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Mosco convergence of sequences of homogeneous polynomials.

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Abstract

In this paper we give a characterization of uniform convergence on weakly compact sets, for sequences of homogeneous polynomials in terms of the Mosco convergence of their level sets. The result is partially extended for holomorphic functions. Finally we study the relationship with other convergences.

Throughout this paper E will be a Banach space over $\mathbf{K}$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Results hold both for the real and complex cases unless one of them is specified. E^* will denote the dual space and B_E, S_E the unit ball and the unit sphere respectively.

$\mathcal{P}(^k\mathrm{E})$ will denote the space of all k-homogeneous polynomials on E. $\mathcal{P}(^k\mathrm{E})$ is a Banach space endowed with the usual norm

$$||P|| = Sup\{|P(x)| : ||x|| \leq 1\}$$

For a general reference on infinite dimensional polynomials see [Ll] or [Mu].

If $\{x_n\}_n$ is a sequence of elements of E and $x \in E$, $x = w - \lim_n x_n$ means that x is the limit of the sequence $\{x_n\}_n$ in the weak topology, meanwhile $x = \lim_n x_n$ means that the limit is in the norm topology.

We start with a definition:

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Definition. *We will say that a sequence of nonempty closed subsets of a Banach space E, $\{A_n\}_n$, converges in the Mosco sense to a closed subset A ($A_n \overset{M}{\to} A$ in short) whenever the following two conditions hold:*

(i) For every $x \in A$ there exists a sequence $\{x_n\}_n$, norm convergent to x such that $x_n \in A_n$ for every n.

(ii) Given $J \subset \mathbf{Z}^+$ cofinal, for every sequence $\{x_{n_j}\}_{j \in J}$ weakly convergent to x, the condition $x_{n_j} \in A_{n_j}$ for every j, implies $x \in A$.

This concept of convergence is closely related with Kuratowski convergence which is defined in the same way changing weak convergence by norm convergence in (ii). (See [K]).

Let us introduce two more concepts of set convergence. We will say that a sequence of closed sets A_n converges in strong (respectively Wijsman) sense to a closed set A, provided that the sequence $\{d_n\}$ converges to d unifomly on bounded sets (respectively pointwise), where d_n and d denote the distance functions to A_n and A respectively. ([M],[W],[B2]).

Kuratowski convergence may be defined in any topological space, while Wijsmann and strong ones require metric spaces. We need duality for Mosco convergence.

Remarks:

(1) It is assumed usually that the sets in the definition of Mosco convergence are convex, and consequently weakly closed. By the moment we do not, but let observe that without that condition a constant sequence may be non-convergent!

(2) Mosco convergence implies Kuratowski convergence. If E is a Schur space (in particular a finite dimensional one) both convegences agree.

(3) If E is a reflexive space and we are dealing with convex sets, then strong convergence implies Mosco convergence implies Wijsman convergence (see [T]). And E may be renormed in such a way that Mosco and Wijsman convergence agree.

(4) If some sets A_n are empty, and $A \neq \emptyset$, we will say that $\{A_n\}_n$ converges in the Mosco sense to A if there exists n_0 such that

$A_n \neq \emptyset$ for every $n \geq n_0$ and $\{A_n\}_{n \geq n_0}$ converges in the Mosco sense to A. If $A = \emptyset$ the definition works even if some, or every, A_n are empty.

Given a $P \in \mathcal{P}(^k\mathrm{E})$ and $\alpha \in \mathbf{K}$, we will denote $\{x \in E : P(x) = \alpha\}$ by $V(P - \alpha)$. Now, let us consider $P, P_n \in \mathcal{P}(^k\mathrm{E})$, then $V(P_n - \alpha) \stackrel{M}{\to} V(P - \alpha)$ reads as:

(i) For every x such that $P(x) = \alpha$ there exists $\{x_n\}_n$ such that $P_n(x_n) = \alpha$ and $x = \lim_n x_n$.

(ii) If $P_{n_j}(x_{n_j}) = \alpha$ for evey $j \in J$ and $x = w - \lim_j x_{n_j}$, then $P(x) = \alpha$

As in [F], the Mosco convergence of the level sets for $\alpha = 1$ (and $\alpha = -1$ in the real case if k is even) give us $V(P_n - \alpha) \stackrel{M}{\to} V(P - \alpha)$ for every $\alpha \neq 0$.

