

## Some results about blow-up and global existence to a semilinear degenerate heat equation.

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### Abstract

In this paper, we are dealing with the following degenerate parabolic problem :

$$(P_t) \begin{cases} \partial_t u - |x|^2 \Delta u = g(u) & \text{in } \mathbb{R}^+ \times B_1 \\ u(t, x) \equiv 0 & \text{in } \mathbb{R}^+ \times \partial B_1 ; u(0, x) = u_0 \geq 0 \end{cases}$$

where  $B_1 = \{x \in \mathbb{R}^N ; \|x\| = 1\}$  and  $g$  is nonlinear.

We are interested in analyzing such questions as local and global existence, blow-up in finite time and convergence to a stationary solution for solutions of  $(P_t)$ .

First, we give some examples of nonlinearities  $g$  where the blow up in  $L^2(\frac{dx}{|x|^2}) \cap L^\infty(B_1)$  occurs. In a second part of this work, we present two cases of global existence of solutions to  $(P_t)$  which converge in  $L^\infty(B_1)$  to a stationary solution of  $(P_t)$  when  $t \rightarrow \infty$ .

## 1 Introduction

In this work, we study the following problem :

$$(P_t) \begin{cases} \partial_t u - |x|^2 \Delta u = g(u) & \text{in } \mathbb{R}^+ \times B_1 \\ u(t, x) \equiv 0 & \text{in } \mathbb{R}^+ \times \partial B_1 ; u(0, x) = u_0 \geq 0, \end{cases}$$

where  $g$  is nonlinear and  $B_1$  is the unit ball in  $\mathbb{R}^N$ .

First, using Hille-Yosida theory, we prove for all  $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})^1$  and  $g \in W_{loc}^{1,\infty}(\mathbb{R}^N)$ , the local existence and the uniqueness of the solution  $u(t) = S(t)u_0$  of  $(P_t)$ , where  $S(t)$  is the semigroup associated to  $(P_t)$ . Then, we are interested in the behaviour of the solution  $u(t)$  as  $t$  increases. Precisely, under different assumptions of  $g$  and  $u_0$ , we give on one hand, some examples of blow-up in finite time and on the other hand, some examples of global existence of solutions to  $(P_t)$  which converge to a stationary solution of  $(P_t)$ .

Throughout this work, we keep in mind the results of [7] and [8] which deal with the stationary problem  $(P)$  :

$$(P) \begin{cases} -|x|^2 \Delta u = g(u) & \text{in } B_1 \\ u \in H_0^1(B_1) / \{0\} ; u \geq 0 \end{cases}$$

Precisely, in [7], the authors prove the nonexistence of nontrivial solutions to  $(P)$  in the case where  $g$  satisfies the following assumptions :

$$(GS1) \quad \lambda - \left(\frac{N-2}{2}\right)^2 + \lim_{s \rightarrow +\infty} \frac{g(s)}{s} > 0.$$

$$(GS2) \quad \forall s > 0, \quad G(s) \leq \frac{g(s)s}{2}.$$

Otherwise, in [8], the authors give some results about the existence of nontrivial solutions of  $(P)$  in the case where  $g$  is sublinear. They treat three cases :

1.  $g(u) \sim \lambda u + u^p - u^q$  where  $1 < p < q$
2.  $g(u) \sim \lambda u - u^p$  where  $p > 1$  and  $\lambda > \left(\frac{N-2}{2}\right)^2$
3.  $g(u) \sim u^\alpha + \lambda u$  where  $0 < \alpha < 1$  and  $\lambda < \left(\frac{N-2}{2}\right)^2$

It is worth noting that in all cases, an unbounded connected branch of positive solutions in either  $H_0^1(B_1)$  or  $L^\infty(B_1)$  exists and in the second and third case, there is uniqueness of the nontrivial solution in  $H_0^1(B_1)$ . Then, it is very natural to see in which cases the nonexistence of nontrivial solutions of  $(P)$  implies the blow-up in finite time for solutions of  $(P_t)$  and when the uniqueness of the solution of  $(P)$  implies the convergence to a stationary solution for solutions to  $(P_t)$  when  $t \rightarrow +\infty$ . In this work, we prove some results in these directions.

So, the outline of the present paper is as follows :

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<sup>1</sup> $L^2(\frac{dx}{|x|^2}) := \{u / \int_{B_1} \frac{|u|^2}{|x|^2} dx < \infty\}$

1. Local existence of solutions to  $(P_t)$  in  $\mathbb{R}^+ \times L^\infty \cap L^2(\frac{dx}{|x|^2})$ .
2. Some examples of blow up in finite time for solutions to  $(P_t)$ 
  - (a) The case  $g(0) = 0$
  - (b) The case  $g(0) > 0$
3. Two examples of existence of global solutions and convergence to a stationary solution.

Precisely, in Section 2, we apply Hille-Yosida theory in  $L^\infty \cap L^2(\frac{dx}{|x|^2})$ . In Section 3, we start adapting a classical spectral method (see for instance [4]) to prove the blow-up in finite time when  $g$  satisfies :

(B1)  $g$  is convex and positive in  $\mathbb{R}^+$ .

(B2)  $(\frac{N-2}{2})^2 < \lim_{s \rightarrow 0^+} \frac{g(s)}{s} = \lambda < +\infty$ .

(B3) There exists  $s_0 > 0$  such that  $\int_{s_0}^{+\infty} \frac{ds}{g(s)-\lambda s} < +\infty$

Next, we use a well known "energy method" (see for instance [4]). For this, we assume the following hypothesis :

(B4)  $\lambda = \lim_{s \rightarrow 0^+} \frac{g(s)}{s} < +\infty$  and there exists  $\alpha > 0$ ,  $C > 0$  such that  $h(s) = g(s) - \lambda s \geq Cs^{\alpha+1}$ , for all  $s \geq 0$ .

(B5) There exists  $\epsilon > 0$  such that  $(2 + \epsilon) \int_0^s h(t) dt \leq sh(s)$ ,  $\forall s > 0$ .

Then, we prove that if  $u_0$  satisfies  $\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} < 0$ , where  $G(s) = \int_0^s g(t) dt$ , the solution  $u(t)$  to  $(P_t)$  blows up in finite time. Finally, we conclude the section with the case  $g(0) > 0$ . Precisely, we apply a method from [3] which links directly the blow-up and the nonexistence of stationary solutions. For this, we assume :

(B6)  $g(0) > 0$ ,  $g \in C^1([0, +\infty[)$ , convex and increasing.

(B7) There exists  $x_0 > 0$  such that  $\int_{x_0}^{+\infty} \frac{ds}{g(s)} < \infty$ .

Then, for any  $u_0 \geq 0$ , the solution  $u(t) = S(t)u_0$  blows up in finite time.

In Section 4, we give some results concerning the existence of global solutions to  $(P_t)$ . First, proving the radial symmetry of the solution to  $(P_t)$  when  $u_0$  is radially symmetric, we exhibit the heat kernel of  $-|x|^2\Delta$  in  $H_0^1(B_1)$ . Then, using a method due to Fujita, we prove the existence of a global solution of  $(P_t)$  for small initial data when  $g(t) \sim \lambda t + t^p$ ,  $p > 1$  and  $\lambda < 0$ . Moreover, we prove that  $u(t)$  converges to 0 in  $L^\infty(B_1)$  with an exponential decay when  $t \rightarrow +\infty$ .

Finally, assuming the following hypothesis :

(G3)  $s \rightarrow \frac{g(s)}{s}$  is continuous and strictly decreasing,

(G4)  $\frac{g(s)}{s} \xrightarrow{s \rightarrow +\infty} -\infty$ ,

(G5)  $\lim_{s \rightarrow 0^+} \frac{g(s)}{s} = \lambda > \left(\frac{N-2}{2}\right)^2$ ,

we show that for any  $u_0 > 0$  satisfying  $u_0 \in L^\infty \cap L^2\left(\frac{dx}{|x|^2}\right)$ ,  $\|u_0\|_{L^\infty} \leq f^{-1}(0)$  and  $u_0 \not\equiv f^{-1}(0)$  where  $f(t) := \frac{g(t)}{t}$ , the solution  $u(t)$  of  $(P_t)$  is global and converges to the unique nontrivial stationary solution of  $(P_t)$  in  $L^\infty(B_1) \cap H_0^1(B_1)$ .

