore G is bounded from above on $[T,+\infty)$ by a constant $D\in\mathbb{R}$. Morever $\overline{V}_t\geq 0$ on $[T,+\infty)$ and we deduce

$$N\overline{V}^{1-q}(t)\leq D\overline{V}^{-q}(t)+C\alpha^{-1}e^{-\alpha T}\quad for\quad all\quad t\geq T.$$

Then we obtain a contradiction as t goes to infinity and $\overline{V}$ is bounded on $[T,+\infty)$. Then the assumptions of proposition 1 are achieved, with $a_1=0$, $a_2=N$, $b_1=b_2=0$, $c_1=N-1$ and $\beta=N-1$ in (24). Thus proposition 1 applies and $V\in L^\infty(Cl)$. Using changes of variables (10), (7) and (5), we obtain (14).

(ii)$\underline{\alpha<0 :}$

If $\overline{V}$ is bounded, then we obtain (14) as above and since $\alpha<0$, that is $q>\frac{N+1+\sigma}{N-1}$, we have $|x|^{1-N}<<|x|^{\frac{2+\sigma}{1-q}}$ near 0 which implies (15).

If $\overline{V}$ is not bounded, then there exist $1<t_0<t_1$ such that $1<\overline{V}(t_0)<\overline{V}(t_1)$. Let $e\in(t_1,+\infty)$. We define

$$s_e=\min\{s\in[t_0,e]/\max_{[t_0,e]}\overline{V}=\overline{V}(s)\}.$$

Then $\overline{V}(t)\leq\overline{V}(s_e)$ for all $t\in[t_0,e]$. We claim that $\overline{V}_t(s_e)\geq 0$. Actually, if $s_e\in]t_0,e[$, then $\overline{V}_t(s_e)=0$. If $s_e=t_0$, then $t_1\in]t_0,e[$ implies $\overline{V}(t_1)\leq\overline{V}(s_e)=\overline{V}(t_0)$ and this is false. If $s_e=e$, then $\overline{V}_t(s_e)<0$ would be a contradiction with $\overline{V}(s_e)=\max_{[t_0,e]}\overline{V}$.

Now integrate (57) on $[t_0,s_e]$, we obtain since $\overline{V}_t(s_e)\geq 0$:

$$\begin{aligned}N\overline{V}(s_e)\;&\leq C\int_{t_0}^{s_e}e^{-\alpha t}\overline{V}^q(t)dt+N\overline{V}(t_0)+\overline{V}_t(t_0)\\&\leq C\overline{V}^q(s_e)\int_{t_0}^{s_e}e^{-\alpha t}dt+C_0\end{aligned}$$

where $C_0>0$ only depends on t_0. Therefore, because $\overline{V}(s_e)\geq\overline{V}(t_0)>1$, we have

$$\overline{V}^{1-q}(s_e)\leq-\frac{C}{\alpha N}e^{-\alpha s_e}+C_0. \tag{59}$$

Since the function $r \mapsto -\frac{C}{\alpha N}e^{-\alpha r} + C_0$ is increasing, we deduce from $s_e \leq e$, $\overline{V}(e) \leq \overline{V}(s_e)$, $q \in (0,1)$ and (59) that (59) holds for e replacing s_e. Therefore there exist $D > 0$ such that for all $t > t_1$:

$$\overline{V}(t) \leq De^{-\frac{\alpha}{1-q}t} = De^{-[N-1+\frac{2+\sigma}{1-q}]t}. \tag{60}$$

Finally we introduce the function U defined on Cl by

$$U(t,\theta) = e^{[N-1+\frac{2+\sigma}{1-q}]t}V(t,\theta) \tag{61}$$

and its average $\overline{U}$ defined in (13). Because of (60), $\overline{U}$ is bounded on $(t_1, +\infty)$ and U satisfies (23) with $a_1 = b_1 = b_2 = 0$, $a_2 = 2 - N - 2\gamma$ and $c_1 = \gamma(\gamma + 2 - N)$ where $\gamma = \frac{2+\sigma}{1-q}$ and $\beta = -\frac{2+\sigma}{1-q} > 0$ in (24) because $\alpha < 0$. Moreover the assumptions of proposition 1 are achieved and then $U \in L^\infty(Cl)$. Using changes of variables (61), (10), (7) and (5), we obtain (15).

