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Magneto-micropolar fluid motion: existence of weak solutions.

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Abstract

By using the Galerkin method, we prove the existence of weak solutions for the equations of the magneto-micropolar fluid motion in two and three dimensions in space. In the two-dimensional case, we also prove that such weak solution is unique. We also prove the reproductive property.

1 Introduction

In this work we study global existence of weak solutions for the equations that describes the motion of a viscous incompressible magneto-micropolar fluid in a bounded domain $\Omega \subseteq \mathbb{R}^n, n = 2$ or 3, in a time interval $[0,T], 0 < T < +\infty$. Such equation are given by (see [1], for instance):

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u.\nabla u - (\mu+\chi)\Delta u + \nabla(p + \frac{1}{2}h.h) = \chi \operatorname{rot} w + rh.\nabla h + f \\
&j\frac{\partial w}{\partial t} + ju.\nabla w - \gamma\Delta w + 2\chi w - (\alpha+\beta)\nabla \operatorname{div} w = \chi \operatorname{rot} u + g \quad (1.1) \\
&\frac{\partial h}{\partial t} - \nu\Delta h + u.\nabla h - h.\nabla u = 0 \\
&\operatorname{div} u = 0, \quad \operatorname{div} h = 0 \text{ in } \Omega.
\end{aligned}$$

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Here, $u(t,x) \in \mathbb{R}^n$ denotes the velocity of the fluid at a point $x \in \Omega$ and time $t \in [0,T]$; $w(t,x) \in \mathbb{R}^n, h(t,x) \in \mathbb{R}^n$ and $p(t,x) \in \mathbb{R}$ denote, respectively, the microrotational velocity, the magnetic field and the hydrostatic pressure; the constants μ, χ, r, α, β, γ, j and ν are constants associated to properties of the material. From physical reasons, these constants satisfy $\min\{\mu,\ \chi,\ r,\ j,\ \gamma,\ \nu,\ \alpha+\beta+\gamma\} > 0$; $f(t,x)$ and $g(t,x) \ \in \mathbb{R}^n$ are given external fields.

We assume that on the boundary $\partial\Omega$ of Ω, the following conditions hold

$$u(t,x) = w(t,x) = h(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega \tag{1.2}$$

(we shall consider homogeneous boundary conditions just for simplicity). The initial conditions are

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad h(0,x) = h_0(x), \quad x \in \Omega. \tag{1.3}$$

Equation (1.1)(i) has the familiar form of the Navier-Stokes equations but it is coupled with equation (1.1)(ii), which essentially describes the motion inside the macrovolumes as they undergo microrotational effects represented by the microrotational velocity vector w. For fluids with no microstructure this parameter vasnishes. For Newtonian fluids, equations (1.1)(i) and (1.1)(ii) decouple since $\chi = 0$.

It is now appropriate to cite some earlier works on the initial boundary-value problem (1.1)-(1.3), which are related to ours and also to locate our contribution therein. When the magnetic field is absent ($h \equiv 0$), the reduced problem was studied by Lukaszewicz [6, 7], Galdi and Rionero [2] and Padula and Russo [8]. Lukaszewicz [6] established the global existence of weak solutions for (1.1) - (1.3) under certain assumptions by using linearization and an almost fixed point theorem. In the same case, by using the same technique, Lukaszewicz [7] also proved the local and global existence, as well as the uniqueness, of strong solutions. Again when $h \equiv 0$, Galdi and Rionero [2] established results similar to the ones of Lukaszewicz [6]. Padula and Russo [8] studied the uniqueness of the solutions for problem (1.1)-(1.3) in unbounded domains.

In a recently work, Kagei and Skowron [3] studied the reduced problem (with $h \equiv 0$) coupled with an equation of thermal convection. They used arguments analogous to the ones by Lukaszewicz [6], [7].

The full system (1.1)-(1.3) was studied by Galdi and Rionero [2], and they stated without proofs results of existence and uniqueness of strong solutions. Rojas-Medar [9] also studied the system (1.1)-(1.3) and established the existence and uniqueness of strong solutions by using the spectral Galerkin method, reaching the same level of knowledge as in the case of the classic Navier-Stokes equations for strong solutions. Ahmadi and Shahinpoor [1] studied the stability of solutions of the system (1.1)-(1.3).

In this work, we use the Galerkin method, as in Lions [5], to prove the global existence of weak solutions for $n = 2$ or 3; in the two-dimensional case, we also prove the uniqueness of solutions.

Let $\{u, w, h\}$ be a weak solution of (1.1)-(1.2) (the exact definition will be given later on). If the functions u, w and h satisfy the following conditions:

$$u(0,x) = u(T,x),\ w(0,x) = w(T,x),\ h(0,x) = h(T,x), \qquad (1.4)$$

then we say that the system has the *reproductive property* (see Kaniel and Shinbrot [4] for the case of Navier-Stokes equations). We observe that the above property is a generalization of the notion of periodicity. We will show that (1.1)-(1.2) has always a weak solution with the reproductive property.

We reach in this way, for weak solutions, basically the same level of knowledge as in the case of the classic Navier-Stokes equations.

Finally, the paper is organized as follows: in Section 2 we state the basic assumptions and results that to used later on in the paper; we also rewrite (1.1) - (1.3) in a more suitable weak form; we describe the approximation method and state our results (Theorems 2.1, 2.2 and 2.3). Each one of the following sections will be devoted to their proofs.