## 2 Local existence

Throughout this section, we assume that  $g \in W_{loc}^{1,\infty}(\mathbb{R})$ . Our goal is to study the local existence of a solution to  $(P_t)$ . Precisely, we show that we can apply Hille-Yosida theory in  $L^\infty(B_1) \cap L^2\left(\frac{dx}{|x|^2}\right)$ . Consequently, for every  $u_0 \in L^\infty \cap L^2\left(\frac{dx}{|x|^2}\right)$ , the uniqueness of solutions of  $(P_t)$  follows. First, we remark :

**Proposition 2.1.** *Let  $A = -|x|^2\Delta$ . Then,  $A$  is a self adjoint maximal monotone operator in  $L^2\left(\frac{dx}{|x|^2}\right)$ . Moreover,  $\mathcal{D}(A) = \{u \in L^2\left(\frac{dx}{|x|^2}\right) / u \in H_0^1(B_1) \text{ and } |x|^2\Delta u \in L^2\left(\frac{dx}{|x|^2}\right)\}$ .*

**Proof.** For this, notice that for every  $u \in \mathcal{D}(A)$  and  $\lambda > 0$  :

$$\begin{aligned} \|u - \lambda |x|^2\Delta u\|_{L^2\left(\frac{dx}{|x|^2}\right)}^2 &= \|u\|_{L^2\left(\frac{dx}{|x|^2}\right)}^2 + 2\lambda \|\nabla u\|_{L^2}^2 \\ &+ |\lambda|^2 \int_{B_1} |x|^2 |\Delta u|^2 \geq \|u\|_{L^2\left(\frac{dx}{|x|^2}\right)}^2 \end{aligned}$$

which implies that  $A$  is dissipative in  $L^2(\frac{dx}{|x|^2})$ . Then, it suffices to show that  $A$  is maximal. Taking  $f \in L^2(\frac{dx}{|x|^2})$ , we consider the following minimization problem :

$$I_\lambda = \inf_{u \in H_0^1(B_1)} \mathcal{E}(u)$$

where  $\mathcal{E}(u) = \frac{1}{2} \int_{B_1} (\frac{|u|^2}{|x|^2} + \lambda |\nabla u|^2) dx - \int_{B_1} \frac{f u}{|x|^2}$

By Cauchy-Schwarz's inequality,

$$I_\lambda \geq \inf_{u \in H_0^1(B_1)} \frac{1}{2} \int_{B_1} (\frac{|u|^2}{|x|^2} + \lambda |\nabla u|^2) dx - (\int_{B_1} \frac{|f|^2}{|x|^2})^{\frac{1}{2}} (\int_{B_1} \frac{|u|^2}{|x|^2})^{\frac{1}{2}} > -\infty$$

then, considering a minimizing sequence  $\{u_n\}_{n \in \mathbf{N}} \subset H_0^1(B_1) \cap L^2(\frac{dx}{|x|^2})$ , it follows that  $\|u_n\|_{H_0^1 \cap L^2(\frac{dx}{|x|^2})} \leq C$ . And by standard compactness arguments, there exists  $u \in H_0^1(B_1) \cap L^2(\frac{dx}{|x|^2})$  such that up to subsequences :

$$u_n \xrightarrow{n \rightarrow \infty} u \text{ weakly in } H_0^1(B_1), u_n \xrightarrow{n \rightarrow \infty} u \text{ weakly in } L^2(\frac{dx}{|x|^2})$$

and

$$\int_{B_1} \frac{f u_n}{|x|^2} \xrightarrow{n \rightarrow \infty} \int_{B_1} \frac{f u}{|x|^2}$$

Therefore,  $I_\lambda$  is achieved by  $u$  and the proof is complete. ■

We deduce immediatly the following corollary :

**Corollary 2.2.** *A is maximal monotone in  $L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$ . Moreover,  $\mathcal{D}(A) = \{u \in H_0^1(B_1) \cap L^\infty/|x|^2 \Delta u \in L^2(\frac{dx}{|x|^2}) \cap L^\infty\}$ .*

**Proof.** Let  $f \in L^2(\frac{dx}{|x|^2}) \cap L^\infty$  and  $\lambda > 0$ . By Proposition 2.1, there exists  $u \in H_0^1 \cap L^2(\frac{dx}{|x|^2})$  such that

$$u - \lambda |x|^2 \Delta u = f \text{ in } B_1 \tag{2.1}$$

Thus, it suffices to show that  $u \in L^\infty(B_1)$ . Multiplying (2.1) by  $(u - \|f\|_{L^\infty})^+$ , we obtain :

$$\begin{aligned} & \int_{B_1} \frac{(u - \|f\|_{L^\infty})^{+2}}{|x|^2} dx + \lambda \int_{B_1} |\nabla(u - \|f\|_{L^\infty})^+|^2 dx \\ &= \int_{B_1} (f - \|f\|_{L^\infty}) \frac{(u - \|f\|_{L^\infty})^+}{|x|^2} \leq 0 \end{aligned}$$

which yields  $(u - \|f\|_{L^\infty})^+ \equiv 0$  and  $u \leq \|f\|_{L^\infty}$ . By the same arguments, we show that  $u \geq -\|f\|_{L^\infty}$ . This ends the proof of Corollary 2.2. ■

**Remark.** For  $N \geq 3$ ,  $L^\infty(B_1) \subset L^2(\frac{dx}{|x|^2})$ . And in this case, to prove Corollary 2.2, it suffices to show the maximality of  $A$  in  $L^\infty$ .

Now, we apply Hille-Yosida theory (see [15]) and we deduce the following proposition :

**Proposition 2.3.** *Let  $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$ . Then, there exists a unique solution  $u(t) = S(t)u_0$  to  $(P_t)$  in a maximal interval  $[0, T[$ ,  $T > 0$  such that*

- (i)  $u(\cdot) \in C^0([0, T[, L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})) \cap C^1([0, T[, L^2(\frac{dx}{|x|^2}))$ .
- (ii) For all  $t$  in  $]0, T[$ ,  $u(t) \in H_0^1(B_1) \cap L^\infty \cap L^2(\frac{dx}{|x|^2})$  and  $|x|^2 \Delta u(t) \in L^2(\frac{dx}{|x|^2})$ .
- (iii) If  $u_0 \geq 0$ , then  $u(t) \geq 0$  for all  $t > 0$ .
- (iv) If  $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$  satisfies  $|x|^2 \Delta u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$ , then  $u(t) \in C^1([0, T[, L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2}))$ .

**Proof.** By Proposition 2.1, Corollary 2.2 and since  $g \in W_{loc}^{1,\infty}$  we can apply Theorems 3.7 and 3.9 of [4]. This proves assertions (i), (ii) and (iv). Now, let us prove assertion (iii). For every  $T_0 < T$ , we multiply the equation in  $(P_t)$  by  $\frac{(-u)^-}{|x|^2}$  and integrate by parts to obtain for every  $t \in [0, T_0]$  :

$$\frac{1}{2} \frac{d}{dt} \int_{B_1} \frac{|u^-|^2}{|x|^2} dx = - \int_{B_1} |\nabla u^-|^2 - \int_{B_1} \frac{g(u)u^-}{|x|^2} \leq C(T_0) \int_{B_1} \frac{|u^-|^2}{|x|^2}$$

which implies by Gronwall's lemma that  $u^- \equiv 0$ . This completes the proof of Proposition 2.3. ■

As a consequence of Hille-Yosida Theory, we have the following alternative for  $u(t) = S(t)u_0$  :

**Corollary 2.4.** *If  $u_0 \in L^2(\frac{dx}{|x|^2}) \cap L^\infty$ , then, either  $T = T(\|u_0\|_{\mathcal{D}(A)}) = +\infty$  and the solution  $u(\cdot) = S(\cdot)u_0$  is global, or  $T < +\infty$  and the solution blows up in finite time which means that*

$$\|u(t)\|_{L^\infty} + \|u(t)\|_{L^2(\frac{dx}{|x|^2})} \xrightarrow{t \rightarrow T^-} +\infty$$

**Proof.** See [4].

**Remarks.** If  $g \equiv 0$  and  $u_0 \in H^1 \cap L^\infty \cap L^2(\frac{dx}{|x|^2})$  then  $u(t) = S(t)u_0$  is global and satisfies :

$$\int_{B_1} \frac{|u(t)|^2}{|x|^2} \leq e^{-(\frac{N-2}{2})^2 t} \|u_0\|_{L^2(\frac{dx}{|x|^2})}^2. \tag{2.2}$$

The proof is based upon Hardy's inequality. First, observe that since  $g \equiv 0$ ,  $(P_t)$  is linear. Therefore,  $u(t) = S(t)u_0$  is global. Moreover, multiplying  $(P_t)$  by  $u(t)e^{(\frac{N-2}{2})^2 t}$  and integrating by parts, we have :

$$\begin{aligned} \frac{d}{dt} \int_{B_1} \frac{|u^-|^2}{|x|^2} e^{(\frac{N-2}{2})^2 t} dx &= 2 \left(\frac{N-2}{2}\right)^2 \int_{B_1} \frac{|u^-|^2}{|x|^2} e^{(\frac{N-2}{2})^2 t} dx - 2 \\ &\int_{B_1} |\nabla u|^2 e^{(\frac{N-2}{2})^2 t} dx \leq 0 \end{aligned}$$

by Hardy's inequality. Thus, integrating on  $[0, t]$ , we deduce (2.2).

