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The topological complexity of sets of convex differentiable functions.

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Abstract

Let $\mathcal{C}(X)$ be the set of all convex and continuous functions on a separable infinite dimensional Banach space X, equipped with the topology of uniform convergence on bounded subsets of X. We show that the subset of all convex Fréchet-differentiable functions on X, and the subset of all (not necessarily equivalent) Fréchet-differentiable norms on X, reduce every coanalytic set, in particular they are not Borel-sets.

1 Introduction

Let X be an infinite dimensional Banach space. We denote by $\mathbb{N}(X)$ the set of all equivalent norms on X. This space is topologically metrizable complete when equipped with the uniform convergence on bounded subsets of X. Recently, the topological nature of some collections of norms has been investigated : In [1], the collection of all uniformly rotund norms in every direction on a separable space with a basis was shown to be coanalytic non Borel for the Effros-Borel structure, as well as the set of all weakly locally uniformly rotund norms (see [2], [3]).

More recently, it is showed in [12] that the set of all Gâteaux-smooth norms reduces any coanalytic subset of a Polish space M through a continuous function in $\mathbb{N}(X)$. This implies of course that this set is coanalytic complete when $\mathbb{N}(X)$ is equippped with the Effros-Borel structure (see [6], [16] for the definition of this notion).

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In a third article [4], the locally uniformly rotund norms are investigated. Recall that an equivalent norm $\|.\|$ on a Banach space X is localy uniformly rotund (in short, L.U.R.) if whenever $x \in X$ and a sequence (x_n) in X satisfy :

$$\lim_{n\to+\infty} 2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 = 0,$$

one has

$$\lim_{n\to+\infty} \|x - x_n\| = 0.$$

In [4] it was shown by combining the methods of [12] with a classical topological theorem due to Hurewicz [11] and its extensions ([13],[14]), that a similar result is obtained for the L.U.R. norms when X is separable. Moreover for the case when X^* is separable we have the following result for L.U.R. dual-norms (see [4]) :

Theorem 1. *Let X be an infinite dimensional Banach space such that X^* is separable. Let M be a Polish space, and A an analytic subset of M. Then there exists a continuous map $\Lambda : M \to \mathbb{N}(X)$ such that:*

(i) If $t \in A$, then the norm $\Lambda(t) = \|.\|_t$ is not everywhere Gâteaux-differentiable.

(ii) If $t \notin A$, then the dual norm $\Lambda^*(t) = \|.\|_t^*$ is L.U.R.

In particular, since a norm whose dual norm is L.U.R. is Fréchet-differentiable ([9], p.43), it follows that if X is a separable infinite dimensional Banach space such that X^* is separable, then the set of all equivalent Fréchet-differentiable norms on $X \setminus \{0\}$ and the set of all convex continuous and Fréchet-differentiable functions on X are not Borel subsets of $\mathcal{C}(X)$.

The aim of this work is to show a similar result for every separable infinite dimensional Banach space (theorem 2, corollary 7 and 8).

2 Notations and definitions

Let X be a Banach space. We denote by $\mathcal{C}(X)$ the set of all convex and continuous functions on X. This set is topologically metrizable and complete when equipped with the topology of uniform convergence on bounded subsets of X. By $S(X)$ (resp. $B(X)$) we denote the unit sphere (resp. the unit ball) of X.

A norm $\|.\|$ on X is Gâteaux-differentiable at a non zero point x of X, if for every $h \in X$,

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t} \quad \text{exists.}$$

An equivalent reformulation for $x \in S(X)$, is that there is a unique $f \in S(X^*)$ such that $f(x) = 1$ ([9], p. 5). In this case we say that f is the differential of the norm $\|.\|$ at x. This norm will be said Fréchet-differentiable at x if the limit above exists uniformly for $h \in S(X)$.

A norm is said Gâteaux-differentiable (resp. Fréchet-differentiable), if it is Gâteaux-differentiable (resp. Fréchet-differentiable) at all non zero points of X.

3 Results

Theorem 2 is the main result of this work, its proof relies in part on the proof of Theorem 1 mentioned above.

Theorem 2. *Let X be a separable infinite dimensional Banach space. Let M be a Polish space and A an analytic subset of M. Then there is a continuous map $\Gamma : M \to C(X)$ such that:*

(i) If $t \in A$, then the function $\Gamma(t)$ is not everywhere Gâteaux differentiable.

