REVISTA MATEMÁTICA COMPLUTENSE
Volumen 11, número 2: 1998

An unknotting theorem for tori in S^4.

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Abstract

Let T be a torus in S^4 and T^* a projection of T. If the singular set $\Gamma(T^*)$ consists of one disjoint simple closed curve, then T can be moved to the standard position by an ambient isotopy of S^4.

1 Introduction

In this paper we will study an embedded torus T in S^4. If the singular set of the projection T^* ($\subset S^3$) of T consists of one double curve, then what can be said about the position of T? The following theorem is the main result.

Main Theorem (Theorem 4.1). *Let T be a torus in S^4. If the singular set $\Gamma(T^*)$ consists of one simple closed curve, then T can be moved to the standard position by an ambient isotopy of S^4.*

We will work in the PL category. All submanifolds are assumed to be locally flat. Let S^4 be the 4-dimensional sphere, S^3 the 3-dimensional sphere, and $p : S^4 \setminus \{\infty\} \longrightarrow S^3 \setminus \{\infty\}$ the projection defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$.

Let $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 1\}$, and $P_i = B \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i = 0\}$. Let F be a closed oriented surface, and $f : F \longrightarrow S^3 \setminus \{\infty\}$ a map. We say that f is in *general position*, if for each element x of $f(F)$, there exist a regular neighborhood N of x in $S^3 \setminus \{\infty\}$ and a homeomorphism $h : N \longrightarrow B$ such that N and h satisfy the following two conditions:

1991 Mathematics Subject Classification: Primary 57Q45, Secondary 57Q35.

Servicio Publicaciones Univ. Complutense. Madrid, 1998.

(1) Under h, $(N, N \cap f(F), x)$ is homeomorphic to either $(B, P_1, (0,0,0))$, $(B, P_1 \cup P_2, (0,0,0))$ or $(B, P_1 \cup P_2 \cup P_3, (0,0,0))$.

(2) Let R be a component of $f^{-1}(f(F) \cap N)$. There exists an integer i such that $h \circ f|R : R \longrightarrow P_i$ is a homeomorphism.

Note. If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_1 \cup P_2, (0,0,0))$, then x is called a *double point*. If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_1 \cup P_2 \cup P_3, (0,0,0))$, then x is called a *triple point*.

Throughout this paper, we assume that $p|F$ is in general position.

With every point P or subset F of $S^4 \setminus \{\infty\}$, we associate the point $P^* = p(P)$ or the subset $F^* = p(F)$. We define $\Gamma(F^*)$ to be the set of all double points and triple points and put $\Gamma(F) = p^{-1}(\Gamma(F^*)) \cap F$.

A solid torus V is said to be *standard* in S^3, if V is a regular neighborhood of a trivial knot in S^3. And the torus $\partial V \subset S^3 \subset S^4$ is said to be a *standard torus* in S^4. In [H-K], they proved that a boundary of a handlebody in S^4 is unique up to ambient isotopies of S^4.

The circle is taken to be the quotient space $S^1 = \mathbb{R}/(\theta \sim \theta + 2\pi$ for all $\theta \in \mathbb{R})$. We will write "$\theta \in S^1$". We denote by (a, b) the greatest common divisor of the integers a and b. Let $p_b : I \times S^1 \longrightarrow I \times S^1$ be the b-fold cyclic cover given by $(x, \theta) \mapsto (x, b\theta)$ for $b \in \mathbb{Z} \setminus \{0\}$. Let $r_\phi : I \times S^1 \longrightarrow I \times S^1$ be the rotation map given by $(x, \theta) \mapsto (x, \theta + \phi)$ for $\phi \in S^1$. Let $\alpha : S^1 \longrightarrow I \times S^1$ be an immersion. Let $i_\theta : I \times S^1 \longrightarrow I \times S^1 \times \theta \subset I \times S^1 \times S^1$ be the inclusion map $(x, \phi) \mapsto (x, \phi, \theta)$. Let a, b be integers satisfying $b \neq 0$. We define immersed surfaces $\alpha(a, b)$ in $I \times S^1 \times S^1$, which satisfies

$$\alpha(a, b) \cap I \times S^1 \times \theta = i_\theta r_{a\theta/b}(p_b^{-1}(\alpha(S^1))).$$

In particular, we denote by $T_1(a, b)$ the immersed tori $\alpha(a, b)$ obtained from α shown in Figure 1.