Now, we deal with the behaviour of the solution to  $(P_t)$ . In the next section, we give some examples of blow-up in finite time of solutions to  $(P_t)$ .

### 3 Blow up in finite time in $L^2(\frac{dx}{|x|^2}) \cap L^\infty$

Throughout this section, we assume that  $g$  belongs to  $W_{loc}^{1,\infty}$ ,  $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$  and  $G(s) = \int_0^s g(t) dt$ .

### 3.1 Main results

We consider three classes of functions  $g$ . First, we adapt a classical “spectral method” (see for instance [4]). Precisely, we prove the following theorem :

**Theorem 3.1.** *Assuming  $N \geq 3$  and*

(B1)  *$g$  is convex and positive in  $\mathbb{R}^+$ ,*

(B2)  $(\frac{N-2}{2})^2 < \lambda := \lim_{s \rightarrow 0^+} \frac{g(s)}{s} < +\infty,$

(B3) *There exists  $s_0 > 0$  such that  $\int_{s_0}^{+\infty} \frac{ds}{h(s)} < \infty$  where  $h(s) = g(s) - \lambda s.$*

*Then, for any  $u_0 \geq 0$  in  $L^\infty \cap L^2(\frac{dx}{|x|^2})$ ,  $u(t) = S(t)u_0$  satisfies :  $\exists T \in \mathbb{R}^+$  such that*

$$\lim_{t \rightarrow T^-} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = +\infty \quad \text{and} \quad \lim_{t \rightarrow T^-} \|u(t)\|_{L^\infty} = +\infty$$

The second blow-up case is based upon an “energy method” (see for instance [4]).

**Theorem 3.2.** *Assume that  $u_0$  satisfies (\*)  $\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} < 0$  and that  $g$  has the following properties :*

(B4)  $\lambda := \lim_{s \rightarrow 0^+} \frac{g(s)}{s} \in \mathbb{R}$  *and there exists  $\alpha > 0$ ,  $C > 0$  such that  $h(s) = g(s) - \lambda s \geq C s^{\alpha+1}$  for all  $s \geq 0,$*

(B5) *There exists  $\epsilon > 0$  such that for all  $s \geq 0,$   $(2 + \epsilon)H(s) \leq sh(s)$  where  $H(t) = \int_0^t h(s) ds.$*

*Then,  $u(t) = S(t)u_0$  satisfies :  $\exists T > 0$  such that  $\lim_{t \rightarrow T^-} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = +\infty.$*

**Remarks.**

1. If  $g(s) = \lambda s + s^p$  with  $\lambda > (\frac{N-2}{2})^2$  and  $p > 1,$  (B1), (B2) and (B3) are satisfied.



2. If  $g(s) = \lambda s + s^p$  with  $p > 1$ , (B4) and (B5) are satisfied.
3. Let  $\phi \in L^\infty \cap H_0^1$ . Then, by (B4), there exists  $M > 0$ , large enough, such that  $u_0 = M \phi$  satisfies (\*).
4. If  $u_0 \geq 0$  is a radially decreasing nontrivial subsolution of (P) and belongs to  $H_0^1(B_1) \cap L^\infty$ , then, a simple computation based upon a "Pohozaev's equality type" shows that (\*) is satisfied for  $N > 2$ . Indeed, multiplying  $-|x|^2 \Delta u_0 \leq g(u_0)$  by  $\frac{x}{|x|^2} \cdot \nabla u_0$  and integrating by parts, we obtain :

$$-\left(\frac{N-2}{2}\right) \int_{B_1} |\nabla u_0|^2 - \frac{1}{2} \int_{\partial B_1} \left|\frac{\partial u_0}{\partial n}\right|^2 ds \geq (2-N) \int_{B_1} \frac{G(u_0)}{|x|^2}$$

which implies :

$$\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} \leq -\frac{1}{2(N-2)} \int_{\partial B_1} \left|\frac{\partial u_0}{\partial n}\right|^2 ds < 0$$

Finally, we deal with the case  $g(0) > 0$ . In this case, we adapt a method from [3]. And we use the results of nonexistence of solutions to the problem (P).

**Theorem 3.3.** Assume that  $N \geq 3$  and the following assumptions on  $g$  :

(B6)  $g > 0$  is convex, increasing and belongs to  $C^1([0, +\infty[)$ ,

(B7) There exists  $s_0 > 0$  such that  $\int_{s_0}^{+\infty} \frac{ds}{g(s)} < \infty$ .

Then, for all  $u_0 \geq 0$  in  $L^\infty \cap L^2(\frac{dx}{|x|^2})$  and nontrivial,  $u(t) = S(t)u_0$  blows up in finite time in  $L^\infty$  and in  $L^2(\frac{dx}{|x|^2})$ .

**Remarks.**

1. It is worth noting that in Theorems 3.1 and 3.3, no additional assumption is required for  $u_0$ . Here, the nonexistence of weak nontrivial solutions of the stationary problem (P) implies the blow-up in finite time for any initial data in  $L^\infty \cap L^2(\frac{dx}{|x|^2})$ .
2. The assumptions (B3) and (B7) prevent the existence of unbounded global solutions (i.e. which blow up when  $t \rightarrow \infty$ ).

Now, we prove Theorem 3.1:

**Proof of Theorem 3.1.** Let us consider  $\psi_\epsilon$  the eigenfunction associated with the first eigenvalue  $\lambda_\epsilon^1$  of  $-(|x|^2 + |\epsilon|^2)\Delta$  in  $H_0^1(B_1)$  such that  $\int_{B_1} \frac{\psi_\epsilon}{|x|^2} = 1$  (for this, notice that  $N \geq 3$  implies that  $L^2(\frac{dx}{|x|^2}) \subset L^1(\frac{dx}{|x|^2})$ ). It is easy to prove that  $\lambda_\epsilon^1 \rightarrow (\frac{N-2}{2})^2$  when  $\epsilon \rightarrow 0$ . Therefore, by (B2), there exists  $\epsilon > 0$  small enough such that  $\lambda_\epsilon^1 < \lambda$ . Thus, multiplying  $(P_t)$  by  $\frac{\psi_\epsilon}{|x|^2}$ , we obtain :

$$\frac{d}{dt} \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2} + \lambda_\epsilon^1 \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2 + |\epsilon|^2} = \int_{B_1} \frac{g(u(t)) \psi_\epsilon}{|x|^2}$$

Since  $g$  is convex (which implies that  $f$  is convex), by Jensen's inequality, we have :

$$\frac{d}{dt} \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2} \geq (\lambda - \lambda_\epsilon^1) \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2} + h \left( \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2} \right)$$

From which it follows :

$$\frac{d}{dt} \left( \int_0^{\phi(t)} \frac{ds}{h(s)} \right) \geq 1 \quad \text{where } \phi(t) = \int_{B_1} \frac{u(t) \psi_\epsilon}{|x|^2} \tag{3.1}$$

Integrating (3.1), one has  $\int_0^{\phi(t)} \frac{ds}{h(s)} \geq t + C$  which together with (B3) implies that  $\phi(\cdot)$  blows up in finite time. Finally, noting that for  $N \geq 3$ , the injection  $L^\infty \hookrightarrow L^2(\frac{dx}{|x|^2})$  is continuous, the proof of Theorem 3.1 is complete. ■

Next, we give the proof of Theorem 3.2:

**Proof of Theorem 3.2.** Suppose that the solution  $u(t) = S(t)u_0$  is global. Let us consider  $E(t) = \frac{1}{2} \int_{B_1} |\nabla u(t)|^2 - \int_{B_1} \frac{G(u(t))}{|x|^2}$ . Then, multiplying  $(P_t)$  by  $\frac{u(t)}{|x|^2}$  and integrating by parts, we obtain :

$$\int_{B_1} \frac{|u_t|^2}{|x|^2} = -\frac{1}{2} \frac{d}{dt} \int_{B_1} |\nabla u(t)|^2 + \frac{d}{dt} \int_{B_1} \frac{G(u(t))}{|x|^2} = -\frac{d}{dt} (E(t))$$

