REVISTA MATEMÁTICA COMPLUTENSE
Volumen 12, número 1: 1999

Quantitative estimates for interpolated operators by multidimensional methods.

Fernando COBOS*, José María CORDEIRO and
Antón MARTÍNEZ*

Abstract

We describe the behaviour of ideal variations under interpolation methods associated to polygons.

0 Introduction

The behaviour of weakly compact operators under interpolation methods for N-tuples defined by means of polygons has been considered by Cobos, Fernández-Martínez and Martínez [5] and by Carro and Nikolova [4]. Among other things, they showed that the interpolated operator acting between two K-spaces or two J-spaces is weakly compact provided that all but two restrictions of T (located in adjacent vertices of the polygon) are weakly compact. Moreover, a similar result holds for other operator ideals sharing certain properties with weakly compact operators (see [5], Remark 2.9).

In this paper we investigate how far the interpolated operator can be from being weakly compact. In a more general way, we estimate the distance of the interpolated operator to a given operator ideal. In the case of the classical real method for Banach couples, this question has been recently studied by Cobos, Manzano and Martínez [9] and Cobos

1991 Mathematics Subject Classification: 46B70,47D50.

*Supported in part by DGES (PB97-0254).

Servicio Publicaciones Univ. Complutense. Madrid, 1999.

and Martínez [10], [11], where they have established estimates for the measures $\gamma_{\mathcal{I}}$, $\beta_{\mathcal{I}}$ related to a given operator ideal $\mathcal{I}$. We consider here similar questions in the multidimensional context of interpolation spaces associated to polygons. Our techniques use some ideas introduced in [9] combined with the geometrical elements which are natural to the interpolation methods that we deal with.

We start by reviewing in Section 1 some basic facts on ideal variations and on J- and K-methods associated to polygons. Then, in Section 2, we establish estimates for $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$ when one of the N-tuples of Banach spaces degenerates into a single space. Finally, in Section 3, we deal with the case of general N-tuples assuming that the operator ideal $\mathcal{I}$ satisfies the Σ_q-condition (see [14]).

1 Preliminaries

Let A and B be Banach spaces. By $\mathcal{L}(A,B)$ we denote the collection of all bounded linear operators from A into B, endowed with the usual operator norm. The closed unit ball of A is designated by U_A, and A^* stands for the dual of A. We put $\ell_1(U_A)$ for the Banach space of all absolutely summable families of scalars $(\lambda_a)_{a\in U_A}$ with U_A as index set. The map $Q_A : \ell_1(U_A) \longrightarrow A$ defined by $Q_A(\lambda_a) = \sum_{a\in U_A} \lambda_a a$ is a metric surjection. The space $\ell_\infty(U_{B^*})$ is formed by all bounded families of scalars indexed by the elements of U_{B^*}. Write $J_B : B \longrightarrow \ell_\infty(U_{B^*})$ for the isometric embedding given by $J_B b = (\langle f, b\rangle)_{f\in U_{B^*}}$.

A class $\mathcal{I}$ of bounded linear operators is said to be an operator ideal if each component $\mathcal{I}\cap\mathcal{L}(A,B) = \mathcal{I}(A,B)$ is a linear subspace of $\mathcal{L}(A,B)$ that contains the finite rank operators and satisfies that $STR \in \mathcal{I}(E,F)$ whenever $R \in \mathcal{L}(E,A)$, $T \in \mathcal{I}(A,B)$ and $S \in \mathcal{L}(B,F)$. The ideal $\mathcal{I}$ is called closed if each component $\mathcal{I}(A,B)$ is closed in $\mathcal{L}(A,B)$. The ideal $\mathcal{I}$ is said to be surjective if for every $T \in \mathcal{L}(A,B)$ it follows from $TQ_A \in \mathcal{I}(\ell_1(U_A), B)$ that $T \in \mathcal{I}(A,B)$. The ideal $\mathcal{I}$ is called injective if for every $T \in \mathcal{L}(A,B)$ it follows from $J_BT \in \mathcal{I}(A, \ell_\infty(U_{B^*}))$ that $T \in \mathcal{I}(A,B)$. Compact operators $\mathcal{K}$ or weakly compact operators $\mathcal{W}$ are examples of closed injective and surjective operator ideals. Strictly singular operators $\mathcal{S}$ is an ideal which is closed and injective but it is not surjective, while strictly cosingular operators $\mathcal{C}$ is closed and surjective but it is not injective (see [17]).

Given an operator ideal $\mathcal{I}$, we put $\bar{\mathcal{I}}^s$ for its closed surjective hull, that is, the smallest closed surjective operator ideal containing $\mathcal{I}$. For $T \in \mathcal{L}(A,B)$, it turns out that T belongs to $\bar{\mathcal{I}}^s(A,B)$ if and only if for every $\varepsilon > 0$ there is a Banach space E and an operator $R \in \mathcal{I}(E,B)$ such that

$$T(U_A) \subseteq R(U_E) + \varepsilon U_B \quad \text{(see [15])}.$$

The characterization for the elements of the closed injective hull $\bar{\mathcal{I}}^i$ of $\mathcal{I}$ is as follows: Let $T \in \mathcal{L}(A,B)$. The operator T belongs to $\bar{\mathcal{I}}^i(A,B)$ if and only if for every $\varepsilon > 0$ there is a Banach space F and an operator $S \in \mathcal{I}(A,F)$ such that

$$\|Tx\|_B \leq \|Sx\|_F + \varepsilon\|x\|_A\ ,\ x \in A.$$

It is natural then to associate with $\mathcal{I}$ the functionals defined for each $T \in \mathcal{L}(A,B)$ by

$$\begin{aligned}\gamma_{\mathcal{I}}(T) = \gamma_{\mathcal{I}}(T_{A,B}) = \inf\{\sigma > 0 : T(U_A) \subseteq \sigma U_B + R(U_E),\\ R \in \mathcal{I}(E,B),\ E \text{ any Banach space}\},\end{aligned}$$

$$\begin{aligned}\beta_{\mathcal{I}}(T) = \beta_{\mathcal{I}}(T_{A,B}) = \inf\{\sigma > 0 : \text{ there is a Banach space } F \text{ and}\\ S \in \mathcal{I}(A,F) \text{ such that } \|Tx\|_B \leq \sigma\|x\|_A + \|Sx\|_F,\ x \in A\}.\end{aligned}$$

The (outer) measure $\gamma_{\mathcal{I}}$ was introduced by Astala in [1], and it shows the deviation of T from $\bar{\mathcal{I}}^s$ in the sense that

$$\gamma_{\mathcal{I}}(T) = 0 \ \text{ if and only if } \ T \in \bar{\mathcal{I}}^s(A,B).$$