Definition. *We will say that a sequence $\{P_n\}_n$ of polynomials in $\mathcal{P}(^k\mathrm{E})$ converges to $P \in \mathcal{P}(^k\mathrm{E})$ in the Mosco sense if $V(P_n - \alpha) \stackrel{M}{\to} V(P - \alpha)$ for every $\alpha \neq 0$*

In [F] we define, in a similar way we do here, Kuratowski, Wijsman and strong convergence of sequences of homogeneous polynomials, and we characterize them. More precisely, Kuratowski convergence is equivalent to uniform convergence on compact sets and strong convergence is equivalent to norm convergence. In order to characterize Wijsmann convergence we need the following condition:

$$\overline{P_n(B)} \to \overline{P(B)} \quad \text{for every ball} \quad B \subset E$$

For a previous study of the linear case see [B1].

As we noted above, if the sets are not convex (which in general is the case for the level sets of a polynomial), we may have problems with the Mosco convergence. The following easy example give us an idea of what kind of problems may arise.

Example. $P : l_2 \to \mathbf{R}$ defined as $P(x) = \sum_{j=1}^{\infty} x_j^2$. P is a 2-homogeneous non-weakly sequentially continuous polynomial, and the constant sequence $V(P - 1)$ does not converge in the Mosco sense to itself.

Of course this pathology does not happen if the polynomial is weakly sequentially continuous. The space of such polynomials will be denoted by $\mathcal{P}_{\text{wsc}}({}^k\text{E})$.

Amazingly, weak sequential continuity is not only convenient but, in some way, necessary.

Lemma. *Let $P, P_n \in \mathcal{P}({}^k\text{E})$. If $V(P_n-\alpha) \stackrel{M}{\to} V(P-\alpha)$ for every $\alpha \neq 0$ then $P \in \mathcal{P}_{\text{wsc}}({}^k\text{E})$.*

Proof. Let us suppose that $P \notin \mathcal{P}_{\text{wsc}}({}^k\text{E})$. Passing to a subsequence if necessary and using boundness of weakly convergent sequences, we may assume that there exists a sequence $\{x_n\}_n$ converging weakly to x, such that $\alpha_n = P(x_n)$ converges to $\alpha \neq P(x)$.
First we consider the case $\alpha \neq 0$. We may assume $\alpha_n \neq 0$ for every n and therefore $(\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n \in V(P-\alpha)$ for every n; using the first condition in Mosco convergence definition we have that: for every n there exists $\{z_{n,m}\}_m$ norm convergent to $(\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n$, and satisfying $P_m(z_{n,m}) = \alpha$. Let us choose $m(n)$ such that $m(n) < m(n+1)$ and $\|z_{n,m(n)} - (\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n\| < \frac{1}{n}$, and define $y_n = z_{n,m(n)}$. For every $x^* \in S_{E^*}$, we have

$$|x^*(x) - x^*(y_n)| \leq |x^*(x) - (\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x^*(x_n)| + \|x^*\| \|(\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n - y_n\| \leq$$

$$\leq |1 - (\frac{\alpha}{\alpha_n})^{\frac{1}{k}}| |x^*(x)| + (\frac{\alpha}{\alpha_n})^{\frac{1}{k}} |x^*(x) - x^*(x_n)| + \|x^*\| \|(\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n - y_n\| \leq$$

$$\leq |1 - (\frac{\alpha}{\alpha_n})^{\frac{1}{k}}| \|x\| + (\frac{\alpha}{\alpha_n})^{\frac{1}{k}} |x^*(x) - x^*(x_n)| + \frac{1}{n}$$

which goes to 0 when n does. Therefore $\{y_n\}_n$ converges weakly to x, and $P_{m(n)}(y_n) = \alpha$. Renaming y_n as $\tilde{y}_{j_n}$ where $j_n = m(n)$, we have that $x = w - \lim_n \tilde{y}_{j_n}$, $P_{j_n}(\tilde{y}_{j_n}) = \alpha$ but $P(x) \neq \alpha$, which is a contradiction.
Let us proceed with the case $\alpha = 0$. That is $\lim_n P(x_n) = 0$ and $P(x) \neq 0$, where $x = w - \lim_n x_n$. First let us observe that if $x = w - \lim_n x_n$, $x = \lim_n y_n$ and $z_n \in [x_n, y_n]$, then the sequence $\{z_n\}_n$ converges weakly to x.
If we denoted $\beta = P(x)$, being $\beta \neq 0$, in the real case we may choose $z_n \in [x_n, x]$ such that $P(z_n) = \frac{\beta}{2}$ (this is possible because $\lim_n P(x_n) = 0$ and $P(x) \neq 0$). The observation above says us that $\{z_n\}_n$ is weakly convergent to x and $P(z_n) = \frac{\beta}{2}$, thus we are in the previous case. In the complex case some small arrangement must be done. (See theorem proof below). ■

Now, we are able to give a characterization of Mosco convergence of sequences of homogeneous polynomials when the limit belongs to $\mathcal{P}_{\text{wsc}}({}^{\text{k}}\text{E})$.