Thus,  $E(t)$  is decreasing and  $E(t) \leq E(0) < 0$ . Now, multiplying the equation in  $(P_t)$  by  $\frac{u(t)}{|x|^2}$  and integrating by parts :

$$\frac{1}{2} \frac{d}{dt} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = - \int_{B_1} |\nabla u(t)|^2 + \int_{B_1} \frac{g(u(t))u(t)}{|x|^2} \tag{3.2}$$

By using (B5), and taking  $H(s) = \int_0^s h(t) dt$ , we prove that :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{B_1} \frac{|u(t)|^2}{|x|^2} &\geq - \int_{B_1} |\nabla u(t)|^2 + (2 + \epsilon) \int_{B_1} \frac{H(u(t))}{|x|^2} + \lambda \int_{B_1} \frac{|u(t)|^2}{|x|^2} \\ &\geq -2 E(t) + \epsilon \int_{B_1} \frac{H(u(t))}{|x|^2} \\ &\geq -2 E(0) + \epsilon \int_{B_1} \frac{H(u(t))}{|x|^2} \end{aligned} \tag{3.3}$$

Thus, (3.3) and (\*) imply that  $\lim_{t \rightarrow \infty} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = +\infty$ . Then, by (3.3) :

$$\frac{1}{2} \frac{d}{dt} \int_{B_1} \frac{|u(t)|^2}{|x|^2} \geq \epsilon \int_{B_1} \frac{H(u(t))}{|x|^2} \geq C \epsilon \int_{B_1} \frac{|u(t)|^{2+\alpha}}{|x|^2} \geq C \epsilon \left( \int_{B_1} \frac{|u(t)|^2}{|x|^2} \right)^{\frac{2+\alpha}{2}}$$

Taking  $\phi(t) = \int_{B_1} \frac{|u(t)|^2}{|x|^2}$ , we have :

$$\frac{d}{dt} \phi(t) \geq 2\epsilon C \phi(t)^{\frac{\alpha+2}{2}} \tag{3.4}$$

Integrating (3.4) on  $[t_0, t]$ , we obtain :

$$\frac{1}{\phi(t)^{\frac{\alpha}{2}}} - \frac{1}{\phi(t_0)^{\frac{\alpha}{2}}} \geq C(t - t_0)$$

which contradicts that  $u(\cdot)$  is a global solution of  $(P_t)$ . This completes the proof of Theorem 3.2. ■

Finally, we prove Theorem 3.3. Here, we use an approach from [3] : the nonexistence of stationary weak solutions implies the nonexistence of global, bounded solution of  $(P_t)$  for every  $u_0 \geq 0$ .

First, we adapt the definition of a weak stationary solution of  $(P_t)$  from [3] :

**Definition 3.1.** A weak stationary solution of  $(P_t)$  is a function  $u \in L^1(B_1)$  such that  $\frac{g(u)}{|x|^2} \delta(x) \in L^1(B_1)$  (where  $\delta(x) = \text{dist}(x, \partial B_1)$ ) and

$$\forall \xi \in C^2(\bar{B}_1) \quad - \int_{B_1} u \Delta \xi dx = \int_{B_1} \frac{g(u)}{|x|^2} \xi dx$$

Then, we have the following result :

**Proposition 3.4.** *Assume that  $g$  satisfies (B6) and (B7). Then, there is no weak stationary solution of  $(P_1)$ . **Proof.***

We apply a method from [3]. Precisely, for all  $\eta$  such that  $0 \leq \eta < 1$ , we define :

$$(P_\eta) \begin{cases} -|x|^2 \Delta u = (1 - \eta)g(u) \text{ in } B_1 \\ u \geq 0, \quad u \in H_0^1(B_1) \end{cases}$$

As in [3], we define  $h(u) = \int_0^u \frac{ds}{g(s)}$ ,  $\hat{h}(u) = \frac{1}{1-\eta}h(u)$  and  $\Phi(u) = \hat{h}^{-1}(h(u))$ . It is easy to prove the following assertions (see [3]) :

- (i)  $\Phi(0) = 0$  and  $0 \leq \Phi(u) \leq u$ .
- (ii)  $\Phi$  is increasing and concave. Moreover,  $\Phi'(u) \leq 1$ .
- (iii)  $\Phi \in L^\infty$  and  $\Phi(u)$  satisfies :

$$\forall \xi \in C^2(B_1) \quad - \int_{B_1} (\Delta \Phi(u)) \xi \geq (1 - \eta) \int_{B_1} \frac{g(\Phi(u))\xi}{|x|^2}$$

which means that  $\Phi(u)$  is a "weak supersolution" of  $(P_\eta)$ .

For all  $\xi \in C_0^2(\bar{B}_1)$ , let us consider the following iterative scheme :

$$\begin{cases} - \int_{B_1} u_{n+1} \Delta \xi = (1 - \eta) \int_{B_1} \frac{g(u_n)\xi}{|x|^2} \text{ in } B_1 \\ u_0 = \Phi(u), \quad u \in H_0^1(B_1) \end{cases}$$

Then, noting that  $\Phi(u) \in L^\infty$  implies that for  $N \geq 3$ ,  $\frac{g(\Phi(u))}{|x|^2} \in L^1$  and by the fact that 0 is a strict subsolution to  $(P_\eta)$ , we prove, by the maximum principle, that in  $L^\infty$ ,  $\{u_n\}_{n \geq 1}$  is a decreasing sequence of weak supersolutions of  $(P_\eta)$  and  $u_n \leq \Phi(u)$ . Thus,  $v_\eta = \lim_{n \rightarrow \infty} u_n \in L^\infty$  is a weak solution of  $(P_\eta)$ . Now, consider for all  $\epsilon$  in  $]0, 1[$ , the following problem :

$$(P_{\epsilon, \eta}) \begin{cases} -(|x|^2 + |\epsilon|^2) \Delta v = (1 - \eta)g(v) \text{ in } B_1 \\ v \geq 0, \quad v \in H_0^1(B_1) \end{cases}$$

As in [1], we prove the existence of a minimal solution of  $(P_{\epsilon, \eta})$ ,  $v_{\epsilon, \eta}$ , such that  $v_{\epsilon, \eta} \leq v_\eta \leq \Phi(u)$ .

Putting  $w_{\epsilon,\eta}(x) := v_{\epsilon,\eta}(\epsilon x)$ , for  $x \in B_{\frac{1}{\epsilon}}$ , we have :

$$\begin{cases} -(|x|^2 + 1)\Delta w_{\epsilon,\eta} = (1 - \eta)g(w_{\epsilon,\eta}) & \text{in } B_{\frac{1}{\epsilon}} \\ w_{\epsilon,\eta} \geq 0, \quad w_{\epsilon,\eta} \in H_0^1(B_{\frac{1}{\epsilon}}) \end{cases}$$

As above, we can show that  $\epsilon \rightarrow w_{\epsilon,\eta}$  is increasing in  $L^\infty$ . Passing to the limit  $\epsilon \rightarrow 0$ , it is easy to prove that  $w := \lim_{\epsilon \rightarrow 0} w_\epsilon$  satisfies  $\|w\|_{L^\infty} \leq \|v_\eta\|_{L^\infty}$  and is the minimal non trivial solution of the following problem :

$$\begin{cases} -(|x|^2 + 1)\Delta w = (1 - \eta)g(w) & \text{in } \mathbb{R}^N \\ w \geq 0 \end{cases}$$

Therefore,  $w(x) = \frac{1}{C_N|x|^{N-2}} * \frac{g(w)}{|x|^2+1}$  where  $C_N = (N - 2)|\sigma_{N-1}|$  and  $|\sigma_{N-1}|$  the surface area of the unit sphere. Thus,

$$\begin{aligned} w(0) &= \int_{\mathbb{R}^N} \frac{C_N g(w)}{|x|^{N-2}(|x|^2 + 1)} dx \geq \inf_{s \in [0, \|w\|_{L^\infty}]} g(s) \\ &\int_{\mathbb{R}^N} \frac{1}{(|x|^2 + 1)|x|^{N-2}} = +\infty \end{aligned}$$

This contradicts the boundedness of  $w$  and the proof of Proposition 3.4 is now complete. ■

