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Lie algebras with a given lattice of ideals.

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Abstract

We give a complete classification of Lie Algebras whose lattice of ideals has the length ≤ 2.

The classification problem is one of the major problems in the theory of finite dimensional Lie algebras over an algebraically closed field of zero characteristic. Levi's theorem and classification of Cartan-Killing of semisimple Lie algebras reduce this problem to classification of the solvable Lie algebras. The study of solvable Lie algebras generally can be reduce to study of nilpotent Lie algebras. In spite of the developed study of nilpotent Lie algebras over the last years the classification problem is still open; we only have the classification of nilpotent Lie algebras in dimension ≤ 7 [1]. Here we consider a different approach to the classification problem: classification of the Lie algebras in which the lattice of ideals has certain properties. This approach has been considered by Benito [2], [3]. In this paper we develope this study; in particular we obtain a complete classification of Lie algebras whose lattice of ideals has the length ≤ 2.

This paper is inspired by papers of Benito and by paper [7]. I thank Professor Khadjiev for his constructive suggestion. I thank also Professor Ayupov for the direction of this work; his suggestions and encouragements have been most helpful.

In this paper we only consider finite-dimensional Lie algebras over a field of complex numbers.

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1 Lattice of ideals of a Lie algebra

A *lattice* is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. A finite lattice may be representing by a diagram. One obtains such a diagram by representing the elements of the lattice by small circles(or dots); if $a \succ b$ and no c exists such that $a \succ c \succ b$ than we place the circle for a above that for b and we connecte these circles by a line. Then $x \succ y$ if and only if there is a descending broken line connecting x to y.

Let g be a Lie algebra and $L(g)$ be the set of ideals of g with $I \succ J$ defined to mean that ideal J is a subset of ideal I and $I \neq J$. Any two elements I and J of $L(g)$ have a least upper bound $\langle I \cup J \rangle$ (the ideal generated by $I \cup J$) and greatest lower bound $I \cap J$. The obtained lattice is called *lattice of ideals* of the Lie algebra g. As g is a finite dimensional Lie algebra, any descending chain of elements of $L(g)$ is finite. Let

$$g \succ I_1 \succ I_2 \succ \cdots \succ I_k = \{0\}$$

be a maximal descending chain connecting g and $\{0\}$. The number of intervals in this chain, which is uniquely determined by g is called the *length* of lattice $L(g)$.

It is very interesting to study the relations between a Lie algebra g and its lattice of ideals $L(g)$. In the papers [2] and [3] the structure of the Lie algebras in which the lattice of ideals is a chain is studied. In this paper we consider the problem of classification of the Lie algebras with a low length.

2 Solvable Lie algebras with a lattice of ideals of length ≤ 2.

Let g be a solvable Lie algebra ($g \neq 0$) and let

$$D^0 g \supset D^1 g \supset \cdots \supset D^{q-1} g \supset D^q g = \{0\}$$

be its derived serie of ideals defined by

$$D^0 g = g, \quad D^i g = [D^{i-1} g, D^{i-1} g], \quad i \geq 1.$$

We suppose that $D^{q-1}\boldsymbol{g} \neq 0$. Then, the length of lattice of ideals of $\boldsymbol{g}$, denoted $l(\boldsymbol{g})$, is $\geq q$. Thus, for description of solvable Lie algebras $\boldsymbol{g}$ with $l(\boldsymbol{g}) \leq 2$, it is sufficient to consider the Lie algebras with $q \leq 2$.

Proposition 1. *Let $\boldsymbol{g}$ be a solvable Lie algebra with $l(\boldsymbol{g}) \leq 2$. Then* $\dim \boldsymbol{g} \leq 2$.

Proof. Suppose $\dim \boldsymbol{g} \geq 3$. Following three cases are possible:

$$(1) \quad \dim D^0\boldsymbol{g} - \dim D^1\boldsymbol{g} \geq 2, \quad D^1\boldsymbol{g} \neq 0,$$

$$(2) \quad \dim D^0\boldsymbol{g} - \dim D^1\boldsymbol{g} = 1, \quad \dim D^1\boldsymbol{g} \geq 2,$$

$$(3) \quad D^1\boldsymbol{g} = 0.$$

In the first case the subspace $\langle X \rangle + D^1\boldsymbol{g}$ is an ideal of $\boldsymbol{g}$ for any non-zero element X of $D^0\boldsymbol{g} \setminus D^1\boldsymbol{g}$ and we have $l(\boldsymbol{g}) \geq 3$.

In the second case we consider a non-zero element X in $D^0\boldsymbol{g} \setminus D^1\boldsymbol{g}$. As $D^1\boldsymbol{g}$ is an ideal of $\boldsymbol{g}$, ad$X$ is a linear operator in $D^1\boldsymbol{g}$. Let $v \in D^1\boldsymbol{g}$ be an eigenvector of ad$X$. The subspace generated by v is an ideal of $\boldsymbol{g}$ and we have $l(\boldsymbol{g}) \geq 3$.