Theorem. *Given $P, P_n \in \mathcal{P}({}^{\text{k}}\text{E})$, the sequence $\{P_n\}_n$ converges to P in the Mosco sense if and only if it does uniformly on weakly compact subsets of E and $P \in \mathcal{P}_{\text{wsc}}({}^{\text{k}}\text{E})$.*

Proof. Let us suppose first that $V(P_n - \alpha) \xrightarrow{M} V(P - \alpha)$ for every $\alpha \neq 0$. If the sequence does not converge uniformly on weakly compact subsets, then there exists an $\epsilon > 0$, a K weakly compact, and a sequence $\{x_n\}_n \subset K$ such that $|P_n(x_n) - P(x_n)| > \epsilon$ for every n. Boundness of both $\{||P_n||\}_n$ and $\{x_n\}_n$ (see [F]), allow us to assume, passing to a subsequence if necessary, that there exist $\alpha \neq \beta$ such that

$$\alpha = \lim_n P_n(x_n) \quad \beta = \lim_n P(x_n)$$

By Eberlein's Theorem, passing to a subsequence again, there exists x such that $x = w - \lim_n x_n$. Weak sequential continuity of P implies that $P(x) = \beta$.

If $\beta \neq 0$, condition (i) in Mosco convergence gives us a sequence $\{y_n\}_n$ norm converging to x, such that $P_n(y_n) = \beta$. In the real case a similar argument that we used in the Lemma above says us that there exists a sequence $\{z_n\}_n$ such that $P_n(z_n) = \frac{\alpha+\beta}{2}$ (if $\frac{\alpha+\beta}{2} = 0$, let choose any other nonzero real number laying between α and β), and it converges weakly to x. Then by condition (ii) in Mosco convergence, it follows that $P(x) = \frac{\alpha+\beta}{2}$, contradicting the weak sequential continuity of P. In the complex case we may choose $\{z_n\}_n$ such that $|P_n(z_n) - \alpha| = r_0$, being $0 < r_0 < |\alpha - \beta|$ and $r_0 \neq |\alpha|$. By compactness of $\{w \in \mathbf{C} : |w - \alpha| = r_0\}$, we may assume that there exists a subsequence, $\{z_{n_j}\}_j$ such that $\lim_j P_{n_j}(z_{n_j}) = \delta$ where $|\delta - \alpha| = r_0$ and hence $\delta \neq 0$. The sequence $\tilde{z}_{n_j} = \delta^{\frac{1}{k}} \left(P_{n_j}(z_{n_j})\right)^{-\frac{1}{k}} z_{n_j}$ works because $P_{n_j}(\tilde{z}_{n_j}) = \delta \neq \beta$.

Now, if $\beta = 0$, we may assume that all the $\alpha_n = P_n(x_n)$ are different to 0, and defining $\tilde{x}_n$ as $(\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n$, we have that $P_n(\tilde{x}_n) = \alpha$ and $\{\tilde{x}_n\}_n$ converges weakly to x, hence $P(x) = \alpha$, which is a contradition again.

Conversely if $\{P_n\}_n$ converges uniformly to P on weakly compact subsets of E, it does on compact subsets and therefore it converges in

the Kuratowski sense ([F]). Consequently we only have to check the second condition in Mosco convergence definition.
So we consider a sequence $x = w - \lim_n x_n$, such that $P_n(x_n) = \alpha$. Let us denote by K the sequence with its limit, K is a weakly compact set and consequently $\lim_n(P_n(y) - P(y)) = 0$ uniformly on $y \in K$. Hence for every j there exists n_j such that $|P_{n_j}(y) - P(y)| < \frac{1}{j}$ if $y \in K$, hence

$$|\alpha - P(x_{n_j})| = |P_{n_j}(x_{n_j}) - P(x_{n_j})| < \frac{1}{j} \quad \forall j$$

And therefore $P(x) = lim_j P(x_{n_j}) = \alpha$, following the first equality from the fact that $P \in \mathcal{P}_{\mathrm{wsc}}({}^k\mathrm{E})$.