**Proof of Theorem 3.3.** First, note that by the maximum principle, it suffices to prove Theorem 3.3 in the case  $u_0 \equiv 0$  (note that since  $g$  is increasing,  $u_0 \leq w_0 \Rightarrow \forall t \geq 0, S(t)u_0 \leq S(t)w_0$ ). Moreover,  $g(0) > 0 \Rightarrow u_t > 0$  for  $t$  small. Then, for  $\delta > 0$  small,

$$u(t + \delta) = S(t + \delta)0 = S(t) \circ S(\delta)0 \geq S(t)0 = u(t) \text{ and } u_t \geq 0, \forall t \geq 0$$

Now, taking  $\phi \in C_0^2(\bar{B}_1)$ , multiplying the equation in  $(P_t)$  by  $\frac{\phi}{|x|^2}$  and integrating by parts, we obtain :

$$\frac{d}{dt} \int_{B_1} \frac{u(t)\phi}{|x|^2} - \int_{B_1} u\Delta\phi = \int_{B_1} \frac{g(u(t))\phi}{|x|^2} \tag{3.5}$$

Therefore, choosing  $\phi = \psi_\epsilon$  (defined in the proof of Theorem 3.1) we have :

$$\frac{d}{dt} \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} + \lambda_\epsilon^1 \int_{B_1} \frac{u\psi_\epsilon}{|x|^2 + |\epsilon|^2} = \int_{B_1} \frac{g(u(t))\psi_\epsilon}{|x|^2}$$

Thus,

$$\frac{d}{dt} \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \geq \int_{B_1} \left( \frac{g(u(t))}{u(t)} - \lambda_\epsilon^1 \right) \frac{u(t)\psi_\epsilon}{|x|^2}$$

which provides the following alternative :

1. either there exists  $M > 0$  such that  $\int_{B_1} \frac{g(u(t))\psi_\epsilon}{|x|^2}, \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \leq M$  for all  $t \geq 0$ , or
2.  $\int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \xrightarrow{t \rightarrow +\infty} +\infty$ .

Let us suppose that the second case holds. Then, by Jensen's inequality, we have for  $t$  large enough :

$$\frac{d}{dt} \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \geq \frac{1}{2} \int_{B_1} \frac{g(u(t))\psi_\epsilon}{|x|^2} \geq \frac{1}{2} g \left( \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \right)$$

Hence,

$$\int_0^{f(t)} \frac{ds}{g(s)} \geq \frac{1}{2} t + C \quad \text{where } f(t) = \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2}$$

which contradicts (B7). And  $u(t) = S(t)0$  blows up in finite time.

Finally, suppose that the first case occurs. And let  $\zeta$  denote the unique solution of the following problem :

$$\begin{cases} -(|x|^2)\Delta\zeta = 1 & \text{in } B_1 \\ \zeta \equiv 0 & \text{in } \partial B_1 \end{cases}$$

For  $N \geq 3$ , it is easy to prove that  $\zeta \in W^{2,p}(B_1)$  for all  $p < \frac{N}{2}$  which by Hardy's inequality and by Sobolev's embedding implies that  $\zeta \in L^2(\frac{dx}{|x|^2}) \cap H_0^1(B_1)$ . Hence, there exists  $\{\zeta_n\}_{n \in \mathbb{N}} \subset C_0^\infty(B_1)$  such that :

$$\Delta\zeta_n \xrightarrow{L^1} \Delta\zeta \quad \text{and} \quad \zeta_n \xrightarrow{L^2(\frac{dx}{|x|^2})} \zeta \tag{3.6}$$

Choosing  $\phi := \zeta_n$  in (3.5) and integrating in  $[t, t + 1]$ , we have :

$$\begin{aligned} \left[ \int_{B_1} \frac{u(s)\zeta_n}{|x|^2} \right]_t^{t+1} + \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta_n) \\ = \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta_n}{|x|^2} \end{aligned} \tag{3.7}$$

Passing to the limit  $n \rightarrow \infty$ , we obtain by (3.6) :

$$\int_{B_1} \frac{u(s)\zeta_n}{|x|^2} \xrightarrow{n \rightarrow \infty} \int_{B_1} \frac{u(s)\zeta}{|x|^2}$$

Moreover, by Lebesgue theorem and by (3.6) :

$$\int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta_n}{|x|^2} \xrightarrow{n \rightarrow \infty} \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2}$$

and

$$\int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta_n) \xrightarrow{n \rightarrow \infty} \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta)$$

Therefore,

$$\left[ \int_{B_1} \frac{u(s)\zeta}{|x|^2} \right]_t^{t+1} + \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta) = \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2}.$$

Now, since  $u_t \geq 0$ ,

$$\begin{aligned} \int_{B_1} \frac{u(t)}{|x|^2} &\leq \int_t^{t+1} ds \int_{B_1} \frac{u(s)\zeta}{|x|^2} = \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta) \\ &= \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2} - \left[ \int_{B_1} \frac{u(s)\zeta}{|x|^2} \right]_t^{t+1} \\ &\leq \int_{B_1} \frac{g(u(t+1))\zeta}{|x|^2} \leq M \end{aligned}$$

Therefore, by monotone convergence, there exists  $w \in L^1(\frac{dx}{|x|^2})$  such that  $u(t) \xrightarrow{t \rightarrow +\infty} w$  in  $L^1(\frac{dx}{|x|^2})$ . It implies that for all  $\phi \in C_0^2(\bar{B}_1)$  :

$$\begin{aligned} \int_{B_1} \frac{u(t)\phi}{|x|^2} \xrightarrow{t \rightarrow +\infty} \int_{B_1} \frac{w\phi}{|x|^2}, \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\phi) \xrightarrow{t \rightarrow \infty} \int_{B_1} w(-\Delta\phi) \text{ and} \\ \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\phi}{|x|^2} \xrightarrow{t \rightarrow \infty} \int_{B_1} \frac{g(w)\phi}{|x|^2} \end{aligned}$$

(For this, note that  $\int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\phi}{|x|^2} \leq 2 \int_{B_1} \frac{w\phi}{|x|^2} < +\infty$ ). Therefore, for all  $\phi \in C_0^2(\bar{B}_1)$  :

$$- \int_{B_1} w \Delta \phi = \int_{B_1} \frac{g(w)\phi}{|x|^2}$$

which contradicts the nonexistence of weak stationary solutions to  $(P_t)$ . This completes the proof of Theorem 3.3. ■

**Remarks.** Consider  $g \in C^1$  convex , increasing function satisfying  $\lim_{s \rightarrow 0^+} \frac{g(s)}{s} > \left(\frac{N-2}{2}\right)^2$ , (B7) and for all  $s \geq 0$ ,  $\frac{sg(s)}{2} \geq G(s) = \int_0^s g(t) dt$ . Then, we can apply the previous method. Precisely, for all  $u_0 > 0$ ,  $u(\cdot) = S(\cdot)u_0$  blows up in finite time in  $L^2(\frac{dx}{|x|^2})$ .

It suffices to modify the proof of Theorem 3.3 as follows :

1. 0 is replaced by  $\epsilon\phi_\epsilon$  which is a subsolution of  $(P)$  and  $\epsilon\phi_\epsilon < u_0$ , for  $\epsilon$  small enough.
2. The nonexistence of stationary solutions of  $(P_t)$  is provided by the results from [7].

## 4 Global existence of solutions to $(P_t)$ and convergence to a stationary solution

### 4.1 Main results

In this section, we give two examples of global existence of solutions to  $(P_t)$  which converge to a stationary solution when  $t \rightarrow \infty$ . In each case, we obtain an exponential control of the convergence either in  $L^\infty$  or in  $H_0^1(B_1)$ . Here, it is worth to underline that the convergence to a stationary solution is related to the uniqueness of the solution to  $(P)$ . First, we prove the following :

**Theorem 4.1.** Assume that  $N \geq 2$  and the following hypothesis :

$$(G1) \quad \lim_{s \rightarrow 0^+} \frac{g(s)}{s} = \lambda < 0,$$



(G2) There exists  $\epsilon > 0$ , such that  $|g(s) - \lambda s| \leq C|s|^{1+\epsilon}$ .

Then, for  $u_0$  such that  $\|u_0\|_{L^\infty}$  small enough,  $u(\cdot) = S(\cdot)u_0$  is global and there exists  $C > 0$  such that  $\|u(t)\|_{L^\infty} \leq Ce^{\lambda t}$  for all  $t \geq 0$ .