(ii) If $t \notin A$, then the function $\Gamma(t)$ is Fréchet-differentiable.

Proof of Theorem 2:

The proof is divided into two cases according to whether X^* has or has not the Schur property. Remember that a Banach space has the Schur property if every weakly convergent sequence is norm convergent.

1$^{\text{st}}$ case : *X^* has the Schur property.*

Then in particular X^* contains an isomorphic subspace to $\ell_1(\mathbb{N})$ ([10], p. 112). Moreover, since X is separable we deduce that $c_0(\mathbb{N})$ is isomorphic to a quotient space of X ([15], p.104). Hence, there exists a surjective operator T from X onto $c_0(\mathbb{N})$.

On the other hand, Theorem 1 applied to the Banach space $c_0(\mathbb{N})$ implies that there exists a continous map $\Lambda : M \to \mathbb{N}(c_0(\mathbb{N}))$ such that:

(i) If $t \notin A$, the norm $\Lambda(t) = \|.\|_t$ is Fréchet-differentiable on $c_0(\mathbb{N}) \setminus \{0\}$.

(ii) If $t \in A$, the norm $\Lambda(t) = \|.\|_t$ is not Gâteaux-differentiable at some points of $c_0(\mathbb{N}) \setminus \{0\}$.

We define the map $\Gamma : M \to C(X)$ by ,

$$\Gamma(t) = \|T(.)\|_t^2 .$$

Fact 1. *If $t \notin A$, the function $\Gamma(t)$ is Fréchet-differentiable on X.*

Indeed, it is the composition of Fréchet-differentiable functions since the function $\|.\|_t^2$ is Fréchet-differentiable on $c_0(\mathbb{N})$.

Fact 2. *If $t \in A$, the function $\Gamma(t)$ is not Gâteaux-differentiable at some points of X.*

Let $t_0 \in A$ and let z_0 be a non zero point of $c_0(\mathbb{N})$ where the norm $\|.\|_{t_0}$ is not Gâteaux-differentiable. T being surjective, we consider $x_0 \in X$ such that $T(x_0) = z_0$. Assume that the function $\Gamma(t_0) = \|T(.)\|_{t_0}^2$ is Gâteaux-differentiable at x_0. Then the function $\|T(.)\|_{t_0}$ is also Gâteaux-differentiable at x_0 since $T(x_0) = z_0$ is non zero. It follows that for all $h \in X$,

$$\lim_{t \to 0} \frac{\|T(x_0 + th)\|_{t_0} - \|T(x_0)\|_{t_0}}{t} \text{ exists.}$$

Hence for all $k \in c_0(\mathbb{N})$

$$\lim_{t \to 0} \frac{\|z_0 + tk\|_{t_0} - \|z_0\|_{t_0}}{t} \text{ exists,}$$

since T is surjective.

In other words, the norm $\|.\|_{t_0}$ is Gâteaux-differentiable at z_0. It is a contradiction , thus the function $\Gamma(t_0)$ is not Gâteaux-differentiable at x_0.

■

2^{nd} **case** :*X^* has not the Schur property.*

In this case, we use the same idea as in the proof of the first case, namely, to find a Banach space Y whose dual is separable and an appropriate operator T from X to Y. Then we consider the functions

$\|T(.)\|_t$ on X where the collection of the norms $\|.\|_t$ on Y is appropriately given by Theorem 1. To have (ii) , T has to be "nearly surjective" (with dense range) and if the norm $\|.\|_t$ is not everywhere Gâteaux-differentiable, it is necessarily not Gâteaux-differentiable at some points of T(X) .

Fact 3. *There exists a separable reflexive Banach sapce Y and a non compact operator T from X to Y with dense range .*

Proof. Since X^* fails the Schur property, there exists a subset C in X which is w-compact but not norm-compact. We put

$$K = \overline{\text{conv}}(C \cup (-C))$$

which is also w-compact ([18], II.c.8) but not norm-compact. Let E_K be the vector space generated by K equipped with the norm j_K (the Minkowski functional of K). Then E_K is a Banach space since K is w-compact. Indeed, let $(x_n)_{n>0}$ be a j_K-Cauchy sequence in E_K. Then for every integer p, there exists an integer $N(p)$ such that $x_p \in (x_{N(p)} + p^{-1}K)$ for every integer $q \geq N(p)$. We put

$$K_p = x_{N(p)} + p^{-1}K.$$

It is clear that every finite intersection of K_p is non empty, and so by compactness, $\bigcap_{p>0} K_p \neq \emptyset$. It is easy to check that $\bigcap_{p\geq 0} K_p = \{x_\infty\}$ and that $j_K - \lim_n x_n = x_\infty$. Thus, (E_K, j_K) is a Banach space.