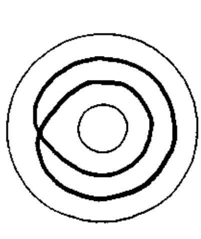

Figure 1

All the homology groups are with coefficients in $\boldsymbol{Z}$.

Example 1.1. If $(a,b)=1$ and $b \neq 0$, then there exists a torus T in S^4 with $T^* = \alpha(a,b)$ (see [T, Theorem 8]).

Example 1.2. There exists an embedded torus T in S^4 with $T^* = T_1(a,b)$ where $(a,b)=1$, $b \neq 0$. We can check that $(S^3, \Gamma(T^*))$ is homeomorphic to (S^3, (a,b)-torus knot) where (a,b)-torus knot is defined in [R] (see p 53). Therefore $T_1(a,b)$ is the immersed torus having the singular set $\Gamma(T^*)$ of one simple closed curve.

2 Solid tori and immersed surfaces in S^3

Lemma 2.1. *Let V be a solid torus, A a properly embedded annulus into V with $[a_i] \neq 0$ in $H_1(V)$ where a_0, a_1 are the components of ∂A, then there exists an embedding map $h : A \times I \longrightarrow V$ with $h(a,0) = a$ for all $a \in A$, and $h(\partial A \times I \cup A \times 1) \subset \partial V$.*

Proof (Only outline). We find a disk E such that $\partial E = l \cup k$, l and k are disjoint arcs, $intE \cap A = \phi$, $l \cap k = \partial l = \partial k$, $l \subset \partial V$, and $k \subset A$. Let B be a component of $\partial V \setminus (a_0 \cup a_1)$ with $B \supset l$. Then $A \cup B$ is a torus. There exists a 3-manifold W with $\partial W = A \cup B$, $W \supset E$. Let $N(E)$ be a regular neighborhood of E in W. We have that $\partial N(E) = D_0 \cup C \cup D_1$ such that D_i is a disk, C is an annulus, and $\partial N(E) \cap \partial W = C$. Then $\partial(\overline{W \setminus N(E)}) = (A \cup \overline{B \setminus C}) \cup D_0 \cup D_1$ is a 2-sphere. By the Schöenflies Theorem ([R] p 34), $\overline{W \setminus N(E)}$ is a 3-ball. W is obtained from $\overline{W \setminus N(E)}$ by attaching a 1-handle $N(E)$. Therefore W is a solid torus. We make a map h by using W.

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Lemma 2.2. *If V_1, V_2 and V_3 are solid tori in S^3 such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ is an annulus and $S^3 = V_1 \cup V_2 \cup V_3$, then there exist integers i, j such that V_i and V_j are standard solid tori in S^3.*

Proof. The set $V_1 \cap V_2 \cap V_3$ consists of two disjoint simple closed curves. Let c be a component of $V_1 \cap V_2 \cap V_3$. We denote $c = p_i l_i + q_i m_i \in H_1(\partial V_i)$ (i=1,2 or 3) where l_i is a preferred longitude of ∂V_i, m_i is a meridian of

∂V_i, and (p_i, q_i) is a pair of relatively prime integers. By van Kampen's theorem, we have $\pi_1(V_i \cup V_j) \cong < l_i, l_j | \ l_i^{p_i} = l_j^{p_j} >$. We get

$$H_1(V_i \cup V_j) \cong \begin{cases} \boldsymbol{Z} & \text{if } (p_i, p_j) = 1 \\ \boldsymbol{Z} \oplus \boldsymbol{Z}_{|d|} & \text{if } (p_i, p_j) = |d| \neq 1 \\ \boldsymbol{Z} \oplus \boldsymbol{Z}_{|p_s|} & p_k = 0, p_s \neq 0, \{k, s\} = \{i, j\} \\ \boldsymbol{Z} \oplus \boldsymbol{Z} & p_i = p_j = 0 \end{cases}$$

Since $V_i \cup V_j$ is the complement of an open regular neighborhood of some knot, $H_1(V_i \cup V_j) \cong \boldsymbol{Z}$. Hence we have to consider the following two cases:

(1) $p_i \neq 0, p_j \neq 0, (p_i, p_j) = 1$ or

(2) $p_k = 0, p_s = \pm 1, \{k, s\} = \{i, j\}$.