∎

Alaouglou's Theorem give us trivially the following

Corollary. *Let E be a reflexive Banach space, $P_n \in \mathcal{P}({}^k\mathrm{E})$ and $P \in \mathcal{P}_{\mathrm{wsc}}({}^k\mathrm{E})$. Then the sequence $\{P_n\}_n$ converges to P uniformly on bounded subsets of E if and only if it converges in the Mosco sense.*

Some authors define Mosco convergence of sequences of functions as the Mosco convergence of their epigraphs. The convergence of the level sets allow us to consider the complex case. In the real case, it is easy to realize that Mosco convergence of the epigraphs follows from Mosco convergence of the level sets,and it is stronger as the following example shows.

Example. Let $E = c_0$, $P_n(x) = e_1^*(x)^2 + e_n^*(x)^2$, and $P(x) = e_1^*(x)^2$. $\{P_n\}_n$ does not converge uniformly on weakly compact sets to P, (because $P_n(e_n) = 1$ and $P(e_n) = 0$ if $n > 1$), hence, by the Theorem above, it does not converge in the Mosco sense (let us observe that $P \in \mathcal{P}_{\mathrm{wsc}}({}^k\mathrm{E})$). On the other hand it is clear that $epiP_n \xrightarrow{M} epiP$, because if $(x, \alpha) \in epiP$ then $(x, \alpha + e_n^*(x)^2) \in epiP_n$ and it is norm convergent to (x, α). The other condition follows from the fact that if $x = w - \lim_n x_n$, $\lim_n \alpha_n = \alpha$ and $P_n(x_n) \leq \alpha_n$, then $P(x_n) \leq \alpha_n$ too. Taking limits we have that $P(x) \leq \alpha$ (we are using that $P \in \mathcal{P}_{\mathrm{wsc}}({}^k\mathrm{E})$ again).

∎

In the complex case we may prove a stronger result than Theorem above. In fact we get a stronger conclusion with weaker hypotheses. Let us denote by $\mathcal{H}(\mathrm{E})$ (respectively $\mathcal{H}_{\mathrm{wsc}}(\mathrm{E})$), the space of all holomorphic (respectively weakly sequentially continuous holomorphic) functions on E. (See [Mu] for a general reference)

Theorem. *Let $f_n \in \mathcal{H}(\mathrm{E})$ for every n and $f \in \mathcal{H}_{\mathrm{wsc}}(\mathrm{E})$ non constant. If $f = \lim_n f_n$ uniformly on weakly compact subsets of E, then $\{V(f_n)\}_n$ converges in the Mosco sense to $V(f)$.*

Proof. Let us check the first condition, if $f(x) = 0$ and it is not true that there exists a norm convergent sequence to x, $\{x_n\}_n$, such that $f_n(x_n) = 0$ for every n, we may assume, passing to a subsequence if necessary, that there exists $\epsilon > 0$ such that $V(f_n) \cap B(x, \epsilon) = \emptyset$ for every n. Let $z \in S_E$ such that f is not constant on $L = \{x + wz : w \in \mathbf{C}\}$. Let us define $g, g_n : D(0, \epsilon) \to \mathbf{C}$ by

$$g(w) = f(x + wz) \qquad g_n(w) = f_n(x + wz)$$

The sequence $\{g_n\}$ converges uniformly to g which is not identically zero, and g_n never vanish, hence g never vanish (by Hurwith's Theorem), but on the other hand we know that $g(0) = 0$, which is a contradition.

In order to prove the second condition, let us consider $x = w-\lim_j x_{n_j}$ and $f_{n_j}(x_{n_j}) = 0$. By uniform convergence on weakly compact subsets it follows that $\lim_j f(x_{n_j}) = 0$, and by weakly sequentially continuity of f we have that $f(x) = \lim_n f(x_{n_j})$. Hence $f(x) = 0$.

■

The fact that the theorem hypotheses are stable by adition of a constant, give us the following

Corollary. *Let $f_n \in \mathcal{H}(\mathrm{E})$ for every n and $f \in \mathcal{H}_{\mathrm{wsc}}(\mathrm{E})$ non constant. If $f = \lim_n f_n$ uniformly on weakly compact subsets of E, then $\{V(f_n - \alpha)\}_n$ converges in the Mosco sense to $V(f - \alpha)$ for every α.*

In the real case we cannot infer the convergence of the 0-level sets even under stronger conditions as the following example shows.