Consider now the canonical injection $i : E_K \to X^*$, which is w-compact and not norm-compact because $i(B(E_K)) = K$. By the interpolation theorem of Davis-Figiel-Johnson-Pelczynski (see [7] ou [18], II.c.5), there exists a reflexive space R and two operators $\alpha : E_K \to R$ and $\beta : R \to X^*$ such that $i = \beta\alpha$.

In particular, β is not norm-compact since the operator $i = \beta\alpha$ is not. Moreover β is w^*-continuous since R is reflexive. Hence, there exists an operator $\beta_0 : X \to R^*$ such that $\beta = \beta_0^*$ and then, like β, β_0 is not norm-compact (see [18], I.A.15).

We put $Y = \beta_0(X)$ and $T = \beta_0 : X \to Y$. It easily checked that Y and T work. This prove the fact 3.

■

Let Z be a fixed closed hyperplane of Y. We write $Y = \mathbb{R} \oplus Z$. Since Z is a separable reflexive Banach space, we may and do assume that Z is equipped with an equivalent L.U.R. norm $|.|$ whose dual norm $|.|^*$ is also L.U.R. (see [9], p. 55).

Fact 4. *$X_0 = T^{-1}(Z)$ is a hyperplane of X.*

Indeed, let Q be the quotient map from Y onto Y/Z. If we consider the operator $\tilde{T} = Q \circ T$ defined from X to Y/Z, then since $\dim(Y/Z) = 1$ and T has dense range, the operator $\tilde{T}$ corresponds to a non zero element of X^* and therefore, $\ker \tilde{T} = T^{-1}(Z) = X_0$ is a hyperplane of X.

■

Now if we call T_0 the restriction of T on the hyperplane X_0, it is clear that T_0 is also a non-compact operator.

From now on, we denote $(0, z) \in Y$ the element $z \in Z$

Fact 5. *There exists a weakly closed subset F in the unit sphere $S(Z)$ such that $F \subsetneq S(Z) \cap T_0(X_0)$ and F is weakly homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

To prove that fact, we need the following theorem (see [17], [14], [13] p. 133) :

Theorem 3. *Let E be a metrizable compact set, A an analytic subset of E and B a subset of E having an empty intersection with A. If there exists no F_σ subset of E containing A and having empty intersection with B, then there exists a subset K of E included in $A \cup B$ and homeomorphic to $\mathbb{N}^{\mathbb{N}}$ such that $(K \cap B)$ is countable and dense in K.*

We put:

. $E = (B(Z), w)$.

. $A = S(Z) \cap T_0(X_0)$.

. $B = B(Z) \setminus S(Z)$.

E is compact metrizable since Z is separable and reflexive. The set A is a Borel subset (hence analytic) of E. Indeed, the sphere $S(Z)$ is w-borelian since it is a G_δ of $(B(Z), w)$. Moreover, $T_0(X_0)$ is a $|.|$-Borel subset since it is the injective and continuous image of $X_0/\ker T_0$ (see [6]). Moreover, since Z is separable, any norm-Borel subset of Z is w-Borel. Thus $T_0(X_0)$ is w-Borel subset.

Assume now that there is a F_σ subset G of E such that $A \subseteq G$ and $B \cap G = \emptyset$, i.e. that $G = \bigcup_{n \geq 0} K_n$ where K_n are closed subsets in $(B(Z), w)$ and $T_0(X_0) \cap S(Z) \subseteq G \subseteq S(Z)$. In particular, for any integer n, K_n is w-compact in $S(Z)$. Moreover, since the norm $|.|$ is L.U.R. on Z, the weak and norm topologies agree on the sphere $S(Z)$ of Z, and hence K_n is $|.|$-compact for any integer n.