Case (1).

We construct a Seifert fibration on S^3 in which each solid torus V_i has c as a fiber. If $|p_i| \neq 1$ for all i, then there are three exceptional fibers. But we can show that in any Seifert fibration of the 3-sphere, there are at most two exceptional fibers (see [J-S] p 181). This is a contradiction. Hence there exists an integer k with $p_k = \pm 1$. We have $\pi_1(V_i \cup V_k) \cong < l_i, l_k | \ l_i^{p_i} = l_k^{\pm 1} > \cong \boldsymbol{Z}$. Therefore V_j is a standard solid torus $(j \neq i, k)$. Similarly, we can show that V_i is a standard solid torus.

Case (2).

Since $c = q_k m_k = \pm l_s + q_s m_s$, we have $q_k = \pm 1$. There exists a disk D in V_k with $c = \partial D \subset \partial V_k$. Hence $[c]$=0 in $H_1(S^3 \setminus intV_s)$ and $q_s = 0$. The solid torus V_s is a regular neighborhood of some knot K. But K is a boundary of some disk in S^3. Hence K is a trivial knot and V_s is a standard solid torus. Let $V = V_k \cup V_s$. Since $c = \pm m_k = \pm l_s$ and $V_k \cap V_s$ is an annulus, then V is a solid torus. Let V_t be the third solid torus with $t \neq k, s$. Then $S^3 = V \cup V_t$, $V \cap V_t = \partial V = \partial V_t$. But up to homeomorphism there is only one way of decomposing S^3 into two solid tori with the same boundary. Therefore V_t is a standard solid torus.

■

Remark. Let V_i, V_j be as above. If $H_1(V_i \cup V_j) \cong \boldsymbol{Z}$ and $[c]$=0 in $H_1(V_i \cup V_j)$, then $p_k = 0, p_s = \pm 1$, $\{k, s\} = \{i, j\}$.

Fact. Let F be a closed surface in S^4 with $p|F$ in general position, and c a simple closed curve in S^3 such that c is transverse to $f(F)$, $c \cap \Gamma(F^*) = \phi$. Then the number of points of $c \cap \Gamma(F^*)$ is even.

Lemma 2.3. *If F is an oriented closed surface in S^4 with $p|F$ in general position, then $F \setminus \Gamma(F)$ is divided into some regions. Then we can color each region black or white so that adjacent regions have different colors.*

Remark. Suppose that $\Gamma(F^*)$ consists of double points, and let n be a number of components in $\Gamma(F)$ which are not contractible in F. By Lemma 2.3, one sees that if F is a torus, then n is even.

Proof. Let $D_1, \ldots, D_s$ be the components of $S^3 \setminus F^*$. We will construct a function $f : \{D_1, \ldots, D_s\} \longrightarrow Z_2$. Let x_0 be a point of $S^3 \setminus F^*$, x_i a point in D_i, and l_i an arc in S^3 such that l_i is transverse to F^* and $\partial l_i = \{x_0, x_i\}$. We define $f(D_i) = 0$ if the number of points of $l_i \cap F^*$ is even, otherwise $f(D_i) = 1$. By Fact, we can show that f does not depend choices of x_i and l_i. And then f satisfies the property that D_i is an adjacent region of D_j (i.e. there exists a path $l \subset S^3$ such that $l(0) \in D_i, l(1) \in D_j, l(I) \cap \Gamma(F^*) = \phi$, and $l(I) \cap F^* = \{\text{one point}\}$), then $f(D_i) \neq f(D_j)$. Let $\mathcal{E} = \{E_1, \ldots, E_t\}$ be the components of $F^* \setminus \Gamma(F^*)$. The orientation of F induces the orientation of E_i. We define a function $h : \mathcal{E} \longrightarrow Z_2$ by $h(E_i) = 1$ if the positive normal vector of E_i points to a white region, otherwise $h(E_i) = 0$. Using h, we color the regions of $F \setminus \Gamma(F)$.