2 Preliminaries and results

Let $\Omega \subseteq \mathbb{R}^n, n = 2$ or 3, be a bounded domain with smooth boundary $\partial\Omega$. We denote by $L^p(\Omega)$ the usual Lebesgue spaces and by $\|.\|_{L^p}$ the L^p-norm on Ω; in the case $p = 2$, we simply denote the L^2-norm by $|.|$ and the corresponding inner product by $(\cdot,\cdot)$. When B is a Banach space, we denote by $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval $(0,T)$ that are L^q-integrables in the sense of Bochner. The Sobolev spaces $H^s(\Omega)$, $H_0^s(\Omega)$ (with $s \in \mathbb{R}$) are defined as usual; we denote by $\|.\|_s$ and $(.,.)_{H^s}$, respectively the norm and the inner product in $H^s(\Omega)$ (or $H_0^s(\Omega)$) when appropriate). We also will use the following solenoidal function spaces

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{v \in (C_0^\infty(\Omega))^n \ / \ \operatorname{div} v = 0 \quad \text{in} \quad \Omega\} \\ H = &\quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \quad \text{under the} \quad (L^2)^n - \text{norm}, \\ V_s = &\quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \text{ under the } (H^s)^n - \text{norm}. \end{aligned}$$

In the special case where $s = 1$, we denote V_1 simply by V. The norm and inner product in H and V_s are

$$(f, g) = \sum_{i=1}^n \int_\Omega f_i \, g_i dx, \quad |f| = (f, f)^{1/2}$$

and

$$(u, v)_s = \sum_{i=1}^n (u_i, v_i)_{H^s} \quad , \|u\|_s = (u, u)_s^{1/2}.$$

If X is a Banach space, X^* will denote its topological dual.

We observe that V is characterized by

$$V = \{v \in (H_0^1(\Omega))^n \ / \ \operatorname{div} v = 0 \ \text{ in } \ \Omega\},$$

and, consequently, the H^1-norm and H_0^1-norm are equivalent for $u \in V$. We denote $\|u\| = |\nabla u|$.

On the other hand, let us denote

$$\begin{aligned} a(v, w) &= \sum_{i,j=1}^n \int_\Omega \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} \, dx, \\ b(u, v, w) &= \sum_{i,j=1}^n \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx, \\ Lw &= -\gamma \Delta w - (\alpha + \beta) \nabla \operatorname{div} w, \end{aligned}$$

which we define for all vector-valued functions u, v, w, for which the integrals are well defined.

To ease the notation, in the following we will denote with the same symbols the scalar and vector valued functional spaces. The distinction will be clear from the context. Also, in what follows, most of the time, we will denote by u_t, w_t and h_t the time derivatives of u, w and h.

We can now define a notion of weak solution for (1.1)-(1.3).

Definition. *Let $u_0, h_0 \in H$ and $w_0 \in L^2(\Omega)$; we will say that a triple of functions (u, w, h) defined on $(0,T)\times\Omega$ is a weak solution of* (1.1)-(1.3) *if only if the functions u, w, h satisfy*

$$u, h \in L^2(0,T;V)\cap L^\infty(0,T;H),$$
$$w \in L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)),$$

and also satisfy the following equations

$$(u_t,\varphi)+(\mu+\chi)a(u,\varphi)+b(u,u,\varphi)-rb(h,h,\varphi)=(f,\varphi)+\chi(\operatorname{rot} w,\varphi) \tag{2.1}$$

$$j(w_t,\phi)+(Lw,\phi)+2\chi(w,\phi)+jb(u,w,\phi)=(g,\phi)+\chi(\operatorname{rot} u,\phi) \tag{2.2}$$

$$(h_t,\psi)+\nu a(h,\psi)+b(u,h,\psi)-b(h,u,\psi)=0 \tag{2.3}$$

for all $\varphi,\psi\in V$ and $\phi\in H_0^1(\Omega)$ and also the initial conditions (1.3) *(as it usual, the above regularity condition is enough to guarantee that* (1.3) *has a meaning).*

If one does not assume (1.3), we will say that (u, w, h) is a weak solution of (1.1)-(1.2).

Remark. Usually, as in the case of the classical Navier-Stokes equations, one should assume $\varphi,\psi\in V\cap L^n(\Omega)$ and $\phi\in H_0^1(\Omega)\cap L^n(\Omega)$. However, since in this paper $n=2$ or 3, this is not necessary.

To prove the existence of solutions of system (1.1)-(1.3) we will use the Galerkin method. We fix $s=n/2, n=2$ or 3, and we consider the special basis $\{\varphi^i(x)\}_{i=1}^\infty$ of $V_{n/2}$ and $\{\phi^i(x)\}_{i=1}^\infty$ of $H_0^{n/2}(\Omega)$, whose elements we choose as the solutions of the spectral problems:

$$(\varphi^i,v)_{n/2}=\lambda_i(\varphi^i,v),\ \forall v\in V_{n/2} \text{ with } |\varphi^i|=1,$$

and

$$(\phi^i,\omega)_{n/2}=\tilde\lambda_i(\phi^i,\omega),\ \forall\omega\in H_0^{n/2}(\Omega) \text{ with } |\phi^i|=1.$$

Let $V^k = span[\varphi^1(x),\dots,\varphi^k(x)]$ and $H_k = span[\phi^1(x),\dots,\phi^k(x)]$; we observe that $V^k\subseteq V_{n/2}$ and $H_k\subseteq H_0^{n/2}(\Omega)$.