We consider the map $\varphi : \mathbb{R} \times Z \to Z$

$$(\lambda, z) \mapsto \lambda z.$$

If for every integers n and p we put $K_{n,p} = \varphi([0,p] \times K_n)$, then the subsets $K_{n,p}$ are $|.|$-compact since φ is continuous and moreover we have:

$$T_0(X_0) \subseteq \bigcup_{n \geq 0, p \geq 0} K_{n,p}.$$

Thus, the subsets $F_{n,p} = T_0^{-1}(K_{n,p})$ of X_0 are closed and,

$$X_0 = \bigcup_{n \geq 0, p \geq 0} F_{n,p}.$$

According to Baire theorem, there exists two integers n_0 and p_0 such that the interior $\overset{\circ}{F}_{n_0,p_0}$ of F_{n_0,p_0} is non-empty. Let B_0 a non-empty open ball included in $\overset{\circ}{F}_{n_0,p_0}$. Then $T(B_0) \subseteq K_{n_0,p_0}$, and hence $T_0(B_0)$ is $|.|$-relatively compact in Z, and finally by translation and homothety, we deduce that $T_0(B(X_0))$ is $|.|$-relatively compact. This contradicts the fact that T_0 is not a compact operator.

Thus, by the theorem 3 above, there exists a subset K of E satisfying:

. $K \subseteq (T(X_0) \cap S(Z)) \cup B$.

. K w-homeomorphic to $2^{\mathbb{N}}$.

. $K \cap B$ is countable and w-dense in K.

It follows that $F = K \cap (T(X_0) \cap S(Z))$ is w-homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and w-closed in $S(Z)$.

■

The next fact can be also seen as a particular case of the lemma 10 in [8].

Fact 6 . *Let $\sigma : (S(Z), w) \to (S(Z^*), w)$ be the map which associate to $x \in S(Z)$ the element $f = \sigma(x)$ of $S(Z^*)$ such that $f(x) = 1$. Then, since Z is reflexive and the norms $|.|$ and $|.|^*$ are L.U.R., the map σ is an homeomorphism.*

Indeed, for $x \in S(Z)$, $\sigma(x)$ is the differential of the norm $|.|$ at x. Since the norm $|.|$ is Fréchet-differentiable, σ is a $(|.|, |.|^*)$-continuous map (see [9], p. 7) and surjective because Z is reflexive. Similarly, for $f \in S(Z^*)$, $\sigma^{-1}(f)$ is the differential of the norm $|.|^*$ at x. Since the norm $|.|$ is Fréchet-differentiable, σ^{-1} is a map from $S(Z^*)$ onto $S(Z^{**}) = S(Z)$ which is moreover $(|.|^*, |.|)$-continuous.

Hence σ is a $(|.|, |.|^*)$-homeomorphism, and thus, a (w, w)-homeomorphism because the weak and the strong topologies agree on the sphere when the norm is L.U.R. .

■

In particular, the subset $\sigma(F) = F^*$ is ω-closed in the sphere $S(Z^*)$ and ω-homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

By using the same arguments as in the proof of the Theorem 1 of Bossard-Godefroy-Kaufman ([4], theorem 3), but applied to our reflexive space Y and for our special subset F^* of Z, we get:

Lemma 4. *Let M be a Polish space and let A be an analytic subset of M. Then there exists a continuous map $\Lambda : M \to \mathbb{N}(Y)$ such that:*

(i) If $t \notin A$, then the norm $\Lambda(t) = \|.\|_t$ is Fréchet-differentiable.

(ii) If $t \in A$, then the norm $\Lambda(t) = \|.\|_t$ is not Gâteaux-differentiable. More precisely, it is not Gâteaux-differentiable at some points of the subset $(\{0\} \times F)$ of $Y = \mathbb{R} \oplus Z$.

Proof of lemma 4: For the proof, we need the following lemma ([4], lemma 2).

Lemma 5. *Let (S, d) be a metric space which contains a closed subset E homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Let A be an analytic subset of a Polish space (M, d'). Then, there exists an uniformly continuous map φ from $(M \times S, d' + d)$ to $[0, 1]$ such that:*

(i) If $t \notin A$, then $\varphi(t, y) < 1$ for all $y \in S$.

(ii) If $t \in A$, then $\varphi(t, y_0) = 1$ for some $y_0 \in E$.