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Lemma 2.4. *Let F, $p|F$ be as above, and γ^* a component of $\Gamma(F^*)$. If γ^* is a simple closed curve, then $p^{-1}(\gamma^*) \cap F$ consists of two disjoint simple closed curves.*

Proof. Let N be a regular neighborhood of γ^* in S^3. Then $p^{-1}(N) \cap F$ consists of either two disjoint annuli, one Möbius band or two disjoint Möbius bands. Since F is an oriented surface, $p^{-1}(N) \cap F$ consists of two disjoint annuli. Therefore $p^{-1}(\gamma^*) \cap F$ is two disjoint simple closed curves. This completes the proof of Lemma 2.4.

∎

3 Local moves of surfaces in S^4

Lemma 3.1. *Let F be an oriented closed surface in S^4 with $p|F$ in general position. Let γ^* be a component of $\Gamma(F^*)$ which is a simple closed curve, c_1, c_2 the components of $p^{-1}(\gamma^*) \cap F$. If γ^* satisfies one of the following conditions, then γ^* can be cancelled by an ambient isotopy of S^4.*

(1) There exist disks D_1, D_2 in F with $\partial D_i = c_i$ and $int D_i \cap \Gamma(F) = \phi$.

(2) There exists an annulus A in F, and a solid torus V in S^3 such that $\partial A = c_1 \cup c_2$, $\partial V = A^*$, $int V \cap F^* = \phi$, and γ^* is a generator of $H_1(V) \cong \boldsymbol{Z}$.

(3) There exists an annulus A in F with $\partial A = c_1 \cup c_2$, $[c_i] = 1$ in $\pi_1(F)$, and $int A \cap \Gamma(F) = \phi$.

Proof. If γ^* satisfies (1), the lemma is proved by [Y, Lemma (4,4)]. If γ^* satisfies (2), the proof is easy.

Suppose γ^* satisfies (3). The surface A^* is an embedded torus in S^3, and γ^* is a simple closed curve on A^*. Since $[c_i] = 1$ in $\pi_1(F)$, there exist disks D_i in F with $\partial D_i = c_i$ (see [E, Theorem 1.7]). Let $D = D_i$ with $A \cap D_i = c_i$. Let V_1, V_2 be the closures of the components of $S^3 \setminus A^*$ with $V_1 \cup V_2 = S^3$, $\partial V_i = A^*$, and $V_1 \supset F^* \cup D^*$. By the solid torus theorem (see [R] p107), either V_1 or V_2 is a solid torus. In general, D^* is an immersed disk. By Dehn's lemma, there exists a non-singular disk E with $int E \cap A^* = \phi$ and $\partial E = \gamma^*$.

Case 1) V_1 is a solid torus.

Move T by an ambient isotopy of S^4, then we may assume that V_1 is a standard solid torus. And V_2 is a standard solid torus, too. We have $\gamma^* = \partial E \subset \partial V_1$, $E \subset V_1$. Then γ^* is a meridian of V_1 and a preferred longitude of V_2. We have $\partial A = c_1 \cup c_2$, $\partial V_2 = A^*$, $int V_2 \cap F^* = \phi$, and $[\gamma^*] = \pm 1$ in $H_1(V_2) \cong \boldsymbol{Z}$. Using Lemma 3.1 (2), we can prove the lemma in Case 1).

Case 2) V_2 is a solid torus.

Let l be a preferred longitude of ∂V_2, m a meridian of ∂V_2. We express $\gamma^* = pl + qm$ where (p, q) is a pair of relatively prime integers. Since $\gamma^* = \partial E \subset \partial V_1$, then $E \subset V_1$ and $[\gamma^*] = 0$ in $H_1(V_1)$. Hence

$|p|=1$ and $q = 0$. We have $\partial A = c_1 \cup c_2$, $\partial V_2 = A^*$, $int V_2 \cap F^* = \phi$, and $[\gamma^*] = \pm 1$ in $H_1(V_2) \cong \boldsymbol{Z}$. Using Lemma 3.1 (2), we can prove the lemma in Case 2).