The (inner) measure $\beta_{\mathcal{I}}$ was introduced by Tylli in [19] and it gives the deviation of T from $\bar{\mathcal{I}}^i$. These funtionals are subadditive

$$\gamma_{\mathcal{I}}(S+T) \leq \gamma_{\mathcal{I}}(S) + \gamma_{\mathcal{I}}(T) \quad , \quad \beta_{\mathcal{I}}(S+T) \leq \beta_{\mathcal{I}}(S) + \beta_{\mathcal{I}}(T)$$

submultiplicative

$$\gamma_{\mathcal{I}}(ST) \leq \gamma_{\mathcal{I}}(S)\gamma_{\mathcal{I}}(T) \quad , \quad \beta_{\mathcal{I}}(ST) \leq \beta_{\mathcal{I}}(S)\beta_{\mathcal{I}}(T)$$

satisfy that

$$\max\{\gamma_{\mathcal{I}}(T)\ ,\ \beta_{\mathcal{I}}(T)\} \leq \|T\|$$

and moreover the following minimal properties hold

$$\gamma_{\mathcal{I}}(J_B T) = \min\{\gamma_{\mathcal{I}}(jT) \ : \ j : B \longrightarrow F \text{ isometric embedding}\} \qquad (1)$$

$$\beta_{\mathcal{I}}(TQ_A) = \min\{\beta_{\mathcal{I}}(T\pi) \ : \ \pi : E \longrightarrow A \text{ metric surjection}\} \qquad (2)$$

(see [1], pag. 21 and [9], § 2).

Let us see now some concrete cases. Choose $\mathcal{I} = \mathcal{K}$, the ideal of compact operators, so $\bar{\mathcal{K}}^i = \bar{\mathcal{K}}^s = \mathcal{K}$. It can be checked that $\gamma_{\mathcal{K}}(T)$ coincides with the (ball) measure of non-compactness of T

$$\gamma_{\mathcal{K}}(T) = \inf\{\sigma > 0 : \text{ there exists a finite number of elements } b_1, \ldots, b_k \in B \text{ such that } T(U_A) \subseteq \textstyle\bigcup_{j=1}^k \{b_j + \sigma U_B\}\}$$

while $\beta_{\mathcal{K}}(T) = \lim_{n\to\infty} c_n(T)$, where $(c_n(T))$ is the sequence of the Gelfand numbers of T. The measures $\gamma_{\mathcal{K}}$ and $\beta_{\mathcal{K}}$ are equivalent. More precisely

$$\frac{1}{2}\gamma_{\mathcal{K}}(T) \leq \beta_{\mathcal{K}}(T) \leq 2\gamma_{\mathcal{K}}(T) \quad \text{(see [16])}.$$

Take next $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators. Again $\bar{\mathcal{W}}^i = \bar{\mathcal{W}}^s = \mathcal{W}$. The measure $\gamma_{\mathcal{W}}(T)$ is equal to the measure of weak non-compactness introduced by De Blasi [13]

$$\gamma_{\mathcal{W}}(T) = \inf\{\sigma > 0 : \text{ there is a weakly compact set } W \text{ in } B \text{ such that } T(U_A) \subseteq W + \sigma U_B\}.$$

As in the previous example, $\beta_{\mathcal{W}}(T) = \gamma_{\mathcal{W}}(T^*)$, but this time $\gamma_{\mathcal{W}}$ and $\beta_{\mathcal{W}}$ are not equivalent (see [2]).

For $\mathcal{I} = \mathcal{S}$, the ideal of strictly singular operators, one has $\bar{\mathcal{S}}^i = \mathcal{S}$ and $\bar{\mathcal{S}}^s = \mathcal{R}$, where $\mathcal{R}$ stands for the ideal of Rosenthal operators (see [17]). The functional $\beta_{\mathcal{S}}$ is the relevant one to show the deviation of an operator from being strictly singular, while $\gamma_{\mathcal{S}} = \gamma_{\mathcal{R}}$ gives the deviation of an operator from being Rosenthal.

Cosingular operators $\mathcal{C}$ satisfy that $\bar{\mathcal{C}}^s = \mathcal{C}$ and $\bar{\mathcal{C}}^i = \mathcal{R}$. The relevant functional to work with $\mathcal{C}$ is then $\gamma_{\mathcal{C}}$.

Next we review the definition and some basic results on interpolation methods defined by means of polygons.

Let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon in the plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$, $j = 1, \ldots, N$. By a Banach N-tuple we mean a family $\bar{A} =$

$\{A_1, \ldots, A_N\}$ of N Banach spaces A_j which are continuously embedded in a common Hausdorff topological space. It will be useful to imagine each space A_j as sitting in the vertex P_j.

By means of the polygon Π, we define the following family of norms on $\Sigma(\bar{A}) = A_1 + \cdots + A_N$

$$K(t,s;a) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j \, , \; a_j \in A_j \right\}, \quad t,s > 0.$$

The corresponding family of norms on $\Delta(\bar{A}) = A_1 \cap \cdots \cap A_N$ is

$$J(t,s;a) = \max_{1 \le j \le N} \left\{ t^{x_j} s^{y_j} \|a\|_{A_j} \right\}, \quad t,s > 0.$$

Given any interior point (α,β) of Π $[(\alpha,\beta) \in \operatorname{Int}\Pi]$ and any $1 \le q \le \infty$, the K-space $\bar{A}_{(\alpha,\beta),q;K}$ consists of all a in $\Sigma(\bar{A})$ which have a finite norm

$$\|a\|_{(\alpha,\beta),q;K} = \left(\sum_{(m,n)\in \mathbf{Z}^2} \left(2^{-\alpha m - \beta n} K(2^m, 2^n; a) \right)^q \right)^{\frac{1}{q}} \qquad (\text{if } q < \infty)$$

$$\|a\|_{(\alpha,\beta),\infty;K} = \sup_{(m,n)\in \mathbf{Z}^2} \left\{ 2^{-\alpha m - \beta n} K(2^m, 2^n; a) \right\}.$$

The J-space $\bar{A}_{(\alpha,\beta),q;J}$ is formed by all those elements a in $\Sigma(\bar{A})$ which can be represented as

$$a = \sum_{(m,n)\in \mathbf{Z}^2} u_{m,n} \qquad (\text{convergence in } \Sigma(\bar{A}))$$

with $u_{m,n} \in \Delta(\bar{A})$ and

$$\left(\sum_{(m,n)\in \mathbf{Z}^2} \left(2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}) \right)^q \right)^{\frac{1}{q}} < \infty$$

(the sum should be replaced by the supremum if $q = \infty$). The norm in $\bar{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J} = \inf\left\{ \left(\sum_{(m,n)\in \mathbf{Z}^2} \left(2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}) \right)^q \right)^{\frac{1}{q}} \right\}$$

where the infimum is taken over all representations $(u_{m,n})$ of a as above.