In the second part of the section, we prove the following theorem :

**Theorem 4.2.** Assume that  $N \geq 3$  and  $g$  satisfies the following assumptions :

(G3)  $s \rightarrow \frac{g(s)}{s}$  is continuous and strictly decreasing,

(G4)  $\frac{g(s)}{s} \xrightarrow{s \rightarrow +\infty} -\infty$ ,

(G5)  $\frac{g(s)}{s} \xrightarrow{s \rightarrow 0^+} \lambda > (\frac{N-2}{2})^2$ ,

Then, for any  $u_0$  such that  $0 < u_0 \leq f^{-1}(0)$  and  $u_0 \not\equiv f^{-1}(0)$ , with  $f(s) := \frac{g(s)}{s}$ ,  $u(t) = S(t)u_0$  is global and converges to the unique non-trivial solution of (P),  $w_\lambda$ , when  $t \rightarrow \infty$ . Moreover, if we suppose, in addition, that  $-g$  is strictly convex, there exists  $K > 0$  such that  $\|u(t) - w_\lambda\|_{H^1_0(B_1)} \leq Ce^{-Kt}$  for all  $t \geq 0$ .

We start by proving a proposition which provides the heat kernel of  $-|x|^2 \Delta$  :

**Proposition 4.3.** Consider  $u = T(t)u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$  solution of

$$\begin{cases} u_t - |x|^2 \Delta u = \lambda u & \text{in } B_1 \\ u(t, x) = 0 & \text{in } \mathbb{R}^+ \times \partial B_1, \quad u(0, x) = u_0 \end{cases}$$

where  $u_0$  is radial. Then,  $u(t)$  is radial and if  $v(t, s) := u(t, x)$  with  $s = -\ln|x|$  and  $\lambda_N = (\frac{N-2}{2})^2$ , then,

$$v(t, s) = \frac{e^{\frac{N-2}{2}s - (\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} * v(0, s).$$

**Proof.** First, we remark that the radial symmetry of  $u$  follows from the uniqueness of the solution to  $(P_t)$ . Then, to compute the heat kernel of

$-|x|^2\Delta$ , we use a method from [7]. Indeed, put  $w(t, s) := e^{-\frac{N-2}{2}s}v(t, s)$ . We show that  $w$  satisfies :

$$(P_w) \begin{cases} w_t - w_{ss} = (\lambda - \lambda_N)w \text{ in } \mathbb{R}^+ \times (0, +\infty) \\ w(t, 0) = 0, w(t, s) \xrightarrow{s \rightarrow +\infty} 0, w(0, s) = v(0, s)e^{-\frac{N-2}{2}s} \end{cases}$$

Taking  $w(t, -s) = -w(t, s)$  for all  $s \geq 0$ , we have that  $(P_w)$  is satisfied in  $\mathbb{R}^+ \times \mathbb{R}$ . And we can apply Fourier transform. Indeed, for  $N \geq 2$ ,  $w(t, \cdot)$  belongs to  $L^2(\mathbb{R})$  (for  $N > 2$ , it is obvious since  $v \in L^\infty$  and for  $N = 2$ , it suffices to remark that  $\int_{B_1} \frac{|u|^2}{|x|^2} < \infty \Rightarrow \int_0^{+\infty} w^2 ds < \infty$ ).

A simple computation shows that  $\hat{w}(t, x) = \hat{w}_0 e^{-(|x|^2 + (\lambda_N - \lambda)t)}$ . Using inverse Fourier transform, one has :

$$w(t, s) = w_0 * \frac{e^{-(\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \quad \text{and} \quad v(t, s) = v_0 * \frac{e^{\frac{N-2}{2}s - (\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}.$$

This completes the proof of Proposition 4.3. ■

Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Here, we apply a method from [4]. First, remark that by the maximum principle, it suffices to prove Theorem 4.1 when  $u_0$  is radially symmetric. Then, by Proposition 4.3,  $T(t)u_0$  is radially symmetric and

$$\begin{aligned} \|T(t)u_0\|_{L^\infty} &= \left\| v_0 * \frac{e^{\frac{N-2}{2}s - (\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \right\|_{L^\infty} \\ &\leq \|v_0\|_{L^\infty} \left\| \frac{e^{\frac{N-2}{2}s - (\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \right\|_{L^1} \end{aligned} \tag{4.1}$$

Now, using Laplace transform  $f(y) := \int_{-\infty}^{+\infty} e^{yx - \frac{|x|^2}{4t}} dx$ , we show that  $\|e^{\frac{N-2}{2}s - \frac{s^2}{4t}}\|_{L^1} = e^{t\lambda_N} (4\pi t)^{\frac{1}{2}}$ . Therefore, by (4.1),  $\|T(t)u_0\|_{L^\infty} \leq e^{\lambda t} \|u_0\|_{L^\infty}$ . Now, we apply a method from [4]. First, we define  $\Theta(\cdot)$  such that  $\Theta(x) = \frac{C}{\epsilon|\lambda|} x^{1+\epsilon} - x$  with  $C$  defined in (G2) and  $\delta > 0$  satisfy

$$\min \Theta(x) + \delta < 0, \quad \Theta(\delta) + \delta \geq 0 \quad \text{and} \quad \Theta'(\delta) \leq 0$$

Let us choose  $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$  such that  $\|u_0\|_{L^\infty} \leq \delta$ . Then,  $u = S(t)u_0$  satisfies :

$$\begin{aligned} \|u(t)\|_{L^\infty} &\leq \|T(t)u_0\|_{L^\infty} + C \int_0^t e^{\lambda(t-s)} \|u(s)\|_{L^\infty}^{1+\epsilon} ds \leq e^{\lambda t} \delta \\ &+ C e^{\lambda t} \int_0^t e^{c\lambda s} (e^{-\lambda s} \|u(s)\|_{L^\infty})^{1+\epsilon} ds \end{aligned}$$

Putting  $\phi(t) = \sup_{[0,t]} e^{-\lambda s} \|u(s)\|_{L^\infty}$  which is an increasing function, we have :

$$\phi(t) \leq \delta + C \int_0^t \phi^{1+\epsilon}(s) e^{c\lambda s} ds \leq \delta + \frac{C}{\epsilon|\lambda|} \phi(t)^{1+\epsilon}.$$

If  $\mu = \inf\{x > 0 / \Theta(x) + \delta \leq 0\} \geq \delta$ , it is easy to prove that  $\phi(t) \leq \mu$  for all  $t \in [0, T[$ , where  $T$  is defined in Proposition 2.3. Moreover,  $\|u(t)\|_{L^\infty} \leq e^{\lambda t} \mu$ , which implies that  $u$  is global and  $T = \infty$ . This completes the proof of Theorem 4.1.

■

**Remarks.**

1. If  $p \in ]1, +\infty[$ , the function  $g : s \rightarrow s^p$  satisfies the hypothesis of Theorem 4.1. Therefore, Theorems 3.1 and 4.1 show that the behaviour of the solution of  $(P_t)$  depends on the initial data.
2. It is worth noticing that for  $N = 2$ , we obtain almost a complete description of the behaviour of solutions of  $(P_t)$ . Precisely,  $\lambda_N = (\frac{N-2}{2})^2 = 0$  is the "blow-up critical parameter" (see [13]) which means that for  $\lambda < \lambda_N$ , there exists global solutions of  $(P_t)$  for small initial data and if  $\lambda > \lambda_N$  then for all  $u_0 \neq 0$ ,  $u(t) = S(t)u_0$  blows up in finite time. However, we do not know what happens in the case  $\lambda = \lambda_N$ . Moreover, since the heat kernel of  $-|x|^2 \Delta$  does not vanish at the boundary, we cannot apply a method due to Fujita (see [9]) which would furnish the answer. For  $N \geq 3$ , we suspect that  $\lambda_N$  still remains the critical blow-up parameter.

Now, we give the proof of Theorem 4.2.

**Proof of Theorem 4.2.**

Since there is a unique nontrivial solution to (P), it suffices to prove Theorem 4.2 when  $u_0$  is radially decreasing. In this case,  $S(t)u_0$  is also radially decreasing. Indeed, choosing  $\epsilon \in ]0, 1[$ , we remark that  $u(t, \epsilon x) := u_\epsilon(t)$  is solution to

$$(P_{\epsilon,t}) \begin{cases} u_t - |x|^2 \Delta u = g(u) & \text{in } \mathbb{R}^+ \times B_{\frac{1}{\epsilon}} \\ u(t, x) = 0 & \text{in } \mathbb{R}^+ \times \partial B_{\frac{1}{\epsilon}}, \quad u(0, x) = u_0(\epsilon x) \end{cases}$$

Since  $u_0(\epsilon x) \geq u_0(x)$ , by the maximum principle, for any  $\epsilon \in ]0, 1[$ ,  $u_\epsilon(t) \geq S(t)u_0$  which proves that  $S(t)u_0$  is radially decreasing.