For every $k \geq 1$, we define approximations u^k, w^k and h^k of u, w and h, respectively by means of the following finite expansions:

$$\begin{aligned} u^k(t,x) &= \sum_{i=1}^{k} c_{ik}(t)\varphi^i(x), w^k(t,x) = \sum_{i=1}^{k} d_{ik}(t)\phi^i(x), \\ h^k(t,x) &= \sum_{i=1}^{k} e_{ik}(t)\varphi^i(x), \end{aligned}$$

where $c_{ik}, d_{ik}, e_{ik} \in W^{1,1}(0,T)$ and satisfy the following equations (a.e. $t \in [0,T]$):

$$\begin{aligned} (u_t^k, \varphi) + (\mu+\chi)a(u^k,\varphi) + b(u^k,u^k,\varphi) - rb(h^k,h^k,\varphi) \\ = (f,\varphi) + \chi(\operatorname{rot} w^k, \varphi), \end{aligned} \tag{2.4}$$

$$j(w_t^k,\phi) + (Lw^k,\phi) + 2\chi(w^k,\phi) + jb(u^k,w^k,\phi) = (g,\phi) + \chi(\operatorname{rot} u^k,\phi), \tag{2.5}$$

$$(h_t^k,\psi) + \nu a(h^k,\psi) + b(u^k,h^k,\psi) - b(h^k,u^k,\psi) = 0, \tag{2.6}$$

for all $\varphi, \psi \in V^k$ and $\phi \in H_k$, and the following initial conditions:

$$u^k(0) = u_0^k, \quad w^k(0) = w_0^k, \quad h^k(0) = h_0^k \quad \text{in } \Omega \tag{2.7}$$

where $(u_0^k)_k$, $(w_0^k)_k$ and $(h_0^k)_k$ are suitable sequences of functions chosen in V^k, H_k and V^k, respectively, such that $u_0^k \to u_0$ and $h_0^k \to h_0$ in H and $w_0^k \to w_0$ in $L^2(\Omega)$ as $k \to \infty$.

By using these approximation, we will prove the following results.

Theorem 2.1. *If $f \in L^2(0,T;V^*), g \in L^2(0,T;H^{-1}(\Omega))$, $u_0, h_0 \in H, w_0 \in L^2(\Omega)$, then there exists a weak solution (u,w,h) of (1.1)-(1.3).*

Theorem 2.2. *Let $n = 2$. The weak solution (u,w,h) of Theorem 2.1 is unique. Moreover, u, h are almost everywhere equal to a continuous functions from $[0,T]$ to H and w is almost everwhere equal to a continuous function from $[0,T]$ to $L^2(\Omega)$.*

Remark. Exactly as in [5, p. 74], it is possible to prove that, after modification in a set of measure zero in $[0,T]$, the solution refered to in Theorem **2.1** satisfy the following: u, h are continuous from $[0,T] \to V^*_{(n-2)/4}$ and w is continuous from $[0,T] \to H^{-(n-2)/4}$ and also weakly continuous from $[0,T] \to H$ and $[0,T] \to L^2(\Omega)$, respectively.

Theorem 2.3. *If* $f \in L^2(0,T;V^*)$, $g \in L^2(0,T;H^{-1}(\Omega))$, *then there exist a weak solution of (1.1)-(1.2) having the reproductive property* (1.4).

Remark. In the case where we have uniqueness of solutions for the initial value problem, as for example the two dimensional case (Theorem 2.2), if the external forces fields are regular and T-periodics in time, the above Theorem 2.3, furnishes a T-periodic weak solution for (1.1)-(1.2). In fact, it is a strong solution and actually very regular. This is so because we can prove that $u(t), w(t), h(t) \in C^\infty(\Omega)$ for $t > 0$ and any initial data $u_0, h_0 \in H, w_0 \in L^2(\Omega)$. Thus, $u_p(t)$, $w_p(t)$, $h_p(t) \in C^\infty(\Omega)$ for $t \in [T, 2T]$, and, by the T-periodicity, we conclude that $u_p(t) = u_p(t+T) \in C^\infty(\Omega), w_p(t) = w_p(t+T) \in C^\infty(\Omega)$ and $h_p(t) = h_p(t+T) \in C^\infty(\Omega)$ for $t \in [0,T]$. In particular, we must have $u_p(0), w_p(0)$ and $h_p(0) \in C^\infty(\Omega)$.

3 Proof of Theorem 2.1

Setting $\varphi = u^k$, $\phi = w^k$ and $\psi = rh^k$ in (2.4), (2.5) and (2.6), respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}|u^k|^2 + (\mu+\chi)a(u^k,u^k) = \chi(\operatorname{rot} w^k, u^k) + rb(h^k,h^k,u^k) + (f,u^k),$$

$$\frac{j}{2}\frac{d}{dt}|w^k|^2 + \gamma a(w^k,w^k) + 2\chi|w^k|^2 + (\alpha+\beta)|\operatorname{div} w^k|^2$$
$$= \chi(\operatorname{rot} u^k, w^k) + (g, w^k),$$

$$\frac{r}{2}\frac{d}{dt}|h^k|^2 + r\nu a(h^k,h^k) = rb(h^k,u^k,h^k),$$

since $b(\zeta,\varphi,\varphi) = 0$ for $\forall \zeta \in V^k$ and $\varphi \in H_k$.