These interpolation spaces were introduced by Cobos and Peetre in [12]. One can find there continuous characterizations of $\bar{A}_{(\alpha,\beta),q;K}$ and $\bar{A}_{(\alpha,\beta),q;J}$, using integrals instead of sums, but they will not be required here. An important difference with the classical real method for couples, where K- and J-spaces coincide to within equivalence of norms (see [3] and [18]), is that in general $\bar{A}_{(\alpha,\beta),q;K} \neq \bar{A}_{(\alpha,\beta),q;J}$. We only have now that $\bar{A}_{(\alpha,\beta),q;J}$ is continuously embedded in $\bar{A}_{(\alpha,\beta),q;K}$ (see [12], Thm. 1.3).

Let $\bar{B} = \{B_1, \ldots, B_N\}$ be another Banach N-tuple which we also imagine as sitting on the vertices of another copy of the polygon Π. By $T \in \mathcal{L}(\bar{A}, \bar{B})$ we mean a linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j, $j = 1, \ldots, N$. Let $M_j = \|T\|_{A_j,B_j}$.

If $T \in \mathcal{L}(\bar{A}, \bar{B})$, then the restriction of T to $\bar{A}_{(\alpha,\beta),q;K}$ gives a bounded linear operator $T : \bar{A}_{(\alpha,\beta),q;K} \longrightarrow \bar{B}_{(\alpha,\beta),q;K}$. The norm of this interpolated operator has been computed in [8], Thm. 1.9. It turns out that

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;K},\bar{B}_{(\alpha,\beta),q;K}} \leq C_1 \max\left\{M_i^{c_i}\, M_k^{c_k}\, M_r^{c_r} \,:\, \{i,k,r\} \in \mathcal{P}\right\}. \quad (3)$$

Here C_1 is a constant depending only on Π and (α, β), $\mathcal{P}$ stands for the set of all those triples $\{i, k, r\}$ such that (α, β) belongs to the triangle with vertices P_i, P_k, P_r, and (c_i, c_k, c_r) are the barycentric coordinates of (α, β) with respect to P_i, P_k, P_r. A similar estimate holds for J-spaces.

When the interpolated operator is considered from a J-space into a K-space then a better estimate is valid. Namely

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;J},\bar{B}_{(\alpha,\beta),q;K}} \leq C_2 \prod_{j=1}^{N} M_j^{\theta_j}. \quad (4)$$

Here $0 < \theta_1, \ldots, \theta_N < 1$ with $\sum_{j=1}^{N} \theta_j = 1$ and $\sum_{j=1}^{N} \theta_j P_j = (\alpha, \beta)$ (that is, $\bar{\theta} = (\theta_1, \ldots, \theta_N)$ are some barycentric coordinates of (α, β) with respect to the vertices $P_1, \ldots, P_N$), and C_2 is a constant depending only on $\bar{\theta}$ (see [8], Thm. 3.2).

Estimate (1.4) implies that

$$\|a\|_{(\alpha,\beta),q;K} \leq C_3 \prod_{j=1}^{N} \|a\|_{A_j}^{\theta_j}, \quad a \in \Delta(\bar{A}). \quad (5)$$

On the other hand, inequality (1.3) in the case of J-spaces yields that

$$\|a\|_{(\alpha,\beta),q;J} \le C_4 \max\left\{\|a\|_{A_i}^{c_i}\ \|a\|_{A_k}^{c_k}\ \|a\|_{A_r}^{c_r}\ :\ \{i,k,r\} \in \mathcal{P}\right\}, a \in \Delta(\bar{A}). \tag{6}$$

2 Estimates for degenerated cases

The following result describes the behaviour of the ideal variations when one of the N-tuples reduces to a single Banach space.

Theorem 2.1. *Let $\mathcal{I}$ be an operator ideal, let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha,\beta) \in$ Int Π and $1 \le q \le \infty$. Define $\mathcal{P}$ and $\bar{\theta} = (\theta_1, \dots, \theta_N)$ as before. Assume that $\bar{A} = \{A_1, \dots, A_N\}$ is a Banach N-tuple and that B is a Banach space.*

If $T \in \mathcal{L}(\Sigma(\bar{A}), B)$ then

a) $\gamma_{\mathcal{I}}(T_{\bar{A}_{(\alpha,\beta),q;K},B})$

$$\le D_1 \max\left\{\gamma_{\mathcal{I}}(T_{A_i,B})^{c_i}\ \gamma_{\mathcal{I}}(T_{A_k,B})^{c_k}\ \gamma_{\mathcal{I}}(T_{A_r,B})^{c_r}\ :\ \{i,k,r\} \in \mathcal{P}\right\}.$$

b) $\gamma_{\mathcal{I}}(T_{\bar{A}_{(\alpha,\beta),q;J},B}) \le D_2 \prod_{j=1}^{N} \gamma_{\mathcal{I}}(T_{A_j,B})^{\theta_j}.$

If $T \in \mathcal{L}(B, \Delta(\bar{A}))$ then

c) $\beta_{\mathcal{I}}(T_{B,\bar{A}_{(\alpha,\beta),q;J}})$

$$\le D_3 \max\left\{\beta_{\mathcal{I}}(T_{B,A_i})^{c_i}\ \beta_{\mathcal{I}}(T_{B,A_k})^{c_k}\ \beta_{\mathcal{I}}(T_{B,A_r})^{c_r}\ :\ \{i,k,r\} \in \mathcal{P}\right\}.$$

d) $\beta_{\mathcal{I}}(T_{B,\bar{A}_{(\alpha,\beta),q;K}}) \le D_4 \prod_{j=1}^{N} \beta_{\mathcal{I}}(T_{B,A_j})^{\theta_j}.$

Here D_1 and D_3 are constants depending only on Π and (α,β), while D_2 and D_4 are other constants that only depend on $\bar{\theta}$.

Proof. Since $\bar{A}_{(\alpha,\beta),q;K} \hookrightarrow \bar{A}_{(\alpha,\beta),\infty;K}$ with norm less than or equal to 1, in order to establish a) it is enough to consider the case $q = \infty$. Observe that there is a constant C, depending only on Π and (α,β), such that

$$\sup_{t,s>0}\left\{t^{-\alpha}s^{-\beta}K(t,s;a)\right\} \le C\|a\|_{(\alpha,\beta),\infty;K}\,, \quad a \in \bar{A}_{(\alpha,\beta),\infty;K}.$$

Hence, given any $\varepsilon, t, s > 0$ and $a \in U_{\bar{A}_{(\alpha,\beta),\infty;K}}$, we can find a decomposition $a = \sum_{j=1}^{N} a_j$ with $a_j \in A_j$ and $\|a_j\|_{A_j} \leq (1+\varepsilon)Ct^{\alpha-x_j}s^{\beta-y_j}$, $1 \leq j \leq N$. So

$$U_{\bar{A}_{(\alpha,\beta),\infty;K}} \subseteq \sum_{j=1}^{N}(1+\varepsilon)Ct^{\alpha-x_j}s^{\beta-y_j}U_{A_j}.$$