Now, as above we prove that :

$$\frac{d}{dt} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = - \int_{B_1} |\nabla u(t)|^2 + \int_{B_1} \frac{g(u(t))u(t)}{|x|^2}$$

Moreover,  $E(u(t)) = \frac{1}{2} \int_{B_1} |\nabla u(t)|^2 - \int_{B_1} \frac{G(u(t))}{|x|^2}$  satisfies

$$\frac{d}{dt} E(u(t)) < 0 \quad \text{and} \quad E(u(t)) < E(u_0). \tag{4.2}$$

Futhermore, multiplying the equation in (P<sub>t</sub>) by  $\frac{(u-f^{-1}(0))^+}{|x|^2}$  we obtain :

$$\begin{aligned} \frac{d}{dt} \int_{B_1} \frac{((u(t) - f^{-1}(0))^+)^2}{|x|^2} + \int_{B_1} |\nabla(u(t) - f^{-1}(0))^+|^2 \\ = \int_{B_1} \frac{((\lambda - f(u(t)))(u - f^{-1}(0))^+)^2}{|x|^2} \leq 0 \end{aligned}$$

which implies that for all  $t \geq 0$ ,  $u(t) \leq f^{-1}(0)$  and therefore  $\cup_{t \geq 0} \{u(t)\}$  is uniformly bounded in  $L^\infty(B_1)$ . By (4.2), for  $N \geq 3$ , it follows that :

$$\int_{B_1} \frac{|\nabla u(t)|^2}{2} \leq E(u_0) - G(f^{-1}(0)) \int_{B_1} \frac{1}{|x|^2} dx \leq C$$

Therefore,  $\cup_{t \geq 0} \{u(t)\}$  is bounded in  $L^\infty(B_1) \cap H_0^1(B_1)$ . Then, for any sequence  $\{t_n\}_{n \in \mathbb{N}}$  such that  $t_n \rightarrow +\infty$ , there is  $w \in L^\infty(B_1) \cap H_0^1(B_1)$  (depending a priori on  $\{t_n\}_{n \in \mathbb{N}}$ ) satisfying

$$u(t_n) \xrightarrow{n \rightarrow \infty} w \text{ in } H_0^1(B_1), \quad u(t_n) \xrightarrow{n \rightarrow \infty} w \text{ in } L^2\left(\frac{dx}{|x|^2}\right)$$

and

$$G(u(t_n)) \xrightarrow{n \rightarrow \infty} G(w) \text{ in } L^1\left(\frac{dx}{|x|^2}\right).$$

For this, notice that on one hand

$$\int_{B_1} \frac{|u(t_n) - w|^2}{|x|^2} \leq \left(\int_{B_1} |u(t_n) - w|^{p'}\right)^{\frac{1}{p'}} \left(\int_{B_1} \frac{1}{|x|^{2p}}\right)^{\frac{1}{p}}$$

where  $p < \frac{N}{2}$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . On the other hand, since  $\cup_{t \geq 0} \{u(t)\}$  and  $w$  are uniformly bounded in  $L^\infty$ ,

$$\int_{B_1} \frac{|G(u(t_n)) - G(w)|^2}{|x|^2} \leq C \int_{B_1} \frac{|u(t_n) - w|^2}{|x|^2}$$

Let us show that

$$u(t_n) \xrightarrow{H_0^1(B_1)} w \text{ when } n \rightarrow \infty$$

For this, it suffices to prove that  $\int_{B_1} |\nabla u(t_n)|^2 \xrightarrow{n \rightarrow \infty} \int_{B_1} |\nabla w|^2$ . Let us prove that  $\int_{B_1} |\nabla u(t_n)|^2$  does not concentrate in  $x = 0$ . First, for any  $\delta < 1$ , multiplying the equation in  $(P_t)$  by  $\frac{u(t)}{|x|^2}$  in  $B_\delta$ , we have :

$$\frac{d}{dt} \int_{|x| \leq \delta} \frac{|u|^2}{|x|^2} + \int_{|x| \leq \delta} |\nabla u(t)|^2 - \int_{|x| = \delta} \frac{\partial u(t)}{\partial n} u(t) ds = \int_{|x| \leq \delta} \frac{g(u(t))u(t)}{|x|^2}$$

Since  $u(t)$  is radially decreasing,

$$\frac{d}{dt} \int_{|x| \leq \delta} \frac{|u|^2}{|x|^2} + \int_{|x| \leq \delta} |\nabla u(t)|^2 \leq \int_{|x| \leq \delta} \frac{g(u(t))u(t)}{|x|^2} \tag{4.3}$$

Integrating (4.3) in  $[t, t + 1]$ , we obtain :

$$\left[ \int_{|x| \leq \delta} \frac{u(s)^2}{|x|^2} \right]_t^{t+1} + \int_t^{t+1} ds \int_{|x| \leq \delta} |\nabla u(s)|^2 \leq C \int_{|x| \leq \delta} \frac{1}{|x|^2}$$

where  $C$  is independent of  $t$ . Then, for all  $\epsilon > 0$ , there is  $\delta(\epsilon) > 0$  small enough such that for all  $\delta \leq \delta(\epsilon)$ , we have :

$$0 \leq \int_t^{t+1} \int_{|x| \leq \delta} |\nabla u(t)|^2 \leq \epsilon \tag{4.4}$$

To conclude the proof, suppose that  $\int_{B_1} |\nabla w|^2 < \lim_{n \rightarrow \infty} \int_{B_1} |\nabla u(t_n)|^2$ . Then, by (4.2)  $E(w) < E_\infty = \lim_{t \rightarrow \infty} E(u(t))$ . However, by (4.4), it is easy to prove that

$$\int_t^{t+1} ds \int_{B_1} |\nabla u(t_n + \tau)|^2 \xrightarrow{n \rightarrow \infty} \int_t^{t+1} \int_{B_1} |\nabla(S(\tau)w)|^2 \tag{4.5}$$

Indeed, by the boundedness of  $\{u(t)\}_{t \geq 0}$  in  $L^\infty \cap H_0^1(B_1)$ ,

$$\begin{aligned} \||x|^2 \Delta(S(t + \tau)u_0)\|_{L^2(\frac{dx}{|x|^2})} &\leq \||x|^2 \Delta(T(t)u_0)\|_{L^2(\frac{dx}{|x|^2})} \\ &+ \int_t^{t+\tau} \||x|^2 \Delta T(t + \tau - s)g(u(s))\|_{L^2(\frac{dx}{|x|^2})} ds \end{aligned}$$

from which it follows :

$$\begin{aligned} \||x|^2 \Delta S(t + \tau)u_0\|_{L^2(\frac{dx}{|x|^2})} &\leq \frac{C}{\tau} \|u(t)\|_{L^\infty} \\ &+ C \int_t^{t+\tau} \frac{ds}{(t + \tau - s)^{\frac{1}{2}} s^{\frac{1}{2}}} \|u_0\|_{L^\infty} \leq \frac{C}{\tau} \end{aligned}$$

(for this, using a method from [4] Lemma 3.10, we prove that

$$\||x|^2 \Delta T(t)u_0\|_{L^2(\frac{dx}{|x|^2})} \leq \frac{1}{t^{\frac{1}{2}}} \|u_0\|_{H_0^1(B_1)}$$

and

$$\|T(t)u_0\|_{H_0^1(B_1)} \leq t^{-\frac{1}{2}} \|u_0\|_{L^2(\frac{dx}{|x|^2})} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^\infty}$$

for  $N \geq 3$ )

Finally, (4.4) and the compactness of the embedding  $H^2(\delta \leq |x| \leq 1) \hookrightarrow H^1(\delta \leq |x| \leq 1)$  imply (4.5). Now, using that  $E(\cdot)$  is decreasing, we have :

$$\int_t^{t+1} E(S(\tau)w) d\tau \leq E(w) < E_\infty$$

which contradicts (4.5). Thus,

$$\int_{B_1} |\nabla u(t_n)|^2 \xrightarrow{n \rightarrow \infty} \int_{B_1} |\nabla w|^2$$

and for any

$$t \geq 0 E(S(t)(w)) = E(w) = E_\infty.$$

This implies that  $w$  is a stationary solution of  $(P_t.u_0)$  and either  $w \equiv 0$ , or  $w \equiv w_\lambda$  which is the nontrivial solution of  $(P)$ .