Adding the above inequalities and observing that $r\, b(h^k,h^k,u^k) + r\, b(h^k,u^k,h^k) = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) + (\mu+\chi)a(u^k,u^k) + \gamma a(w^k,w^k)\\
&\quad + r\nu a(h^k,h^k) + 2\chi|w^k|^2 + (\alpha+\beta)|\operatorname{div} w^k|^2\\
&\quad = \chi(\operatorname{rot} w^k, u^k) + \chi(\operatorname{rot} u^k, w^k) + (f,u^k) + (g,w^k),\\
&\quad \le \frac{1}{2}\gamma a(w^k,w^k) + \frac{1}{2}(\mu+\chi)a(u^k,u^k) + c(\|f\|^2_{V^*} + \|g\|^2_{H^{-1}})\\
&\qquad + c(|w^k|^2 + |u^k|^2).
\end{aligned} \tag{3.1}$$

The above differential inequality implies for any $t \in [0,T]$ the integral inequality

$$|u^k(t)|^2 + j|w^k(t)|^2 + r|h^k(t)|^2 + \int_0^t ((\mu+\chi)a(u^k,u^k) \\ +\gamma a(w^k,w^k) + 2r\nu a(h^k,h^k) + 4\chi|w^k|^2 + 2(\alpha+\beta)|\text{div} w^k|^2)ds \\ \leq |u^k(0)|^2 + j|w^k(0)|^2 + r|h^k(0)|^2 + c\int_0^t (|w^k(s)|^2 + |u^k(s)|^2)ds \\ +c\int_0^t (\|f(s)\|^2_{V^*} + \|g\|^2_{H^{-1}})ds.$$

Due to our choice of $u^k(0), w^k(0)$ and $h^k(0)$, there exists $c > 0$ independent of k such that $|u^k(0)| \leq c|u_0|, |w^k(0)| \leq c|w_0|$ and $|h^k(0)| \leq c|h_0|$. Consequently, by using Gronwall inequality we obtain the global existence in t for the approximations (u^k, w^k, h^k) and also the following

$$u^k, h^k \text{are uniformly bounded in} L^\infty(0,T;H) \cap L^2(0,T;V), \quad (3.2)$$

$$w^k \text{is uniformly bounded in} L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \quad (3.3)$$

The next step in the proof consists in proving that (u_t^k) and (h_t^k) are uniformly bounded in $L^2(0,T,V_s^*)$ and (w_t^k) is uniformly bounded in $L^2(0,T;H^{-s}(\Omega))$. To do this, we observe that

$$\begin{aligned} u_t^k &= P_k^*(\chi \text{rot}\, w^k + rh^k.\nabla h^k + f - u^k.\nabla u^k - (\mu+\chi)\Delta u^k), && (3.4) \\ jw_t^k &= R_k^*(\chi \text{rot}\, u^k + g - Lw^k - ju^k.\nabla w^k - 2\chi w^k), && (3.5) \\ h_t^k &= P_k^*(h^k.\nabla u^k - u^k.\nabla h^k - \nu\Delta h^k), && (3.6) \end{aligned}$$

where P_k and R_k are the projections $P_k : H \to V^k$ and $R_k : L^2(\Omega) \to H_k$ defined by

$$P_k u = \sum_{i=1}^k (u,\varphi^i)\varphi^i \qquad \text{and} \qquad R_k w = \sum_{i=1}^k (w,\phi^i)\phi^i.$$

Since $V_s \hookrightarrow H$ and $V^k \hookrightarrow V_s$, we can consider the restriction $P_k : V_s \to V_s$. It is easy to see that P_k is linear and continuous in this case, hence $P_k^* : V_s^* \to V_s^*$ defined by

$$\langle P_k^*(v), w\rangle = \langle v, P_k w\rangle \ \ \forall\, v \in V_s^*, \ \forall\, w \in V_s,$$

and $\|P_k^*\| \leq \|P_k\| \leq 1$. We also observe that the functions $\varphi^i(x)$ are invariant by P_k, i.e,

$$P_k(\varphi^i) = \varphi^i.$$

Analogously, $R_k : H_0^s(\Omega) \to H_0^s(\Omega)$ is linear and continuous, then $R_k^* : H^{-s}(\Omega) \to H^{-s}(\Omega)$ is well defined by

$$\langle R_k^*(u), \theta\rangle = \langle u, R_k\theta\rangle \ \ \forall\, u \in H^{-s}(\Omega),\ \forall\, \theta \in H_0^s(\Omega),$$

and $\|R_k^*\| \leq \|R_k\| \leq 1$. Moreover, the functions ϕ^i are invariant by R_k, i.e,

$$R_k(\phi^i) = \phi^i.$$

Next, we observe that

$$\|P_k^*(\operatorname{rot} w^k)\|_{V_s^*} \leq \|\operatorname{rot} w^k\|_{V_s^*} = \sup_{\|v\|_{V_s} \leq 1} |\langle \operatorname{rot} w^k, v\rangle| \leq c|\nabla w^k|.$$

Consequently,

$$\int_0^t \|P_k^*(\operatorname{rot} w^k(t)\|_{V_s^*}^2 \leq c \int_0^t |\nabla w^k(t)|^2 dt \leq c$$

thanks to the estimate (3.2).

Also we have

$$\|P_k^*\Delta u^k\|_{V_s^*} \leq \|\Delta u^k\|_{V_s^*} = \sup_{\|v\|_{V_s}\leq 1} |\langle \Delta u^k, v\rangle| = \sup_{\|v\|_{V_s}\leq 1} |(\nabla u^k, \nabla v)| \leq c|\nabla u^k|,$$

and thus, by using the estimate (3.2), we get

$$\int_0^T \|P_k^*\Delta u^k(t)\|_{V_s^*}^2 dt \leq c\int_0^T |\nabla u^k(t)|^2 dt \leq C.$$

To prove the boundedness of $P_k^*(h^k.\nabla h^k)$ and $P_k^*(u^k.\nabla u^k)$ in the space $L^2(0,T;V_s^*)$ we will use the following interpolation result whose proof can be found in Lions [5].

Lemma. *If (u^k) is a bounded sequence in $L^2(0,T;V) \cap L^\infty(0,T;H)$, then (u^k) is also bounded in $L^4(0,T;L^p(\Omega))$, where* $\dfrac{1}{p} = \dfrac{1}{2} - \dfrac{1}{2n}$, i.e., $p = 4$ if $n = 2$, $p = 3$ if $n = 3$.