In the case (3) any subspace of $\boldsymbol{g}$ is an ideal of $\boldsymbol{g}$ and we have $l(\boldsymbol{g}) \geq 3$.

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As all nilpotent Lie algebras of dimension ≤ 2 are abelian, we have

Corollary 1. *Let $\boldsymbol{g}$ be a nilpotent Lie algebra. The lattice of ideals of $\boldsymbol{g}$ has a length ≤ 2 if and only if $\boldsymbol{g}$ is an abelian Lie algebra of dimension ≤ 2.*

Corollary 2. *There exists only one (up to isomorphism) solvable but non-nilpotent Lie algebra with the lattice of ideals of length ≤ 2. In the basis $\{X_1, X_2\}$ this Lie algebra is defined by the following non-zero brackets: $[X_1, X_2] = X_2$.*

3 Semisimple Lie algebras with lattice of ideals of length ≤ 2.

Let g be a semisimple Lie algebra and

$$g = g_1 \oplus \cdots \oplus g_k$$

its decomposition as direct sum of simple Lie algebras. Every simple ideals of g coincides with one of g_i. Moreover, each ideal of g is a sum of certain simple ideals g_i [6]. Thus, $l(g) = k$ and we have.

Proposition 2. *The lattice of ideals of a semisimple Lie algebra g has a length ≤ 2 if and only if g is simple or g is a direct sum of two simple Lie algebras.*

4 Nonsolvable and nonsemisimple Lie algebras with lattice of ideals of length ≤ 2.

Let g be a nonsolvable and nonsemisimple Lie algebra. We consider a Levi's decomposition [6] :

$$g = s + r$$

where s is a semisimple Lie algebra and r is the radical of g (that is maximal solvable ideal of g).

Proposition 3. *Let g be a nonsolvable and nonsemisimple Lie algebra whose lattice of ideals has a length ≤ 2. Then the radical r of g is abelian.*

Proof. Suppose that r is not abelian, that is $[r, r] \neq \{0\}$. We consider an operator $\mathrm{ad}X$, where $X \in s$. This operator is a derivation of the radical r and we have

$$\mathrm{ad}X\,([Y, Z]) = [[X, Y], Z] + [Y, [X, Z]] \in [r, r]$$

for all $Y, Z \in r$. Hence $[r, r]$ is an ideal of g belonging to r (and different of r). Consequently $l\,(g) \geq 3$ in contradiction with the hypothesis.

■

Proposition 4. *Let g be a nonsolvable and nonsemisimple Lie algebra whose lattice of ideals has a length ≤ 2 and s be a Levi's subalgebra of g. Then s is simple.*

Proof. Let $s = s_1 \oplus \cdots \oplus s_k$ is a decomposition of s as direct sum of its simple ideals. Each subspace $s_i + r$, where r is the radical of g, is an ideal of g containing r.

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Proposition 5. *Let g be a nonsolvable and nonsemisimple Lie algebra and let $g = s \oplus r$ its a Levi's decomposition. The lattice of ideals of g has a length 2 if and only if s is simple and the adjoint representation of s in r is irreducible.*

Proof. According to propositions 3 and 4 it remains to show that the adjoint representation of s in r is irreducible, that is the s-module r is simple. Suppose that the s-module r is not simple. From Weyl's theorem on complete reducibility we have a decomposition

$$r = r_1 \oplus \cdots \oplus r_m$$

where r_i, $\quad i = 1, \ldots, m, \quad m \geq 2$, are the simple s-modules. Then each $r_i$is an ideal of g different to r and the length of g is ≥ 3 in contradiction with hypothesis of lemma.

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Corollary 3. *The classification of nonsolvable and nonsemisimple Lie algebras whose lattice of ideals has a length 2 reduce to classifications of simple Lie algebras and their irreducible representations which is well know* [5] .

5 Classification's theorem

From the propositions $1 - 5$ we have following theorem.

Theorem. *Let g be a Lie algebra whose ideals form a lattice of length ≤ 2. Then g is isomorphic to one of the Lie algebras described in the table 1.*

Table 1

Lie algebra	Dimension	Description of the Lie algebra	Lattice
$\{0\}$			$\{0\}$
a_1	1	Abelian Lie algebra	a_1 $\{0\}$
s		An arbitrary simple Lie algebra	s $\{0\}$
a_2	2	Abelian Lie algebra	Infinite lattice of length 2 where any subspace of a_2 is an ideal
$s+V$		s is a arbitrary simple Lie algebra V is an arbitrary simple s−module $[s,v]=s\cdot v$	$s+V$ V $\{0\}$
r	2	$[X,Y]=Y$	r $\langle Y\rangle$ $\{0\}$
$s_1\oplus s_2$		Direct sum of two simple Lie algebras	$s_1\oplus s_2$ s_1 s_2 $\{0\}$

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