We apply lemma 5 to $S = E = (F^*, d)$ where d is the restriction to F^* of a distance on $B(Z^*)$ which defines the w-topology ($(B(Z^*), w)$ is metric compact since Z is reflexive separable) . There exists a uniformly continuous function $\varphi : (M \times F^*, d' + d) \to [0,1]$ such that:

(*i*) If $t \notin A$, then $\varphi(t,f) < 1$ for all $f \in F^*$.

(*ii*) If $t \in A$, then $\varphi(t,f) = 1$ for some $f_0 \in F^*$.

We put $L = \overline{F^*}^w$ which is a compact of $(B(Z^*), w)$. Observe that since φ is $(d'+d)$-uniformly continuous, it has an unique uniformly continuous extension Φ to the completion $(M \times L)$ of $(M \times F^*)$. Moreover, $t \in A$ if and only if there is $f_0 \in F^*$ such that $\Phi(t, f_0) = 1$.

We consider the following w-compact subsets of $Y^* = \mathbb{R} \oplus Z^*$:

$$R(t) = (\{0\} \times B(Z^*)) \cup \{\pm(1, \Phi(t,f).f) : f \in L\}$$

and

$$K(t) = \overline{\text{conv}}^w(R(t)).$$

Clearly, $K(t)$ is the unit ball of an equivalent dual norm on Y^* which we denote by $|.|_t^*$. Finally, we define an equivalent dual norm $\|.\|_t^*$ by ,

$$\|(s,f)\|_t^{*2} = |(s,f)|_t^{*2} + |f|^{*2},$$

whose predual norm is denoted by $\Lambda(t) = \|.\|_t$. This give us the continuous map $\Lambda : M \to \mathbb{N}(Y)$, which satisfies (*i*) and (*ii*) of the lemma 4. Indeed ,

(*i*) Let $t \notin A$. Assume that the norm $\|.\|_t$ is not Fréchet-differentiable. Then the dual norm $\|.\|_t^*$ is not L.U.R. ([9], p.43), and hence, there exists $f_0 \in S(Z^*)$ and $s_0 > 0$ such that ([4], proof of theorem 3),

$$|(0, f_0)|_t^* = 1 = |(s_0, f_0)|_t^*.$$

Since $t \notin A$, we have $\phi(t,f) < 1$ for all $f \in F^*$. Since F^* is w-closed in $S(Z^*)$, $L \cap S(Z^*) = F^*$. It follows that $|\Phi(t,f).f)|^* < 1$ for all $f \in L$.

Let μ a probability measure on $R(t)$ such that (s_0, f_0) is the barycenter of μ. The function $h((s_0, g)) = |g|^*$ is convex and w^*-lower semicontinuous on Y^*, hence (see [5] proposition 26-19) :

$$1 = |f_0|^* = h((s_0, f_0)) \leq \int_{R(t)} |g|^* d\mu((s,g)).$$

Since $|\Phi(t,f).f|^* < 1$ for all $f \in L$, it follows that μ is supported by $(\{0\} \times B(Z^*))$ and hence $s_0 = 0$. This contradiction shows (*i*).

(*ii*) Let $t \in A$ and let then $f_0 \in F^*$ such that $\Phi(t, f_0) = 1$. We consider $z_0 = \sigma^{-1}(f_0) \in F$, so $f_0(z_0) = |z_0| = 1$. For any $f \in Z^*$ and $s \in \mathbb{R}$, we have :

$$\|(s, f)\|_t^* \geq \|(0, f)\|_t^* = \sqrt{2}|f|^*.$$

Hence for all $z \in Z$,

$$\|(0, z)\|_t = \frac{|z|}{\sqrt{2}}.$$

On the other hand, since $\Phi(t, f_0) = 1$, we have for all $s \in [-1, 1]$,

$$|(s, f_0)|_t^* = 1.$$

Hence for all $s \in [-1, 1]$,

$$< (s, f_0), (0, z_0) >= 1 = \|(s, f_0)\|_t^* \cdot \|(0, z_0)\|_t \, .$$

And thus, the norm $\|.\|_t$ is not Gâteaux-differentiable at a point $(0, z_0)$ of $(\{0\} \times F)$. This finishes the proof of lemma 4 .

■

We define the map $\Gamma : M \to C(X)$ where for $t \in M$ and $x \in X$, $\Gamma(t)(x) = \|T(x)\|_t^2$. Let us check that Γ is the map we are looking for.