Let $\sigma_j > \gamma_{\mathcal{I}}(T_{A_j,B})$. According to the definition of $\gamma_{\mathcal{I}}$, there exists a Banach space E_j and an operator $R_j \in \mathcal{I}(E_j, B)$ so that

$$T\left(U_{A_j}\right) \subseteq \sigma_j U_B + R_j\left(U_{E_j}\right), \quad 1 \leq j \leq N.$$

Therefore

$$T\left(U_{\bar{A}_{(\alpha,\beta),\infty;K}}\right)$$

$$\subseteq \sum_{j=1}^{N}(1+\varepsilon)C\sigma_j t^{\alpha-x_j}s^{\beta-y_j}U_B + \sum_{j=1}^{N}(1+\varepsilon)Ct^{\alpha-x_j}s^{\beta-y_j}R_j(U_{E_j})$$

$$\subseteq (1+\varepsilon)C\left(\sum_{j=1}^{N}t^{\alpha-x_j}s^{\beta-y_j}\sigma_j\right)U_B + R_{\varepsilon,t,s}(U_E).$$

Here $E = \{(z_1,\ldots,z_N) : z_j \in E_j\}$ normed by $\|(z_1,\ldots,z_N)\|_E = \max\{\|z_j\|_{E_j} : 1 \leq j \leq N\}$ (i.e., $E = (\oplus_{j=1}^{N} E_j)_{\ell_\infty}$), and $R_{\varepsilon,t,s} : E \longrightarrow B$ is the operator defined by $R_{\varepsilon,t,s}(z_1,\ldots,z_N) = (1+\varepsilon)C\sum_{j=1}^{N} t^{\alpha-x_j}s^{\beta-y_j}R_j z_j$. Ideal property of $\mathcal{I}$ implies that $R_{\varepsilon,t,s} \in \mathcal{I}(E,B)$. Hence

$$\gamma_{\mathcal{I}}(T_{\bar{A}_{(\alpha,\beta),q;K},B}) \leq C\inf_{t,s>0}\left\{\sum_{j=1}^{N}t^{\alpha-x_j}s^{\beta-y_j}\gamma_{\mathcal{I}}(T_{A_j,B})\right\}$$

$$\leq NC\inf_{t,s>0}\left\{\max_{1\leq j\leq N}\{t^{\alpha-x_j}s^{\beta-y_j}\gamma_{\mathcal{I}}(T_{A_j,B})\}\right\}$$

$$= NC\max\left\{\gamma_{\mathcal{I}}(T_{A_i,B})^{c_i}\,\gamma_{\mathcal{I}}(T_{A_k,B})^{c_k}\,\gamma_{\mathcal{I}}(T_{A_r,B})^{c_r} : \{i,k,r\} \in \mathcal{P}\right\}$$

where we have used [8], Thm. 1.9, in the last equality. This establishes a).

To prove b) let again $\sigma_j > \gamma_{\mathcal{I}}(T_{A_j,B})$, and consider the following norm on $\Sigma(\bar{A})$

$$|\!|\!| a |\!|\!| = \inf\left\{\sum_{j=1}^{N} \sigma_j \|a_j\|_{A_j} \ : \ a = \sum_{j=1}^{N} a_j \, , \, a_j \in A_j\right\}.$$

Take any $a \in U_{\bar{A}_{(\alpha,\beta),q;J}}$ and $\varepsilon > 0$. Using the Hahn-Banach theorem, we can find $f \in \left(\Sigma(\bar{A}), |\!|\!| \cdot |\!|\!|\right)^*$ such that $f((1+\varepsilon)^{-1}a) = |\!|\!|(1+\varepsilon)^{-1}a|\!|\!|$ and $\|f\|_{A_j^*} \le \sigma_j \, , \, 1 \le j \le N$. By (4), the norm $\|f\|_{(\bar{A}_{(\alpha,\beta),q;J})^*}$ of the restriction of f to $\bar{A}_{(\alpha,\beta),q;J}$ is less than or equal to $C\prod_{j=1}^{N} \sigma_j^{\theta_j}$. Whence

$$\begin{aligned} |\!|\!| a |\!|\!| &= (1+\varepsilon)|f((1+\varepsilon)^{-1}a)| \\ &\le (1+\varepsilon)C\prod_{j=1}^{N} \sigma_j^{\theta_j}\|(1+\varepsilon)^{-1}a)\|_{(\alpha,\beta),q;J} < (1+\varepsilon)C\prod_{j=1}^{N} \sigma_j^{\theta_j}. \end{aligned}$$

This allows us to find a representation $a = \sum_{j=1}^{N} a_j$ of a with $\|a_j\|_{A_j} \le (1+\varepsilon)C\sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j - 1} \ldots \sigma_N^{\theta_N} \, , \, 1 \le j \le N$. Choosing again Banach spaces E_j and operators $R_j \in \mathcal{I}(E_j, B)$ with

$$T\left(U_{A_j}\right) \subseteq \sigma_j U_B + R_j\left(U_{E_j}\right), \quad 1 \le j \le N,$$

it follows that

$$T\left(U_{\bar{A}_{(\alpha,\beta),q;J}}\right) \subseteq (1+\varepsilon)C\sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j - 1} \ldots \sigma_N^{\theta_N} T(U_{A_j})$$

$$\subseteq (1+\varepsilon)CN\sigma_1^{\theta_1} \ldots \sigma_N^{\theta_N} U_B + (1+\varepsilon)C\sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j - 1} \ldots \sigma_N^{\theta_N} R_j(U_{E_j})$$

$$\subseteq (1+\varepsilon)CN\sigma_1^{\theta_1} \ldots \sigma_N^{\theta_N} U_B + R(U_E)$$

where $E = \left(\oplus_{j=1}^{N} E_j\right)_{\ell_\infty}$ and $R \in \mathcal{I}(E, B)$ is the operator defined by

$$R(z_1, \ldots, z_N) = (1+\varepsilon)C\sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j - 1} \ldots \sigma_N^{\theta_N} R_j z_j.$$

Consequently

$$\gamma_{\mathcal{I}}(T_{\bar{A}_{(\alpha,\beta),q;J},B}) \leq CN \prod_{j=1}^{N} \gamma_{\mathcal{I}}(T_{A_j,B})^{\theta_j}.$$