(iii)$\underline{\alpha = 0 :}$

If $\overline{V}$ is bounded, we use the fact that $|x|^{1-N} << |x|^{1-N}|ln|x||^{\frac{1}{1-q}}$ near 0 and we obtain (16). If $\overline{V}$ is not bounded, then in the same way as above, we prove the following inequality which is similar to (60) :

$$\overline{V}(t) \leq Dt^{\frac{1}{1-q}}. \tag{62}$$

Finally, we use a function W defined on Cl by :

$$W(t,\theta) = t^{\frac{1}{1-q}}V(t,\theta).$$

It satisfies (23) with $a_1 = 2/(1-q)$, $a_2 = N$, $b_1 = 2/(1-q)(2/(1-q)-1)$, $b_2 = N$, $c_1 = N - 1$ and $\beta = (N-1)/2$ in (24) for example. Then the assumptions of proposition 1 are achieved, we still obtain (16).

4 Convergence results

In this last section, we prove theorem 2. We distinguish two cases.

First case : we assume $q \leq \min\left(\frac{N+\sigma+1}{N-1}, 1\right)$.

Consider the function V introduced in (10). Because of (11), V satisfies (46) with $A = N$ and $f = g$. Moreover V is bounded from theorem 1 on an set $Cl = [2,+\infty) \times S_+^{N-1}$ and theorem 2 assumptions imply (47). Then lemma 3 ensures that (19) holds.

If $l = 0$, then we introduce $\overline{V}$ defined in (13). Lemma 3 and $l = 0$ imply $\lim_{t\to+\infty} \overline{V}(t) = \lim_{t\to+\infty} \overline{V}_t(t) = 0$. On the other hand, because of (11) , the function $\overline{V}_t + N\overline{V}$ is nondecreasing and then it is nonpositive in $[2,+\infty)$. Therefore the function $t \mapsto e^{Nt}\overline{V}(t)$ is nonincreasing and then

$$\overline{V}(t) = O(e^{-Nt}) \quad at \quad infinity\ . \tag{63}$$

(i) Assume $2+\sigma > 0$.
If $\sigma + 1 + q \leq 0$, then we introduce the function Y defined on Cl by

$$Y(t,.) = e^{(N-1)t}V(t,.) \tag{64}$$

and we will prove that $Y(t,.) = O(e^{-\gamma t})$ to obtain (20). Because of (11), Y satisfies in Cl :

$$Y_{tt} + (2-N)Y_t + \Delta_{S^{N-1}}Y = h(t,\theta)Y^q \tag{65}$$

where from (2) there exists $C > 0$ such that :

$$h(t,\theta) \sim Ce^{-(2+\sigma)t} \tag{66}$$

near $+\infty$ and uniformly on S_+^{N-1}. The average $\overline{Y}$ of Y satisfies in $[2,+\infty)$:

$$\overline{Y}_{tt} + (2-N)\overline{Y}_t - (N-1)\overline{Y} = \int_{S_+^{N-1}} h(t,\theta)Y^q(t,\theta)\Phi_1(\theta)d\theta. \tag{67}$$

We claim that $\overline{Y}$ is nonincreasing. Actually, if $\overline{Y}$ is not monotone, there exists a sequence (t_n) of strict maxima of $\overline{Y}$ which tends to $+\infty$ and we have a contradiction from (66) and the fact $\overline{Y}(t_n) > 0$ when we take (67) at large t_n. Because of (63), $\overline{Y}(t)$ tends to 0 at infinity and since it is nonnegative, we deduce that $\overline{Y}$ is nonincreasing in an interval $[T,+\infty)$ with $T \geq 2$. Now, from (66), there exists $K > 0$ such that (67) implies in $[T,+\infty)$

$$\overline{Y}_{tt} + (2-N)\overline{Y}_t - (N-1)\overline{Y} \leq Ke^{-(2+\sigma)t}\overline{Y}^q. \tag{68}$$

If we consider the function E defined by

$$E(t)=\frac{\overline{Y}_t^2}{2}-(N-1)\frac{\overline{Y}^2}{2}-Ke^{-(2+\sigma)t}\frac{\overline{Y}^{q+1}}{q+1} \tag{69}$$

then (68) ensures that E is nondecreasing in $[T,+\infty)$. Therefore there exists $\tilde{l}=\lim_{t\to+\infty}E(t)\in I\!R\cup\{+\infty\}$. Since $\lim_{t\to+\infty}\overline{Y}(t)=0$, we deduce from (69) that $\lim_{t\to+\infty}\frac{\overline{Y}_t^2(t)}{2}=\tilde{l}$. Moreover $\overline{Y}$ is bounded and thus $\tilde{l}=0$. It implies that E is nonpositive and we get

$$\begin{aligned}-\overline{Y}_t &\le \overline{Y}^{\frac{q+1}{2}}e^{-\frac{(2+\sigma)}{2}t}\left[2K+(N-1)e^{(\sigma+1+q)t}\right]\\ &\le \overline{Y}^{\frac{q+1}{2}}e^{-\frac{(2+\sigma)}{2}t}\tilde{K}\end{aligned} \tag{70}$$