We have that, for all $v \in V_s$,

$$\begin{aligned} |\langle u^k.\nabla u^k, v\rangle| &\leq \sum_{i,j=1}^{n} |u_i^k(t)|_{L^p}|u_j^k(t)|_{L^p}|\frac{\partial v_j}{\partial x_i}|_{L^n} \\ &\leq c\sum_{i,j=1}^{n} |u_i^k(t)|_{L^p}|u_j^k(t)|_{L^p}\|\frac{\partial v_j}{\partial x_i}\|_{s-1} \\ &\leq c|u^k(t)|_{L^p}^2\|v\|_s \end{aligned}$$

since $\frac{1}{p}+\frac{1}{p}+\frac{1}{n}=1$, using the Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^n(\Omega)$. This implies

$$\int_0^T \|P_k^*(u^k.\nabla u^k)\|_{V_s^*}^2 dt \leq c\int_0^T |u^k(t)|_{L^p}^4 dt.$$

Thus, thanks to the above lemma, $P_k^*(u^k.\nabla u^k)$ is uniformly bounded in $L^2(0,T;V_s^*)$.

Similarly, we prove that $P_k^*(h^k.\nabla h^k)$ is uniformly bounded in $L^2(0,T;V_s^*)$.

The above estimates together with the equality (3.4) implies that (u_t^k) is uniformly bounded in $L^2(0,T;V_s^*)$. Analogously, we can prove that (h_t^k) is uniformly bounded in $L^2(0,T;V_s^*)$. Now, to prove that (w_t^k) is uniformly bounded in $L^2(0,T;H^{-s}(\Omega))$, we estimate the right-hand side of the equality (3.5). We observe that

$$\begin{aligned} j^2\|w_t^k(s)\|_{-n/2}^2 \leq\ & c(\chi^2\|R_k^*(\mathrm{rot}\, u^k)\|_{-n/2}^2+\|R_k^*g\|_{-n/2}^2+\gamma\|R_k^*Lw^k\|_{-n/2}^2 \\ & +j^2\|R_k^*(u^k.\nabla w^k)\|_{-n/2}^2+2\chi\|w^k\|_{-n/2}^2). \end{aligned}$$

Consequently,

$$\begin{aligned} j^2\int_0^T \|w_t^k(s)\|_{-n/2}^2 ds \leq\ & c\int_0^T (\|R_k^*(\mathrm{rot}\, u^k(s)\|_{-n/2}^2+\|R_k^*g(s)\|_{-n/2}^2 \\ & +\|R_k^*(u^k.\nabla w^k)(s)\|_{-n/2}^2+\|w^k(s)\|_{-n/2}^2)ds \\ & +c\int_0^T \|R_k^*Lw^k(s)\|_{-n/2}^2 ds \\ & \equiv F^k(T). \end{aligned}$$

Thus, it is sufficient to estimate $F^k(T)$ independently of k. The first integral in $F^k(T)$ is estimated analogously as before. We estimate the second integral; we have

$$\|R_k^* Lw^k\|_{-n/2}^2 \le \|Lw^k\|_{-n/2}^2 \le c\|w^k\|_{-n/2+2}^2$$

since L is a strongly elliptic operator. As $n = 2$ or 3, we have $H^1 \hookrightarrow H^{-n/2+2}$, and, consequently,

$$\|R_k^* Lw^k\|_{-n/2}^2 \le c\|w^k\|_{H^1}^2 \le c|\nabla w^k|^2.$$

Therefore, estimate (3.3) implies that there exist a constant $c > 0$, independent of k, such that

$$c\int_0^t \|R_k^* Lw^k(s)\|_{-n/2}^2 ds \le c\int_0^t |\nabla w^k(s)|^2 ds \le c.$$

Therefore, arguing as in Lions [5, p.76] and making use of the Aubin-Lions Lemma, with $B_0 = V, p_0 = 2, B_1 = V_s^*, p_1 = 2$ and $B = H$ (see Theorem 1.5.1 and Lemma 1.5.2 in [8, p. 58]), we can conclude that there exists $u, h \in L^2(0,T;V)$ and subsequences, which we keep denoting $(u^k), (h^k)$ to simplify the notation, satisfying

$$\begin{aligned} &u^k \to u \text{ and } h^k \to h \text{ weakly in } L^2(0,T;V),\\ &u^k \to u \text{ and } h^k \to h \text{ weakly} * \text{ in } L^\infty(0,T;H),\\ &u^k \to u \text{ and } h^k \to h \text{ strongly in } L^2(0,T;H),\\ &u_t^k \to u_t \text{ and } h_t^k \to h_t \text{ weakly in } L^2(0,T;V_s^*), \end{aligned}$$

as $k \to \infty$.

Also by the Aubin-Lions Lemma, with $B_0 = H_0^1(\Omega), p_0 = 2, B_1 = H^{-s}(\Omega), p_1 = 2$ and $B = L^2(\Omega)$, we conclude that there exists $w \in L^2(0,T;H_0^1(\Omega))$ and a subsequence, which we shall denote again by (w^k), such that

$$\begin{aligned} &w^k \to w \text{ weakly in } L^2(0,T;H_0^1(\Omega)),\\ &w^k \to w \text{ weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\ &w^k \to w \text{ strongly in } L^2(0,T;L^2(\Omega)),\\ &w_t^k \to w_t \text{ weakly in } L^2(0,T;H^{-s}(\Omega)), \end{aligned}$$

as $k \to \infty$.

Once these later convergences are established, it is a standard procedure to take the limit along the previous subsequences in (2.4)-(2.6) (see [5, p. 76-77] to conclude that (u, w, h) is a weak solution of (1.1)-(1.3).