To proceed to c) and d), assume that $T \in \mathcal{L}(B, \Delta(\bar{A}))$ and let $\sigma_j > \beta_{\mathcal{I}}(T_{B,A_j})$, $1 \leq j \leq N$. By the definition of $\beta_{\mathcal{I}}$, we can find Banach spaces F_j and operators $S_j \in \mathcal{I}(B, F_j)$ so that

$$\|Tb\|_{A_j} \leq \sigma_j \|b\|_B + \|S_j b\|_{F_j}, \quad b \in B.$$

Put $F = \left(\oplus_{j=1}^{N} F_j\right)_{\ell_1}$, $\sigma = \min\{\sigma_1, \dots . \sigma_N\}$ and let $S \in \mathcal{I}(B, F)$ be the operator defined by

$$Sb = \max\left\{\sigma_i^{c_i}\sigma_k^{c_k}\sigma_r^{c_r} \; : \; \{i,k,r\} \in \mathcal{P}\right\} \sigma^{-1}(S_1 b, \dots, S_N b).$$

Using (6) we get that

$$\begin{aligned} \|Tb\|_{(\alpha,\beta),q;J} &\leq C \max\left\{\|Tb\|_{A_i}^{c_i} \, \|Tb\|_{A_k}^{c_k} \, \|Tb\|_{A_r}^{c_r} \; : \; \{i,k,r\} \in \mathcal{P}\right\} \\ &\leq C \max\left\{\sigma_i^{c_i}\sigma_k^{c_k}\sigma_r^{c_r} \; : \; \{i,k,r\} \in \mathcal{P}\right\} \|b\|_B + C\|Sb\|_F, \end{aligned}$$

and c) follows.

Finally, working with the operator $V \in \mathcal{I}(B, F)$ given by

$$Vb = \sigma^{-1}\left(\prod_{j=1}^{N} \sigma_j^{\theta_j}\right)(S_1 b, \dots, S_N b)$$

and using (5), we derive that

$$\begin{aligned} \|Tb\|_{(\alpha,\beta),q;K} &\leq C \prod_{j=1}^{N} \|Tb\|_{A_j}^{\theta_j} \leq C \prod_{j=1}^{N} \left(\sigma_j \|b\|_B + \|S_j b\|_{F_j}\right)^{\theta_j} \\ &\leq C \prod_{j=1}^{N} \sigma_j^{\theta_j} \left(\|b\|_B + \frac{1}{\sigma}\|R_j b\|_{F_j}\right)^{\theta_j} \leq C\left(\prod_{j=1}^{N} \sigma_j^{\theta_j}\right)\|b\|_B + C\|Vb\|_F. \end{aligned}$$

This implies d) and completes the proof.

■

Writing down Theorem 2.1 for the case $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators, we get a quantitative version of Thms 2.3 and 2.4 in [5]. For $\mathcal{I} = \mathcal{K}$, the ideal of compact operators, we obtain estimates for the measure of non-compactness of the interpolated operator that are analogous to those proved in [7], Prop. 3.1 and 3.3 for entropy numbers. Recall that the measure of non-compactness is the limit of the sequence of entropy numbers. Theorem 2.1 can be also applied to derive results on strict singularity and cosingularity.

3 Estimates for the general case

We deal now with the case of non-degenerated N-tuples. It is not difficult to show by means of examples that Theorem 2.1 fails in this general case. However, assuming an extra condition on the operator ideal $\mathcal{I}$, we shall be able to describe the behaviour of the ideal variations.

Given any sequence of Banach spaces $(Z_{m,n})_{(m,n)\in \mathbf{Z}^2}$, any sequence of non-negative numbers $(\lambda_{m,n})_{(m,n)\in \mathbf{Z}^2}$ and $1 < q < \infty$, we denote by $\ell_q(\lambda_{m,n}Z_{m,n})$ the vector-valued ℓ_q space defined by

$$\ell_q(\lambda_{m,n}Z_{m,n}) = \Big\{ z = (z_{m,n}) : z_{m,n} \in Z_{m,n} \quad \text{and}$$

$$\|z\|_{\ell_q(\lambda_{m,n}Z_{m,n})} = \left(\textstyle\sum_{(m,n)\in \mathbf{Z}^2} (\lambda_{m,n}\|z_{m,n}\|_{Z_{m,n}})^q\right)^{\frac{1}{q}} < \infty \Big\}.$$

Any operator $T \in \mathcal{L}\left(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})\right)$ between two vector-valued ℓ_q spaces can be imagined as an infinite matrix with entries $Q_{r,s}TP_{u,v}$. Here $P_{u,v} : \lambda_{u,v}Z_{u,v} \longrightarrow \ell_q(\lambda_{m,n}Z_{m,n})$ is the embedding $P_{u,v}z = (\delta^{u,v}_{m,n}z)$, where

$$\delta^{u,v}_{m,n} = \begin{cases} 1 & \text{if } m = u, n = v \\ 0 & \text{otherwise} \end{cases}, \text{ and } Q_{r,s} : \ell_q(\mu_{m,n}Y_{m,n}) \longrightarrow \mu_{r,s}Y_{r,s} \text{ is the}$$

projection $Q_{r,s}(y_{m,n}) = y_{r,s}$.

For $1 < q < \infty$, we say that the operator ideal $\mathcal{I}$ satisfies the Σ_q-condition if for any sequences of Banach spaces

$$(\lambda_{m,n}Z_{m,n}),\ (\mu_{m,n}Y_{m,n}) \text{ and any } T \in \mathcal{L}\left(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})\right),$$

it follows from $Q_{r,s}TP_{u,v} \in \mathcal{I}\left(\lambda_{u,v}Z_{u,v}, \mu_{r,s}Y_{r,s}\right)$ for any r, s, u, v that

$$T \in \mathcal{I}\left(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})\right).$$

Weakly compact operators, Rosenthal operators, Banach-Saks operators or dual Radon-Nikodym operators are examples of ideals satisfying the Σ_q-condition (see [14]). All of them are also injective surjective and closed.

The following result shows the behaviour of the measure $\gamma_{\mathcal{I}}$ with K-spaces.

Theorem 3.1. *Let* $\Pi = \overline{P_1 \dots P_N}$ *be a convex polygon with vertices* $P_j = (x_j, y_j)$, *let* $(\alpha,\beta) \in$ *Int* Π, $1 < q < \infty$, *and let* $\mathcal{I}$ *be an operator ideal which satisfies the* Σ_q*-condition. Assume that* $\bar{A} = \{A_1, \dots, A_N\}$ *and* $\bar{B} = \{B_1, \dots, B_N\}$ *are Banach* N*-tuples and let* $T \in \mathcal{L}(\bar{A}, \bar{B})$. *Then for the interpolated operator we have*

$$\gamma_{\mathcal{I}}\left(\left[J_{\bar{B}_{(\alpha,\beta),q;K}}T\right]_{\bar{A}_{(\alpha,\beta),q;K},\ell_\infty(U_{\bar{B}^*_{(\alpha,\beta),q;K}})}\right)$$

$$\leq D\ \max\left\{\gamma_{\mathcal{I}}(T_{A_i,B_i})^{c_i}\ \gamma_{\mathcal{I}}(T_{A_k,B_k})^{c_k}\ \gamma_{\mathcal{I}}(T_{A_r,B_r})^{c_r}\ :\ \{i,k,r\} \in \mathcal{P}\right\}$$

where D *is a constant depending only on* Π *and* (α,β).