First, if $t \notin A$, according to lemma 4, the function $\|\cdot\|_t^2$ is Fréchet-differentiable on Y, and then $\Gamma(t)$ is also Fréchet-differentiable as composition of Fréchet-differentiable functions.

Fact 7. *IF $t \in A$, the map $\Gamma(t)$ is not Gâteaux-differentiable at some points of X.*

Let $t_0 \in A$. The lemma 4 provides the existence of $z_0 \in F$ such that the norm $\|.\|_{t_0}$ is not Gâteaux-differentiable at $(0, z_0)$. Since $F \subseteq S(Z) \cap T_0(X_0)$ (see Fact 5), we consider $x_0 \in X_0$ such that $T_0(x_0) = z_0$. Recall that T_0 is the restriction of the operator $T : X \to Y$ to the hyperplane X_0 and that an element z of the hyperplane Z of Y is written in Y as $(0, z)$. Then since $z_0 \in Z$, we have :

$$T(x_0) = (0, z_0) \in Y.$$

Let us check that $\Gamma(t_0)$ is not Gâteaux-differentiable at x_0. Indeed, if not, since $\Gamma(t_0)(x_0) = \|T(x_0)\|_{t_0}^2 = \|(0, z_0)\|_{t_0}^2 = \frac{1}{2} \neq 0$, the semi-norm

$\|T(.)\|_{t_0}$ will be also Gâteaux-differentiable at x_0. Thus, the norm $\|.\|_{t_0}$ will be Gâteaux-differentiable at $T(x_0) = (0, z_0)$ along the directions subset $T(X)$. It follows that the norm $\|.\|_{t_0}$ is Gâteaux-differentiable at $(0, z_0)$ since $T(X)$ is dense in Y . With this contradiction we finish the proof of theorem 2 .

■

Now we turn to the norm case. Let $\mathcal{N}(X)$ be the set of all continuous norms on a Banach space X, equipped with the topology induced by $\mathcal{C}(X)$.

Corollary 6. *Let X be a separable infinite dimensional Banach space. Let M be a Polish space and A an analytic subset of M. Then, there exists a continuous map $\Psi : M \to \mathcal{N}(X)$ such that:*

(i) If $t \in A$, then the norm $\Psi(t)$ is not everywhere Gâteaux-differentiable on $X \setminus \{0\}$.

(ii) If $t \notin A$, then the norm $\Psi(t)$ is Fréchet-differentiable on $X \setminus \{0\}$.

Proof. We consider the semi-norms $(\Gamma(t))^{\frac{1}{2}} = \|T(.)\|_t$, defined in the proof of theorem 2 satisfying :

. If $t \in A, \Gamma(t) = \|T(.)\|_t^2$ is not everywhere Gâteaux-differentiable.

. If $t \notin A, \Gamma(t) = \|T(.)\|_t^2$ is Fréchet-differentiable on X.

X being separable infinite dimensional, we have $\overline{S(X^*)}^{w^*} = B(X^*)$ and $(B(X^*), w^*)$ is metrizable. Hence, there exists a sequence $(x_n^*)_{n\geq 1}$ in $S(X^*)$ which is w^*-dense in the unit ball $B(X^*)$. Then we define the injective operator J from X in $c_0(\mathbb{N})$ by ,

$$J(x) = \left(\frac{x_n^*(x)}{n}\right)_{n\geq 1}.$$

Let $\|.\|$ be a Fréchet-differentiable norm on $c_0(\mathbb{N})$. we define $|||.|||$ a Fréchet-differentiable norm on X by ,

$$|||x||| = \|J(x)\|.$$

Let now the norm $|||.|||_t$ defined on X by :

$$|||x|||_t^2 = |||x|||^2 + \|T(x)\|_t^2.$$

It is easy to check that the map $\Psi : M \to \mathcal{N}(X)$, defined by $\Psi(t) = |||\cdot|||_t$, works. ∎

Since any uncountable Polish space M (for example $\mathbb{N}^{\mathbb{N}}$)) contains an analytic non-Borel subset A, theorem 2 and corollary 6 imply :

Corollary 7. *Let X be a separable infinite dimensional Banach space. Then, the set of continuous convex and Fréchet-differentiable functions on X and the set of all Fréchet-differentiable (and not necessarily equivalent) norms on $X \setminus \{0\}$, are non Borel subsets of $C(X)$.*

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