in $[T_0,+\infty)$ with $T_0\ge T$ and $\tilde{K}>0$. Without loss of generality, we can assume $\overline{Y}>0$ in $[T_0,+\infty)$ and (70) implies that the function $\phi:t\mapsto -\overline{Y}^{-\frac{1-q}{2}}+\frac{2\tilde{K}}{2+\sigma}e^{-\frac{(2+\sigma)}{2}t}$ is nonincreasing in $[T_0,+\infty)$. Since $\lim_{t\to+\infty}\phi(t)=0$, we deduce that ϕ is nonnegative and we obtain $\overline{Y}(t)=O(e^{-\gamma t})$ near $+\infty$. Finally, using the function U defined by $U(t,\theta)=e^{\gamma t}Y(t,\theta)$, its average and proposition 1, we obtain $Y(t,.)=O(e^{-\gamma t})$ which implies (20).

On the other hand, the assumptions of lemma 4 are fulfilled and we obtain the inclusion property of (i).

If $\sigma+1+q>0$, then we introduce the function Z defined on Cl by $Z(t,\theta)=e^{Nt}V(t,\theta)$. Because of (63), $\overline{Z}$ is bounded and satisfies from (11)

$$Z_{tt}-NZ_t+(N-1)Z+\Delta_{S^{N-1}}Z=h(t,\theta)Z^q \tag{71}$$

in Cl with $h(t,\theta)\sim e^{-(\sigma+1+q)t}$ near $+\infty$. Proposition 1 applies, Z is bounded in Cl and lemma 3 implies (22). If $k=0$, then we proceed as in case $\sigma+1+q<0$: we introduce the function E defined by $E(t)=\frac{1}{2}\overline{Z}_t^2(t)-e^{-(\sigma+1+q)t}\frac{\overline{Z}^{q+1}}{q+1}(t)$ to prove that $\overline{Z}(t)=O(e^{-\gamma t+t})$ near $+\infty$ which implies (20) because of proposition 1. We end this case as above.

(ii) Assume $2+\sigma\le 0$. Then [1] ensures the result.

Second case : we assume $\frac{N+\sigma+1}{N-1} < q < 1$. From theorem 1, (20) holds and the proof of the end is similar to the one of first case.

Now assume $a(x) = |x|^\sigma$ and $\lim_{n\to\infty} r_n^{-\gamma} w(r_n, .) = 0$ for some sequence $r_n \to 0$, it remains to prove that $u \equiv 0$ near 0. The function U defined as above satisfies in Cl

$$U_{tt} + AU_t + BU + \Delta_{S^{N-1}} U = h(t,\theta) U^q \tag{72}$$

where $A = 2 - N - 2\gamma < 0$, $B = \gamma(\gamma + N - 2) > 0$ and h is defined by

$$h(t,\theta) = e^{-\beta t} |z + |z|^2 e_N|^{-\beta} \tag{73}$$

with $\beta = N + 2 - (N-2)q > 0$, (r,θ) denotes the spherical coordinates of z and $t = -lnr$. We introduce the energy function E defined in $[2, +\infty)$ by

$$E(t) = \int_{S_+^{N-1}} \left(\frac{1}{2} U_t^2 - \frac{1}{2} |\nabla_{S^{N-1}} U|^2 + \frac{B}{2} U^2 - \frac{1}{q+1} U^{q+1} h \right) d\theta. \tag{74}$$

We claim that E is nondecreasing. Actually, because of (72), we have

$$E'(t) = -A \int_{S_+^{N-1}} U_t^2 d\theta - \int_{S_+^{N-1}} \frac{1}{q+1} U^{q+1} h_t d\theta.$$

Denote by $e^{-t}\phi(\theta)$ the first coordinate of z and remark that $\phi \geq 0$ on S_+^{N-1}. From (73), $h_t(t,\theta) = \beta e^{-\beta t} [e^{-2t} + 2e^{-3t}\phi(\theta) + e^{-4t}]^{-\frac{\beta}{2}-1} [e^{-3t}\phi(\theta) + e^{-4t}] > 0$ and then, E is nondecreasing. On the other hand, since there exists a sequence $r_n \to 0$ such that $\lim_{n\to+\infty} r_n^{-\gamma} w(r_n, .) = 0$, we deduce that $0 \in \Gamma(U) = \bigcap_{t\geq 2} \overline{\bigcup_{\tau \geq t} U(\tau, .)}^{C^2(S_+^{N-1})}$. Therefore, using the fact that E is nondecreasing, we obtain as in [4] that $\Gamma(U) = \{0\}$. Thus, [1] implies that $u \equiv 0$ near 0.

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