Proof. Let $F_{m,n} = (B_1 + \ldots + B_N, K(2^m, 2^n; \cdot))$, $(m,n) \in \boldsymbol{Z}^2$, and form the vector-valued space $\ell_q(2^{-\alpha m - \beta n} F_{m,n})$. The map $j : \bar{B}_{(\alpha,\beta),q;K} \longrightarrow \ell_q(2^{-\alpha m-\beta n}F_{m,n})$ defined by $jb = (\dots, b, b, b, \dots)$ is an isometric embedding. By (1.1), it is then enough to show the inequality for jT.

Let $\sigma_j > \gamma_{\mathcal{I}}(T_{A_j,B_j})$ and find Banach spaces E_j and operators $R_j \in \mathcal{I}(E_j, B_j)$ so that

$$T\left(U_{A_j}\right) \subseteq \sigma_j U_{B_j} + R_j\left(U_{E_j}\right), \quad j = 1, \dots, N. \tag{7}$$

Put

$$W_{m,n} = (E_1 \oplus \ldots \oplus E_N)_{\ell_\infty}, \quad (m,n) \in \boldsymbol{Z}^2$$

and, for $\delta > 0$ and $(r,s) \in \boldsymbol{Z}^2$, consider the operator
$R : \ell_q(W_{m,n}) \longrightarrow \ell_q\left(2^{-\alpha m - \beta n} F_{m,n}\right)$ defined by

$$R\left(z_1^{m,n}, \dots, z_N^{m,n}\right) = \left(\sum_{j=1}^{N} (1+\delta) 2^{(\alpha - x_j)(m+r)} 2^{(\beta - y_j)(n+s)} R_j z_j^{m,n}\right).$$

This operator is bounded because

$$\|R\left(z_1^{m,n},\ldots,z_N^{m,n}\right)\|_{\ell_q\left(2^{-\alpha m-\beta n}F_{m,n}\right)}$$

$$\leq \Big(\sum_{(m,n)\in \mathbf{Z}^2} \Big(2^{-\alpha m-\beta n}\sum_{j=1}^{N}(1+\delta)2^{mx_j+ny_j}2^{(\alpha-x_j)(m+r)}$$

$$.2^{(\beta-y_j)(n+s)}\|R_j\|_{E_j,B_j}\,\|z_j^{m,n}\|_{E_j}\Big)^q\Big)^{\frac{1}{q}}$$

$$\leq (1+\delta)\,N\max_{1\leq j\leq N}\left\{2^{(\alpha-x_j)r}2^{(\beta-y_j)s}\|R_j\|_{E_j,B_j}\right\}\|\left(z_1^{m,n},\ldots,z_N^{m,n}\right)\|_{\ell_q(W_{m,n})}.$$

Moreover, since each entry
$Q_{t,w}RP_{u,v}(z_1,\cdots,z_N)=$

$$\begin{cases} 0 & \text{if } (t,w)\neq(u,v)\\ \displaystyle\sum_{j=1}^{N}(1+\delta)2^{(\alpha-x_j)(t+r)}2^{(\beta-y_j)(w+s)}R_jz_j & \text{if } (t,w)=(u,v)\end{cases}$$

belongs to $\mathcal{I}(W_{u,v},2^{-\alpha t-\beta w}F_{t,w})$, the Σ_q-property implies that

$$R\in\mathcal{I}\left(\ell_q(W_{m,n}),\ell_q(2^{-\alpha m-\beta n}F_{m,n})\right).$$

We claim that
$jT\left(U_{\bar{A}_{(\alpha,\beta),q;K}}\right)$

$$\subseteq\left[N(1+\delta)\max_{1\leq j\leq N}\left\{2^{r(\alpha-x_j)+s(\beta-y_j)}\right\}\right]U_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}+R\left(U_{\ell_q(W_{m,n})}\right).$$

Indeed, given any $a\in U_{\bar{A}_{(\alpha,\beta),q;K}}$ we can choose $d_{m,n}=d_{m,n}(a)>0$ with

$$2^{-\alpha m-\beta n}K(2^m,2^n;a)<d_{m,n}\quad\text{and}\quad\sum_{(m,n)\in\mathbf{Z}^2}d_{m,n}^q\leq(1+\delta)^q.$$

Since

$$K\left(2^{m+r},2^{n+s};a\right)<2^{\alpha(m+r)}2^{\beta(n+s)}d_{m+r,n+s}$$

we can find a decomposition $a=\sum_{j=1}^N a_j^{m,n}$ with $a_j^{m,n}\in A_j$ and

$$2^{(m+r)x_j}2^{(n+s)y_j}\|a_j^{m,n}\|_{A_j}\leq 2^{\alpha(m+r)}2^{\beta(n+s)}d_{m+r,n+s}.$$

Put

$$\rho_j^{m,n} = 2^{(m+r)x_j} 2^{(n+s)y_j}\,,\ 1 \le j \le N\,; \quad \rho_0^{m,n} = 2^{\alpha(m+r)} 2^{\beta(n+s)} d_{m+r,n+s}.$$

By (7), we can choose $z_j^{m,n} \in U_{E_j}$ such that

$$\|T(\frac{\rho_j^{m,n}}{\rho_0^{m,n}} a_j^{m,n}) - R_j z_j^{m,n}\|_{B_j} \le \sigma_j.$$

In other words,

$$\|T a_j^{m,n} - \frac{\rho_0^{m,n}}{\rho_j^{m,n}} R_j z_j^{m,n}\|_{B_j} \le \frac{\rho_0^{m,n}}{\rho_j^{m,n}} \sigma_j = 2^{(m+r)(\alpha-x_j)} 2^{(n+s)(\beta-y_j)} \sigma_j d_{m+r,n+s}.$$

Let

$$z = \left((1+\delta)^{-1} d_{m+r,n+s}\, z_1^{m,n}, \ldots, (1+\delta)^{-1} d_{m+r,n+s}\, z_N^{m,n}\right).$$