Now, let us prove that  $u(t) \xrightarrow{t \rightarrow \infty} w$  in  $L^\infty(B_1)$ . By a bootstrap argument (see [12]), it is easy to prove that for any  $\delta > 0$ ,

$$\|u(t) - w\|_{L^\infty(|x| \geq \delta)} \xrightarrow{t \rightarrow \infty} 0 \tag{4.6}$$

We consider  $u_0 := \epsilon \psi_\epsilon$  which satisfies

$$|x|^2 \Delta u_0 + g(u_0) \geq 0 \text{ if } \epsilon \text{ is small enough} \tag{4.7}$$

We recall that  $\psi_\epsilon$  is the eigenfunction of  $-(|x|^2 + |\epsilon|^2)\Delta$  defined in the proof of Theorem 3.1. Note that by (G5) and (G4), (4.7) is satisfied for  $\epsilon$  small enough. Then,  $u_0$  is a strict subsolution of  $(P)$  and as above, it implies that  $\frac{d}{dt} S(t)u_0 \geq 0$  for all  $t \geq 0$ . Hence,  $u(\cdot)$  is increasing. Hence,  $w = v_\lambda$ . Then, by Dini's theorem and (4.6), we have for all  $\delta > 0$  :

$$\|u(t) - w_\lambda\|_{L^\infty(|x| \geq \delta)} \xrightarrow{t \rightarrow \infty} 0 \tag{4.8}$$

Moreover, from [8] we know that  $w_\lambda(0) = f^{-1}(0)$ . Therefore, since  $\{u(t)\}_{t \geq 0} \cup \{w_\lambda\}$  are radially decreasing

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|w_\lambda - u(t)\|_{L^\infty(B_1)} &\leq \limsup_{t \rightarrow \infty} \|u(t) - w_\lambda\|_{L^\infty(|x| \leq \delta)} \\ &+ \limsup_{t \rightarrow \infty} \|u(t) - w_\lambda\|_{L^\infty(|x| \geq \delta)} \\ &= \limsup_{t \rightarrow \infty} \|u(t) - w_\lambda\|_{L^\infty(|x| \leq \delta)} \leq f^{-1}(0) \\ &- \lim_{t \rightarrow \infty} \lim_{|x| \rightarrow 0} u(t, x) \end{aligned}$$

Thus, suppose that  $c_\lambda := \lim_{t \rightarrow \infty} \lim_{|x| \rightarrow 0} u(t, x) < f^{-1}(0)$ . Then, since  $u(t, \cdot)$  is radially decreasing, for any  $x_\delta$  such that  $|x| = \delta > 0$ ,  $w_\lambda(x_\delta) = \lim_{t \rightarrow \infty} u(t, x_\delta) \leq c_\lambda$ . This contradicts that  $w_\lambda$  is continuous.

Now, considering any  $u_0$  such that  $0 < u_0 \leq f^{-1}(0)$  and  $u_0 \neq f^{-1}(0)$ , there exists  $\epsilon > 0$  small enough such that  $\epsilon \psi_\epsilon < u_0$ . It implies that

$$S(t)(\epsilon \psi_\epsilon) < S(t)(u_0) \leq f^{-1}(0) \text{ and } \|S(t)u_0 - w_\lambda\|_{L^\infty(B_1)} \xrightarrow{t \rightarrow \infty} 0 \tag{4.9}$$

To conclude the proof of Theorem 4.2, let us prove that if we suppose, in addition, that  $-g$  is strictly convex, then, there exists  $K > 0$  such that  $\|u(t) - w_\lambda\|_{H_0^1(B_1)} \leq C e^{-Kt}$  for all  $t \geq 0$ . First, note that

$$\begin{aligned} \frac{d}{dt} \int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} + \int_{B_1} \frac{|\nabla(w_\lambda - u(t))|^2}{|x|^2} \\ = \int_{B_1} \frac{(g(w_\lambda) - g(u(t)))(w_\lambda - u(t))}{|x|^2} \end{aligned}$$

By (4.9), for  $t$  large enough, we have :

$$\frac{d}{dt} \int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} + \frac{\lambda_1}{2} \left(-\Delta - \frac{g'(w_\lambda)}{|x|^2}\right) \int_{B_1} \frac{(w_\lambda - u(t))^2}{|x|^2} \leq 0 \quad (4.10)$$

where  $\frac{\lambda_1}{2} \left(-\Delta - \frac{g'(w_\lambda)}{|x|^2}\right)$  is the first eigenvalue of  $\left(-\Delta - \frac{g'(w_\lambda)}{|x|^2}\right)$  in  $H_0^1(B_1)$ . Then, the strict convexity of  $-g$  implies (see [8]) :

$$\lambda_1 \left(-\Delta - \frac{g'(w_\lambda)}{|x|^2}\right) > \lambda_1 \left(-\Delta - \frac{1}{|x|^2} \frac{g(w_\lambda)}{w_\lambda}\right) = 0$$

Thus, from (4.10), it is easy to prove that :

$$\int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} \leq C e^{-\frac{\lambda_1 t}{2}}$$

Using (4.10) and putting  $K = \frac{\lambda_1}{2}$ , we have for all  $t$  :

$$\int_{B_1} |\nabla(w_\lambda - u(t))|^2 \leq C e^{-Kt} \quad (4.11)$$

This completes the proof of Theorem 4.2. ■

**Remarks.**

1. If  $g(s) := \lambda s - |s|^{p-1}s$  where  $\lambda > \left(\frac{N-2}{2}\right)^2$  and  $p > 1$ , then,  $g$  satisfies the assumptions of Theorem 4.2.



- Suppose that  $-g$  is strictly convex and satisfies the assumptions (G3) to (G5). Then, taking  $\delta > 0$ , (4.11) and a bootstrap argument show that for all  $t \geq 0$  :

$$\|u(t) - w_\lambda\|_{L^\infty(|x| \geq \delta)} \leq C(\delta)e^{-Kt}.$$

However, we don't know if that remains valid for  $\delta = 0$ .

- The assumption (G5) and the second part of Assumption (G4) suffice to prevent that  $u(t) = S(t)u_0$  converges to 0 in  $L^\infty(B_1)$  when  $t \rightarrow \infty$  and when  $u_0 \neq 0$ .

Indeed, suppose that  $\|S(t)u_0\|_{L^\infty(B_1)} \xrightarrow{t \rightarrow \infty} 0$ . Then, adapting a method from [14], we consider  $\epsilon$  small enough such that  $\lambda_\epsilon^1 < \lambda$ . Then, multiplying the equation in  $(P_t)$  by  $\psi_\epsilon$  :

$$\frac{d}{dt} \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} + \lambda_\epsilon^1 \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2 + |\epsilon|^2} = \int_{B_1} \frac{g(u(t))\psi_\epsilon}{|x|^2}$$

from which it follows for  $t$  large enough :

$$\begin{aligned} \frac{d}{dt} \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} &\geq -\lambda_\epsilon^1 \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} + \int_{B_1} \frac{g(u(t))\psi_\epsilon}{|x|^2} \\ &\geq \frac{1}{2}(g'(0) - \lambda_\epsilon^1) \int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \end{aligned} \tag{4.12}$$

Moreover, assumption (G5) implies that for  $\epsilon$  small enough,  $g'(0) = \lambda > \lambda_\epsilon^1$ . Thus, by (4.12), we have  $\int_{B_1} \frac{u(t)\psi_\epsilon}{|x|^2} \geq Ce^{\frac{(\lambda - \lambda_\epsilon^1)t}{2}} \xrightarrow{t \rightarrow \infty} +\infty$  which contradicts the uniform boundedness of  $\{u(t)\}_{t \geq 0}$ .

- In [8], the authors show the existence and the uniqueness of the solution,  $u_\epsilon$ , to the following pertubed problem :

$$(P_\epsilon) \begin{cases} -|x|^2 \Delta u = g(u) + \epsilon f(u) & \text{in } B_1 \\ u \in H_0^1(B_1) , u \geq 0 \end{cases}$$

where  $g$  satisfies (G3) to (G5),  $f$  is a positive function in  $\mathbb{R}^+$  and belongs to  $C^1(\mathbb{R}^+)$  such that  $\lim_{s \rightarrow +\infty} f(s) + g(s) = -\infty$  and  $\epsilon > 0$  small enough. Moreover,  $\lambda_1(-\Delta - \frac{g'(u_\epsilon) + \epsilon f'(u_\epsilon)}{|x|^2}) > 0$ . Then, Theorem 4.2 holds for  $(P_\epsilon)$ .

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