4 Proof of Theorem 2.2

We first prove the result of regularity. We observe that the proof of the previous theorem shows that, in case $n = 2$, $u_t, h_t \in L^2(0,T;V^*)$ and $w_t \in L^2(0,T;H^{-1}(\Omega))$, since $V_{n/2} = V$ and $H_{n/2} = H_0^1$. Consequently, by applying Lemma 1.2 in Temam [10, p. 260], we obtain that u and h are almost everywhere equals to a continuous functions from $[0,T]$ into H, and w is almost everywhere equal to a continuous function from $[0,T]$ into $L^2(\Omega)$.

We also recall Lemma 1.2 in Temam [10, p. 260-261] which asserts that the following identity holds:

$$\frac{d}{dt}|\varphi(t)|^2 = 2(\varphi_t(t), \varphi(t)) \quad \text{for all} \quad \varphi.$$

This result will be used in the following proof of uniqueness.

Consider that (u_1, w_1, h_1) and (u_2, w_2, h_2) are two solutions of problem (2.1) - (2.2) and (2.3) corresponding to the same f, g, u_0, w_0 and h_0. Define differences

$$\theta = u_1 - u_2, \ \tau = w_1 - w_2 \quad \text{and} \quad \zeta = h_1 - h_2.$$

They satisfy

$$\begin{gathered}
(\theta_t, \varphi) + (\mu + \chi)a(\theta, \varphi) + b(\theta, u_1, \varphi) + b(u_2, \theta, \varphi) \\
-rb(\zeta, h_1, \varphi) - rb(h_2, \zeta, \varphi) - \chi(\operatorname{rot}\tau, \varphi) = 0, \\
j(\tau_t, \phi) + (L\tau, \phi) + 2\chi(\tau, \phi) + jb(\theta, w_1, \phi) + jb(u_2, \tau, \phi) - \chi(\operatorname{rot}\theta, \phi) = 0 \\
(\zeta_t, \psi) + \nu a(\zeta, \psi) + b(\theta, h_1, \psi) + b(u_2, \zeta, \psi) - b(\zeta, u_1, \psi) - b(h_2, \theta, \psi) = 0
\end{gathered}$$

for any $\varphi, \psi \in V$ and $\phi \in H_0^1(\Omega)$; also we have $\theta(0) = \tau(0) = \zeta(0) = 0$.

Setting $\varphi = \theta, \phi = \tau$ and $\psi = r\zeta$ and integrating in t, we obtain

$$\begin{gathered}
\frac{1}{2}|\theta(t)|^2 + (\mu + \chi)\int_0^t a(\theta, \theta)ds, + \int_0^t [b(\theta, u_1, \theta) - rb(\zeta, h_1, \theta) \\
-rb(h_2, \zeta, \theta) - \chi(\operatorname{rot}\tau, \theta)]ds = 0, \\
\frac{j}{2}|\tau(t)|^2 + \gamma\int_0^t |\nabla\tau|^2 ds + (\alpha + \beta)\int_0^t |\operatorname{div}\tau|^2 ds + 2\chi\int_0^t |\tau|^2 ds \\
+ \int_0^t [jb(\theta, w_1, \tau) - \chi(\operatorname{rot}\theta, \tau)]ds = 0 \\
\frac{r}{2}|\zeta(t)|^2 + r\nu\int_0^t a(\zeta, \zeta)ds + r\int_0^t [b(\theta, h_1, \zeta) - b(\zeta, u_1, \zeta) \\
-b(h_2, \theta, \zeta)]ds = 0.
\end{gathered}$$

Adding the above equalities, we have

$$\frac{1}{2}|\theta(t)|^2+\frac{j}{2}|\tau(t)|^2+\frac{r}{2}|\zeta(t)|^2+(\mu+\chi)\int_0^t\|\theta\|^2ds+\gamma\int_0^t\|\tau\|^2ds$$
$$+r\nu\int_0^t\|\zeta\|^2ds+(\alpha+\beta)\int_0^t|\operatorname{div}\tau|^2ds+2\chi\int_0^t|\tau|^2ds$$
$$=-\int_0^t[b(\theta,u_1,\theta)-rb(\zeta,h_1,\theta)-rb(h_2,\zeta,\theta)-\chi(\operatorname{rot}\tau,\theta)+jb(\theta,w_1,\tau)$$
$$-\chi(\operatorname{rot}\theta,\tau)+rb(\theta,h_1,\zeta)-rb(\zeta,u_1,\zeta)-rb(h_2,\theta,\zeta)]ds. \tag{4.1}$$

Now, we observe that $rb(h_2,\zeta,\theta)+rb(h_2,\theta,\zeta)=0$,

$$\begin{aligned}\int_0^t b(\theta,u_1,\theta)ds &\le \int_0^t c\,\|\theta\|^2_{L^4}\|u_1\|ds\\ &\le c\int_0^t\|\theta\|\,|\theta|\,\|u_1\|ds\\ &\le \frac{\mu+\chi}{10}\int_0^t\|\theta\|^2ds+c\int_0^t|\theta|^2\|u_1\|^2ds,\end{aligned}$$

where we used Lemma 3.3 in Temam [10], p.261, together with Hölder and Young inequalities.