Then $z \in U_{\ell_q(W_{m,n})}$ and
$\|(jT)a - Rz\|^q_{\ell_q(2^{-\alpha m-\beta n} F_{m,n})}$

$$\le \sum_{(m,n)\in Z^2} \left[2^{-\alpha m-\beta n} \left(\sum_{j=1}^{N} 2^{m x_j + n y_j} \|T a_j^{m,n} - \frac{\rho_0^{m,n}}{\rho_j^{m,n}} R_j z_j^{m,n}\|_{B_j}\right)\right]^q$$

$$\le \sum_{(m,n)\in Z^2} \left[2^{-\alpha m-\beta n} \left(\sum_{j=1}^{N} 2^{m x_j + n y_j} 2^{(m+r)(\alpha-x_j)+(n+s)(\beta-y_j)} \sigma_j d_{m+r,n+s}\right)\right]^q$$

$$\le \left[N \max_{1\le j\le N} \left\{2^{r(\alpha-x_j)+s(\beta-y_j)} \sigma_j\right\}\right]^q \sum_{(m,n)\in Z^2} d^q_{m+r,n+s}$$

$$\le \left[N(1+\delta) \max_{1\le j\le N} \left\{2^{r(\alpha-x_j)+s(\beta-y_j)} \sigma_j\right\}\right]^q.$$

Whence

$$\gamma_{\mathcal{I}}(jT) \le N(1+\delta) \max_{1\le j\le N} \left\{2^{r(\alpha-x_j)+s(\beta-y_j)} \sigma_j\right\}.$$

Here $\delta > 0$ and $(r,s) \in \boldsymbol{Z}^2$ are arbitrary. Therefore we derive that

$$\gamma_{\mathcal{I}}(jT) \le N \inf_{(r,s)\in \boldsymbol{Z}^2} \left[\max_{1\le j\le N} \left\{2^{r(\alpha-x_j)+s(\beta-y_j)} \sigma_j\right\}\right]$$

$$\leq D \inf_{t,s>0} \left[\max_{1\leq j\leq N} \left\{ t^{\alpha - x_j} s^{\beta - y_j} \sigma_j \right\} \right]$$
$$= D \max \{ \sigma_i^{c_i}\, \sigma_k^{c_k} \sigma_r^{c_r} \ : \ \{i,k,r\} \in \mathcal{P} \}$$

where we have used [8], Thm. 1.9, in the last equality. This implies that

$$\gamma_{\mathcal{I}}(jT) \leq D \max \{ \gamma_{\mathcal{I}}(T_{A_i,B i})^{c_i}\, \gamma_{\mathcal{I}}(T_{A_k,B_k})^{c_k}\, \gamma_{\mathcal{I}}(T_{A_r,B_r})^{c_r} \ : \ \{i,k,r\} \in \mathcal{P} \}$$

and completes the proof. ■

The operator $J_{\bar{B}_{(\alpha,\beta),q;K}}$ is essential in Theorem 3.1 as we show next by means of an example. We adapt an idea of [9], Remark 3.4.

Let $\mathcal{I} = \mathcal{W}$ the ideal of weakly compact operators. According to [2], Thm. 4, there is a Banach space E and a sequence of operators $(R_n)_{n=1}^{\infty} \subseteq \mathcal{L}(E, c_0)$ such that

$$\gamma_{\mathcal{W}}(R_n^{**}) \leq \gamma_{\mathcal{W}}(R_n) \leq 1/n, \tag{8}$$
$$\gamma_{\mathcal{W}}(R_n^{*}) = 1. \tag{9}$$

Put

$$T_n = Q_E^* R_n^* \quad , \quad F = Q_E^*(E^*) \, ,$$

choose Π as the simplex $\{(0,0),(1,0),(0,1)\}$ and consider the 3-tuples

$$\bar{A} = \{\ell_1, \ell_1, \ell_1\} \quad , \quad \bar{B} = \{F, F, \ell_\infty(U_E)\} \, .$$

Let $\alpha > 0$, $\beta > 0$ with $\alpha + \beta < 1$ (i.e. $(\alpha,\beta) \in$ Int Π) and $1 < q < \infty$. It is clear that $\bar{A}_{(\alpha,\beta),q;K} = \ell_1$ with equivalence of norms. Moreover $\bar{B}_{(\alpha,\beta),q;K} = F$ (equivalent norms) because F is a closed subspace of $\ell_\infty(U_E)$. Hence, if Theorem 3.1 would be true without $J_{\bar{B}_{(\alpha,\beta),q;K}}$, there would exist a constant $D > 0$ such that for any $n \in \mathbb{N}$

$$\gamma_{\mathcal{W}}\left([T_n]_{\ell_1,F}\right) \tag{10}$$
$$\leq D \gamma_{\mathcal{W}}\left([T_n]_{\ell_1,F}\right)^{1-\alpha-\beta} \gamma_{\mathcal{W}}\left([T_n]_{\ell_1,F}\right)^{\alpha} \gamma_{\mathcal{W}}\left([T_n]_{\ell_1,\ell_\infty(U_E)}\right)^{\beta} .$$

But $Q_E^* : E^* \longrightarrow F$ is an isometry onto, so (9) yields

$$\gamma_{\mathcal{W}}\left([T_n]_{\ell_1,F}\right) = \gamma_{\mathcal{W}}\left([R_n^*]_{\ell_1,E^*}\right) = 1.$$

On the other hand, by (8) and [1], Cor. 5.3, we get

$$\gamma_{\mathcal{W}}\left([T_n]_{\ell_1,\ell_\infty(U_E)}\right) = \gamma_{\mathcal{W}}(T_n^*) = \gamma_{\mathcal{W}}(R_n^{**}) \leq 1/n.$$

Whence (10) reads

$$1 \leq D n^{-\beta} \quad \text{for any } n \in \boldsymbol{N}$$

which is impossible.

Our last result describe the behaviour of $\beta_{\mathcal{I}}$ with J-spaces.

Theorem 3.2. *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in$ Int Π, $1 < q < \infty$, and let $\mathcal{I}$ be an operator ideal which satisfies the Σ_q-condition. Assume that $\bar{A} = \{A_1, \dots, A_N\}$ and $\bar{B} = \{B_1, \dots, B_N\}$ are Banach N-tuples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Then for the interpolated operator we have*

$$\beta_{\mathcal{I}}\left(\left[TQ_{\bar{A}_{(\alpha,\beta),q;J}}\right]_{\ell_1(U_{\bar{A}_{(\alpha,\beta),q;J}}),\bar{B}_{(\alpha,\beta),q;J}}\right)$$

$$\leq D \max\left\{\beta_{\mathcal{I}}(T_{A_i,B_i})^{c_i}\, \beta_{\mathcal{I}}(T_{A_k,B_k})^{c_k}\, \beta_{\mathcal{I}}(T_{A_r,B_r})^{c_r} \;:\; \{i,k,r\} \in \mathcal{P}\right\}$$

where D is a constant depending only on Π and (α, β).

Proof. Put $G_{m,n} = (A_1 \cap \dots \cap A_N, J(2^m, 2^n; \cdot))$, $(m,n) \in \boldsymbol{Z}^2$, and let

$$\pi : \ell_q\left(2^{-\alpha m - \beta n} G_{m,n}\right) \longrightarrow \bar{A}_{(\alpha,\beta),q;J}$$

be the metric surjection $\pi(u_{m,n}) = \sum_{m,n \in \boldsymbol{Z}^2} u_{m,n}$. Taking into account (2), it suffices to establish the inequality for $T\pi$.