■

We will define a symmetry-spun torus in S^4 (see [T]). Let $D^2 \times S^1$ be a solid torus, and K a knot in $D^2 \times S^1$. Let $\tilde{p}_b : D^2 \times S^1 \longrightarrow D^2 \times S^1$ be the b-fold cyclic cover given by $(x, \theta) \mapsto (x, b\theta)$ for $b \in \boldsymbol{Z} \setminus \{0\}$. Let $\tilde{r}_\phi : D^2 \times S^1 \longrightarrow D^2 \times S^1$ be the rotation map given by $(x, \theta) \mapsto (x, \theta + \phi)$ for $\phi \in S^1$. Let $\tilde{i}_\theta : D^2 \times S^1 \longrightarrow D^2 \times S^1 \times \theta \subset D^2 \times S^1 \times S^1$ be the inclusion map $(x, \phi) \mapsto (x, \phi, \theta)$. Let a, b be integers satisfying $b \neq 0$. We define an embedded torus $T^a(K_b)$ in $D^2 \times S^1 \times S^1$, which satisfies

$$T^a(K_b) \cap D^2 \times S^1 \times \theta = \tilde{i}_\theta \tilde{r}_{a\theta/b}(\tilde{p}_b{}^{-1}(K)).$$

And we identify $D^2 \times S^1 \times S^1$ with a regular neighborhood of a standard torus in S^4. Then the torus $T^a(K_b)$ is called a *symmetry-spun torus* in S^4.

Let T be a torus in S^4, $\alpha : S^1 \longrightarrow I \times S^1$ an immersion. Suppose $T^* = \alpha(a, b)$ where $(a, b) = 1$, and $b \neq 0$. Then there exists a knot $\tilde{\alpha}$ in $D^2 \times S^1$ such that T is ambient isotopic to $T^a(\tilde{\alpha}_b)$.

Remark. Let T be as above. There exists a symmetry-spun torus $T^a(\tilde{\alpha}_b)$ in S^4 such that $(T^a(\tilde{\alpha}_b))^* = \alpha(a, b)$ and T is ambient isotopic to $T^a(\tilde{\alpha}_b)$.

Lemma 3.2. *Let T be a torus in S^4, and α an immersion from S^1 to $I \times S^1$ with $T^* = \alpha(a, b)$ where $(a, b) = 1$, and $b \neq 0$. Let $\tilde{\alpha}$ be a knot in $D^2 \times S^1$ obtained from as above. If $\tilde{\alpha}$ is a trivial knot in S^3, then T can be moved to the standard position by an ambient isotopy of S^4.*

Proof. We may assume that T is ambient isotopic to $T^a(\tilde{\alpha}_b)$. By [T,Theorem 8], then there exists a homeomorphism $f : S^4 \longrightarrow S^4$ with $f(T^a(\tilde{\alpha}_b)) = T^0(\tilde{\alpha}_1)$ or $T^1(\tilde{\alpha}_1)$. We easily check that $T^0(\tilde{\alpha}_1)$ and $T^1(\tilde{\alpha}_1)$ can be moved to the standard position by an ambient isotopy of S^4. Then there exists a solid torus V in S^4 with $\partial V = T^0(\tilde{\alpha}_1)$ or $T^1(\tilde{\alpha}_1)$. Hence $\partial f^{-1}(V) = T^a(\tilde{\alpha}_b)$, and $f^{-1}(V)$ is a solid torus. By [H-K, Theorem 1.7], $T^a(\tilde{\alpha}_b)$ can be moved to the standard position by an ambient isotopy of S^4.

■

4 Main Theorem

Theorem 4.1. *Let T be a torus in S^4 with $p|T$ in general position. If $\Gamma(T^*)$ consists of one simple closed curve, then T can be moved to the standard position by an ambient isotopy of S^4.*

Proof. We distinguish four cases according to the position of $\Gamma(T)$. See Figure 2.