Analogously, we can prove

$$r\int_0^t b(\zeta,u_1,\zeta)ds\le\frac{r\nu}{6}\int_0^t\|\zeta\|^2ds+c\int_0^t|\zeta|^2\|u_1\|^2ds.$$

Now, we have

$$\int_0^t\chi(\operatorname{rot}\theta,\tau)ds\le\frac{\mu+\chi}{10}\int_0^t\|\theta\|^2ds+c\int_0^t|\tau|^2ds,$$
$$\int_0^t\chi(\operatorname{rot}\tau,\theta)ds\le\frac{\gamma}{4}\int_0^t\|\tau\|^2ds+c\int_0^t|\theta|^2ds$$

and,

$$\begin{aligned}r\int_0^t b(\zeta,h_1,\theta)ds &\le r\int_0^t\|\zeta\|_{L^4}\|h_1\|\|\theta\|_{L^4}ds\\ &\le c\int_0^t\|h_1\|\|\zeta\|^{1/2}|\zeta|^{1/2}\|\theta\|^{1/2}|\theta|^{1/2}ds\\ &\le C_\varepsilon\int_0^t\|h_1\|^2|\zeta||\theta|ds+\varepsilon\int_0^t\|\zeta\|\|\theta\|ds\end{aligned}$$

$$\begin{aligned} &\leq c\int_0^t \|h_1\|^2[\frac{r}{2}|\zeta|^2+\frac{1}{2}|\theta|^2]ds \\ &+\frac{r\nu}{6}\int_0^t \|\zeta\|^2 ds+\frac{\mu+\chi}{10}\int_0^t \|\theta\|^2 ds. \end{aligned}$$

Analogously, we can prove

$$\begin{aligned} j\int_0^t b(\theta,w_1,\tau)ds &\leq c\int_0^t \|w_1\|^2[\frac{j}{2}|\tau|^2+\frac{1}{2}|\theta|^2]ds \\ &+\frac{\gamma}{4}\int_0^t \|\tau\|^2 ds+\frac{\mu+\chi}{10}\int_0^t \|\theta\|^2 ds, \\ r\int_0^t b(\theta,h_1,\zeta)ds &\leq c\int_0^t \|h_1\|^2[\frac{r}{2}|\zeta|^2+\frac{1}{2}|\theta|^2]ds \\ &+\frac{\mu+\chi}{10}\int_0^t \|\theta\|^2 ds+\frac{r\nu}{6}\int_0^t \|\zeta\|^2 ds. \end{aligned}$$

The constantes depend only on the fixed parameters of problem. By using the above inequalities in (4.1), we get

$$|\theta(t)|^2+j|\tau(t)|^2+r|\zeta(t)|^2\leq c\int_0^t(|\theta(s)|^2+j|\tau(s)|^2+r|\zeta(s)|^2)\mathcal{L}(s)ds$$

where $\mathcal{L}(.)=\|u_1(.)\|^2+\|w_1(.)\|^2+\|h_1(.)\|^2+1\in L^1(0,T)$.

Now, the use of Gronwall inequality implies for every $t\in[0,T]$ that

$$|\theta(t)|^2+j|\tau(t)|^2+r|\zeta(t)|^2\leq(|\theta(0)|^2+j|\tau(0)|^2+r|\zeta(0)|^2)e^c,$$

where $c=\int_0^T\mathcal{L}(s)ds<+\infty$. This last inequality, implies that $\theta(t)\equiv\tau(t)\equiv\zeta(t)\equiv 0$, and hence $u_1=u_2, w_1=w_2$ and $h_1=h_2$. Thus the uniqueness is proved and this completes the proof of Theorem 2.2.

5 Proof of Theorem 2.3

We estimate part of the right-hand side of the equality (3.1) as follows

$$\begin{aligned} \chi(\operatorname{rot} w^k,u^k)+\chi(\operatorname{rot} u^k,w^k) &= 2\chi(w^k,\operatorname{rot} u^k) \\ &\leq 2\chi|w^k|\,|\operatorname{rot} u^k| = 2\chi|w^k|\,|\nabla u^k| \\ &\leq \chi|w^k|^2+\chi a(u^k,u^k) \end{aligned}$$

since $|\text{rot}\, u^k| = |\nabla u^k|$ as in Lukaszewicz [6], p.87. Also,

$$(f, u^k) \le \|f\|_{V^*}|\nabla u^k| \le \frac{1}{2\mu}\|f\|^2_{V^*} + \frac{\mu}{2}a(u^k, u^k),$$

$$(g, w^k) \le \|g\|_{H^{-1}}|\nabla w^k| \le \frac{\gamma}{2}a(w^k, w^k) + \frac{1}{2\gamma}\|g\|^2_{H^{-1}}.$$

Consequently, in (3.1), we have

$$\begin{aligned}\frac{d}{dt}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) + \mu a(u^k, u^k) + \gamma a(w^k, w^k)\\ +2r\nu a(h^k, h^k) + 2\chi|w^k|^2 + 2(\alpha+\beta)|\text{div}\, w^k|^2\\ \le \frac{1}{\mu}\|f\|^2_{V^*} + \frac{1}{\gamma}\|g\|^2_{H^{-1}}\end{aligned}$$

Also, we have,

$$\begin{aligned}\frac{d}{dt}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) + \mu a(u^k, u^k) + \gamma a(w^k, w^k)\\ +2r\nu a(h^k, h^k) \le \frac{1}{\mu}\|f\|^2_{V^*} + \frac{1}{\gamma}\|g\|^2_{H^{-1}}.\end{aligned}$$

Recalling that

$$a(v, v) \ge C_2|v|^2 \quad \forall v \in V \quad \text{and} \quad a(w, w) \ge C_3|w|^2 \quad \forall w \in H^1_0(\Omega),$$

we conclude that

$$\begin{aligned}\frac{d}{dt}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) + \mu C_2|u^k|^2 + \gamma C_3|w^k|^2 + 2r\nu C_2|h^k|^2\\ \le \frac{1}{\mu}\|f\|^2_{V^*} + \frac{1}{\gamma}\ \|g\|^2_{H^{-1}}.\end{aligned}$$