Let $\sigma_j > \beta_{\mathcal{I}}(T_{A_j,B_j})$. There exist Banach spaces Z_j and operators $S_j \in \mathcal{I}(A_j, Z_j)$ such that

$$\|Tx\|_{B_j} \leq \sigma_j \|x\|_{A_j} + \|S_j x\|_{Z_j}\,,\; x \in A_j\,,\; 1 \leq j \leq N. \tag{11}$$

For each $(m,n) \in \boldsymbol{Z}^2$, let $V_{m,n} = (E_1 \oplus \dots \oplus E_N)_{\ell_1}$. Take any $(r,s) \in \boldsymbol{Z}^2$ and let $S : \ell_q\left(2^{-\alpha m - \beta n} G_{m,n}\right) \longrightarrow \ell_q(V_{m,n})$ be the operator defined by $S(u_{m,n}) =$

$$\left(2^{(x_1-\alpha)(m-r)} 2^{(y_1-\beta)(n-s)} S_1 u_{m,n}, \dots, 2^{(x_N-\alpha)(m-r)} 2^{(y_N-\beta)(n-s)} S_N u_{m,n}\right).$$

Since

$\|S(u_{m,n})\|_{\ell_q(V_{m,n})} =$

$$\left(\sum_{(m,n)\in \boldsymbol{Z}^2} \left(\sum_{j=1}^{N} 2^{(x_j-\alpha)(m-r)} 2^{(y_j-\beta)(n-s)} \|S_j u_{m,n}\|_{Z_j}\right)^q\right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{j=1}^{N} 2^{(\alpha-x_j)r} 2^{(\beta-y_j)s} \|S_j\|_{A_j,Z_j} \right) \|(u_{m,n})\|_{\ell_q(2^{-\alpha m-\beta n}G_{m,n})},$$

the operator S is bounded. Now, by the Σ_q-property, it is easy to check that $S \in \mathcal{I}\left(\ell_q\left(2^{-\alpha m-\beta n}G_{m,n}\right), \ell_q\left(V_{m,n}\right)\right)$. A direct computation using (11) shows that
$\|T\pi(u_{m,n})\|_{\bar{B}_{(\alpha,\beta),q;J}}$

$$\leq \max_{1\leq j\leq N} \left\{\sigma_j 2^{(\alpha-x_j)r} 2^{(\beta-y_j)s}\right\} \|(u_{m,n})\|_{\ell_q(2^{-\alpha m-\beta n}G_{m,n})} + \|S(u_{m,n})\|_{\ell_q(V_{m,n})}.$$

This implies that

$$\beta_{\mathcal{I}}(T\pi) \leq \max_{1\leq j\leq N} \left\{\sigma_j 2^{(\alpha-x_j)r} 2^{(\beta-y_j)s}\right\}.$$

Since $(r,s) \in \boldsymbol{Z}^2$ is arbitrary, taking infimum and using [8], Thm. 1.9, the result follows.

■

Theorems 3.1 and 3.2 comprise Thm. 2.6 and Remark 2.9 of [5]. In particular, they give quantitative estimates for the weak compactness results mentioned in the Introduction.

Note that Theorems 3.1 and 3.2 do not apply to compact operators because this ideal fails the Σ_q-condition. This problem has been studied in [6] and [7].

References

[1] K. Astala, *On measures of non-compactness and ideal variations in Banach spaces*, Ann. Acad. Sci. Fenn. Ser. A. I Math. Dissertationes **29** (1980) 1-42.

[2] K. Astala and H.-O. Tylli, *Seminorms related to weak compactness and to Tauberian operators*, Math. Proc. Camb. Phil. Soc. **107** (1990) 367-375.

[3] J. Bergh and J. Löfström, "*Interpolation Spaces. An Introduction*", Springer, Berlin-Heidelberg-New York, 1976.

[4] M. J. Carro and L. I. Nikolova, *Interpolation of limited and weakly compact operators on families of Banach spaces*, Acta Appl. Math. **49** (1997) 151-177.

[5] F. Cobos, P. Fernández-Martínez and A. Martínez, *On the behavior of weak compactness under certain interpolation methods*, Collectanea Math. (to appear).

[6] F. Cobos, P. Fernández-Martínez and A. Martínez, *Interpolation of the measure of non-compactness by the real method*, Studia Math. (to appear).

[7] F. Cobos, P. Fernández-Martínez and A. Martínez, *Measure of non-compactness and interpolation methods associated to polygons*, Glasgow Math. J. **41** (1999) 65-79.

[8] F. Cobos, P. Fernández-Martínez and T. Schonbek, *Norm estimates for interpolation methods defined by means of polygons*, J. Approx. Theory **80** (1995) 321-351.

[9] F. Cobos, A. Manzano and A. Martínez, *Interpolation theory and measures related to operator ideals*, Quarterly J. Math. (to appear).

[10] F. Cobos and A. Martínez, *Remarks on interpolation properties of the measure of weak non-compactness and ideal variations*, Math. Nachr. (to appear).

[11] F. Cobos and A. Martínez, *Extreme estimates for interpolated operators by the real method*, J. London Math. Soc. (to appear).

[12] F. Cobos and J. Peetre, *Interpolation of compact operators: The multidimensional case*, Proc. London Math. Soc. **63** (1991) 371-400.

[13] F. S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **21** (1977) 259-262.

[14] S. Heinrich, *Closed operator ideals and interpolation*, J. Funct. Anal. **35** (1980) 397-411.

[15] H. Jarchow and U. Matter, *Interpolative constructions for operator ideals*, Note di Mat. **8** (1988) 45-56.

[16] A. Lebow and M. Schechter, *Semigroups of operators and measures of non-compactness*, J. Funct. Anal. **7** (1971) 1-26.

[17] A. Pietsch, *"Operator Ideals"*, North-Holland, Amsterdam 1980.

[18] H. Triebel, *"Interpolation Theory, Function Spaces, Differential Operators"*, North-Holland, Amsterdam, 1978.

[19] H.-O. Tylli, *The essential norm of an operator is not self-dual*, Israel J. Math. **91** (1995) 93-110.

Fernando Cobos
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
e-mail addres: `cobos@eucmax.sim.ucm.es`

José María Cordeiro
Departamento de Matemática Aplicada
E.T.S. Ingenieros Industriales
Universidad de Vigo
36200 Vigo, Spain
e-mail addres: `cordeiro@uvigo.es`

Antón Martínez
Departamento de Matemática Aplicada
E.T.S. Ingenieros Industriales
Universidad de Vigo
36200 Vigo, Spain
e-mail addres: `antonmar@uvigo.es`

Recibido: 14 de Octubre de 1998