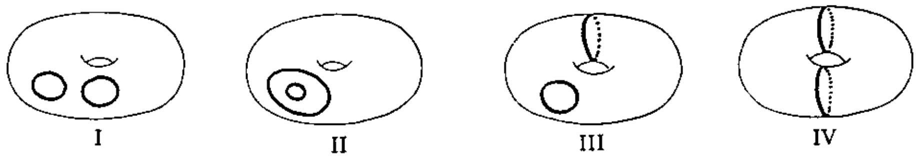

Figure 2

If the position of $\Gamma(T)$ is either I or II, then T can be moved to the standard position by Lemma 3.1. The case III cannot happen by Lemma 2.3. We will consider the case IV. Let A_1, A_2 be the closures of the components of $T \setminus \Gamma(T)$, and $\gamma^* = \Gamma(T^*)$. Then $T_i = p(A_i)$ is an embedded torus, and $T_1 \cap T_2 = \gamma^*$. By the solid torus theorem, there exist solid tori V_1, V_2 with $\partial V_i = T_i$. We distinguish two cases: (1) $T_i \subset V_j$ or (2) $V_i \cap T_j = \gamma^*$ ($\{i,j\} = \{1,2\}$).

Case 1) $T_1 \subset V_2$ or $T_2 \subset V_1$.

We may assume $T_1 \subset V_2$. Move T by an ambient isotopy of S^4, and we suppose that V_2 is a standard solid torus.

(1-i) $[\gamma^*]=0$ in $H_1(V_2)$.

The simple closed curve γ^* is a meridian of V_2. Let $V = S^3 \setminus intV_2$. Then A_2 is an annulus satisfying $\partial A_2 = c_1 \cup c_2$, $\partial V = A_2^*$, $intV \cap F^* = \phi$, and $[\gamma^*]$ is a generator of $H_1(V) \cong \boldsymbol{Z}$. By Lemma 3.1 (2), γ^* can be cancelled.

(1-ii) $[\gamma^*] \neq 0$ in $H_1(V_2)$.

Let N be a regular neighborhood of γ^* in V_2, $A = cl(\partial N \cap intV_2)$, and a_0, a_1 the components of ∂A. Then A is an annulus, and $[a_i] \neq 0$ in $H_1(V_2)$. Cut V_2 by a meridian disk. We obtain Figure 3 (1) by Lemma

2.1. In Figure 3 the curve γ^* is coiled four times to a preferred longitude of V_2. Let $V = \overline{V_2 \setminus N}$, and $B = T_1 \setminus intN$. Then V is a solid torus, and B is an annulus. Let b_0, b_1 be the components of ∂B, then $[b_i] \neq 0$ in $H_1(V)$. We obtain Figure 3 (2) or (3) by Lemma 2.1. By Lemma 3.1 (2), we cancel γ^* of Figure 3 (2). We see in Figure 3 (3) that T^* is an immersed torus $T_1(a,b)$ with $(a,b)=1$, $b \neq 0$. By Lemma 3.2, T can be moved to the standard position. We completed the proof in Case 1).

Case 2) $V_1 \cap T_2 = \gamma^*$ or $V_2 \cap T_1 = \gamma^*$.

If $V_2 \supset V_1$ or $V_1 \supset V_2$, then we can use the method of Case 1). Therefore, we may assume $V_1 \cap V_2 = \gamma^*$. Let N be a regular neighborhood of γ^* in S^3, and $W = V_1 \cup N \cup V_2$. Then ∂W is a torus.

(2-i) $[\gamma^*] = 0$ in $H_1(W)$.

We denote $\gamma^* = p_i l_i + q_i m_i \in H_1(\partial V_i)$ where l_i is a preferred longitude of ∂V_i and m_i is a meridian of ∂V_i. We calculate $H_1(V_1 \cup V_2)$ in a similar way to Lemma 2.2. Since $H_1(W) \cong \boldsymbol{Z}$ and $[\gamma^*] = 0$ in $H_1(W)$, we have $p_j = 0, |p_i| = 1$ where $\{i,j\} = \{1,2\}$ (see Remark after Lemma 2.2). Moreover, we get $|q_j| = 1$, and $\gamma^* = \pm l_i + q_i m_i$. Since γ^* is a boundary of a meridional disk of ∂V_j, V_i is a standard solid torus and $\gamma^* = \pm l_i$. By Lemma 3.1 (2), γ^* can be cancelled.

(2-ii) $[\gamma^*] \neq 0$ in $H_1(W)$.