Let $C_0 = \min\{\mu C_2, \frac{C_3\gamma}{j}, 2r\nu C_2\} > 0$, we have

$$\begin{aligned}\frac{d}{dt}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) + C_0(|u^k|^2 + j|w^k|^2 + r|h^k|)\\ \le \frac{1}{\mu}\|f\|^2_{V^*} + \frac{1}{\gamma}\|g\|^2_{H^{-1}},\end{aligned}$$

or, equivalently,

$$\frac{d}{dt}e^{C_0t}(|u^k|^2 + j|w^k|^2 + r|h^k|^2) \le c\,[\frac{e^{C_0t}}{\mu}\|f\|^2_{V^*} + \frac{e^{C_0t}}{\gamma}\|g\|^2_{H^{-1}}].$$

Integrating from 0 to T, we obtain

$$\begin{aligned} e^{C_0T}(|u^k(T)|^2 + j|w^k(T)|^2 + r|h^k(T)|^2) \\ \leq |u^k(0)|^2 + j|w^k(0)|^2 + r|h^k(0)|^2 \\ +\frac{c}{\mu}\int_0^T e^{C_0t}\|f\|^2_{V^*}dt + \frac{c}{\gamma}\int_0^T e^{C_0t}\|g\|^2_{H^{-1}}dt. \end{aligned}$$

We denote by $\theta^k(t)$ the vector (u^k, w^k, h^k) and $\|\theta^k(t)\|^2 = |u^k(t)|^2 + j|w^k(t)|^2 + r|h^k(t)|^2$.

With this notation, the above inequality is rewritten as

$$e^{C_0T}\|\theta^k(T)\|^2 \leq \|\theta^k(0)\|^2 + c\int_0^T e^{C_0t}(\|f(t)\|^2_{V^*} + \|g(t)\|^2_{H^{-1}})dt \quad (5.1)$$

Now, let's define the mapping $S^k : [0,T] \to \mathbb{R}^{3k}$ as

$$S^k(t) = (c_{1k}(t), ..., c_{kk}(t), j^{1/2}d_{1k}(t), ..., j^{1/2}d_{kk}(t), r^{1/2}e_{1k}(t), ..., r^{1/2}e_{kk}(t))$$

where $c_{ik}(t), d_{ik}(t)$ and $e_{ik}(t), i = 1, ..., k$ are respectively the coefficient of the expansion of $u^k(t), w^k(t)$ and $h^k(t)$, as defined in Section 2.

To be used later on, we observe that

$$\|S^k(t)\|_{\mathbb{R}^{3k}} = \|\theta^k(t)\|, \quad (5.2)$$

since we have chosen the spectral basis $\{\varphi^i(x)\}_{i=1}^{\infty}$ and $\{\phi^i(x)\}_{i=1}^{\infty}$ to be orthonormal in $(L^2(\Omega))^n$.

Now, we define the mapping $\Phi^k : \mathbb{R}^{3k} \to \mathbb{R}^{3k}$ as follows: given $S_0 \in \mathbb{R}^{3k}$ we define $\Phi^k(S_0) = S^k(T)$, where $S^k(t)$ corresponds to the solution of problem (1.1), (1.2) with initial value corresponding to L_0. It is easy to see that Φ^k is continuous and we want to prove that Φ^k has a fixed point. For this, as a consequence of the fixed point theorem of Brouwer, it is enough to prove that for any $\lambda \in [0,1]$, a possible solution of the equation

$$S_0^k(\lambda) = \lambda\Phi^k(S_0^k(\lambda)) \quad (5.3)$$

is bounded independent of λ.

Since $S_0^k(0) = 0$, by (5.3), it is enough to prove this fact for $\lambda \in (0,1]$. In this case, (5.3) is equivalent to $\Phi^k(S_0^k(\lambda)) = S_0^k(\lambda)/\lambda$, and therefore by definition of Φ^k and (5.2), inequality (5.1) implies that

$$e^{C_0T}\|S_0^k(\lambda)/\lambda\|^2_{\mathbb{R}^{3k}} \leq \|S_0^k(\lambda)\|^2_{\mathbb{R}^{3k}} + c\int_0^T e^{C_0t}(\|f(t)\|^2_{V^*} + \|g(t)\|^2_{H^{-1}})dt,$$

which implies that

$$\|S_0^k(\lambda)\|^2_{\mathbb{R}^{3k}} \leq \frac{c\int_0^T e^{C_0 t}(\|f(t)\|^2_{V^*} + \|g(t)\|^2_{H^{-1}})dt}{e^{C_0 T} - 1} = M, \tag{5.4}$$

since $\lambda \in (0, 1]$. This bound is independent of $\lambda \in [0, 1]$ and, therefore, Φ^k has a fixed point $S_0^k(1)$ satisfying the same bound as (5.4).

This corresponds to the existence of a solution $u_p^k(t), w_p^k(t), h_p^k(t))$ of (1.1), (1.2) satisfying $u_p^k(0) = u_p^k(T), w_p^k(0) = w_p^k(T)$ and $h_p^k(0) = h_p^k(T)$, that is periodic approximate solution.

Moreover, $|u^k(0)| + j|w^k(0)|^2 + r|h^k(0)|^2 = \|S_0^k(1)\|^2_{\mathbb{R}^{3k}} \leq M$, which is also independent of k. Thus, the arguments in the proof of Theorem 2.1 can be repeated for the approximate solutions (u_p^k, w_p^k, h_p^k), and this furnishes exactly the same kind of uniform in k estimates for them, and therefore the convergence of a subsequence to a solution (u_p, w_p, h_p) of (1.1)-(1.2) satisfying $u(0) = u(T); w(0) = w(T)$ and $h(0) = h(T)$.

■

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