Suppose that W is a solid torus. Let $A_i = V_i \cap \partial N$, and a_0^i, a_1^i be the components of ∂A_i. Then A_i is an annulus, and $[a_i^k] \neq 0$ in $H_1(W)$. Cut W by a meridional disk D. Using Lemma 2.1, we get Figure 4 (1). Drawing the picture of $T^* \cap N \cap D$, then we get Figure 4 (2). Then we see $T^* \cap D$ in Figure 4 (3). Moreover, γ^* satisfies Lemma 3.1 (2). Thus γ^* can be cancelled.

Suppose that W is not a solid torus. Let $V = S^3 \setminus intW$. By the solid torus theorem, V is a solid torus. We find an annulus A with $N \supset A \supset \gamma^*$, $\partial N \supset \partial A$, $A \cap (V_1 \cup V_2) = \gamma^*$, and $a_i \subset J_i$ where J_1 and J_2 are components of $\partial N \setminus (intV_1 \cup intV_2)$ and a_1, a_2 are the components of ∂A. Let N_i be the closure of the component of $N \setminus A$ with $N_i \cap intV_i \neq \phi$. Then $V_i \cup N_i$ is a solid torus. Let $Z_1 = V_1 \cup N_1$, $Z_2 = V_2 \cup N_2$ and $Z_3 = V$. Then Z_i is a solid torus, $Z_i \cap Z_j = \partial Z_i \cap \partial Z_j$ is the annulus, and $S^3 = Z_1 \cup Z_2 \cup Z_3$. By Lemma 2.2 and the fact that W is not a solid torus, we have that Z_1 and Z_2 are standard tori. Let $W_1 = V_1$, and $W_2 = S^3 \setminus intV_2$. Then W_i is a solid torus, $\partial W_i = \partial V_i = T_i$, and $W_2 \supset W_1$. We can reduce the argument to Case 1).

This completes the proof of Theorem 4.1. ∎

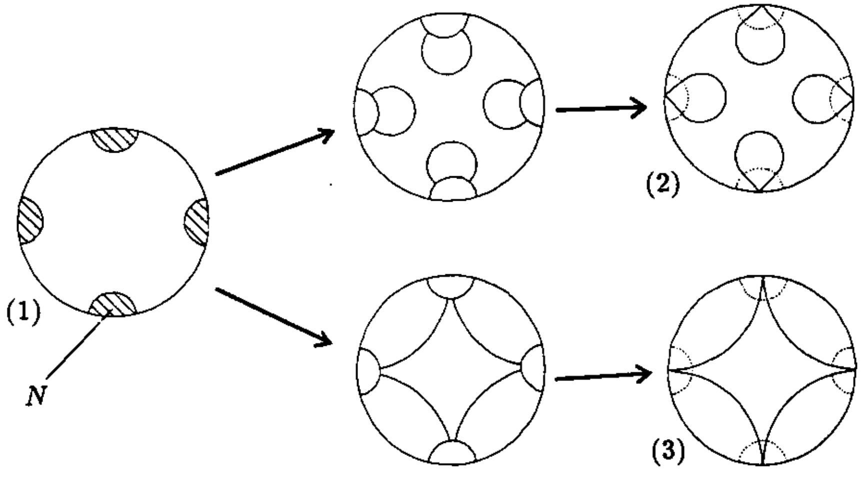

Figure 3

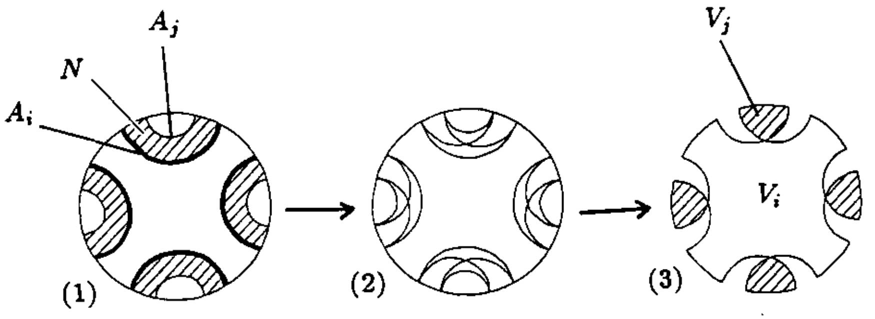

Figure 4

Acknowledgement. The author would like to express her sincere gratitude to Professor Yukio Matsumoto for his valuable advice and encouragement.

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Recibido: 16 de Enero de 1997