Example. Let us suppose that k is odd (the even case is easier). Let us take $\varphi_1, \varphi_2 \in E^*$ linearly independent, (we are only assuming that $dim E > 1$) let us define P and P_n as $\varphi_1^{k-1}(\varphi_1 + \varphi_2)$ and

$(\varphi_1^{k-1} + \frac{1}{n}\varphi_2^{k-1})(\varphi_1 + \varphi_2)$ respectively. $P \in \mathcal{P}_{wsc}(^k E)$, $P = \lim_n P_n$, but

$$V(P) = Ker\varphi_1 \cup Ker(\varphi_1 + \varphi_2) \quad V(P_n) = Ker(\varphi_1 + \varphi_2)$$

and consequently the sequence $\{V(P_n)\}_n$ does not converge, even in the Kuratowski sense, to $V(P)$.

■

However, in the real case, we have the following

Proposition. *If $P \in \mathcal{P}_{wsc}(^k E)$ and $dP(x) \equiv 0$ only if $x = 0$, then the uniform convergence on weakly compact sets of the sequence $\{P_n\}$ to P, implies $V(P_n - \alpha) \xrightarrow{M} V(P - \alpha)$ for every α.*

Proof. The proof of the second condition is similar to that of the complex case. To establish the first condition we have to prove that for every α and for every $x \in V(P - \alpha)$, there exists a norm convergent to x sequence $\{x_n\}_n$ such that $P_n(x_n) = \alpha$. If $x = 0$ (therefore $\alpha = 0$) the constant sequence $x_n = 0$ works. Hence we may assume $x \neq 0$, let us consider $z \in S_E$ such that $dP(x)(z) \neq 0$. The following one-dimensional polynomials:

$$g_n(t) = P_n(x + tz) \qquad g(t) = P(x + tz)$$

verifies that $\{g_n\}_n$ converges to g uniformly on the compact interval $[-1, 1]$, $g(0) = \alpha$, and $g'(0) = dP(x)(z) \neq 0$. Consequently, there exists a sequence $\{t_n\}_n$ such that $\lim_n t_n = 0$, and $g_n(t_n) = \alpha$ eventually. If we define $x_n = x + t_n z$, the sequence $\{x_n\}_n$ fulfils the required conditions.

■

Mosco convergence is related with other convergences in the following sense: it is implied by norm convergence (if the limit is weakly sequentially continuous), and implies Kuratowski convergence. If the Banach space E is a Schur space, then Kuratowski and Mosco convergences are equivalent ($\mathcal{P}(^k E) = \mathcal{P}_{wsc}(^k E)$ for Schur spaces). Moreover, for spaces whose dual unit ball is w^*-sequentially compact (WCG or $l_1 \not\subset E^*$ for example) this property is also necessary. In fact we have the following

Proposition. *If E is Banach space with w^*-sequentially compact dual unit ball, and it is not a Schur space, then there exists a sequence $\{P_n\}_n \subset \mathcal{P}(^k\mathrm{E})$ which is Kuratowski convergent to a weakly sequentially continuous polinomial P, but it does not converge in the Mosco sense.*

Proof. It is enough to consider a weakly null sequence $\{x_n\}_n$ such that $\|x_n\| = 2$ (that sequence exists if E is not Schur), we may choose a bounded sequence $\{x_n^*\}_n \subset B_{E^*}$ such that $x_n^*(x_n) = 1$. Being the unit ball of E^* w^*-sequentially compact, we may assume that the sequence is w^*-convergent to a x^*. Defining $P_n(y) = x_n^*(y)^k$, and $P(y) = x^*(y)^k$ we have that the sequence $\{P_n\}_n$ converges in the Kuratowski sense to P. On the other hand it does not converge in the Mosco sense because $x_n \in V(P_n - 1)$ for every n, but $0 \notin V(P - 1)$.

■

Let us observe that from an example for $k = 1$ it follows an example for any k, because a finite type polynomial is allways weakly sequentially continuous.

If the Banach space is reflexive, then Mosco convergence and norm convergence are trivially equivalent provided that the limit is weakly sequentially continuous. On the other hand if $l_1 \subset E$ we have that there exists a normalized $\tau(E^*, E)$-null sequence (see [B-V]) and therefore norm convergence does not follow from Mosco convergence. In fact, in the linear case this is a characterization; we do not know if it is true in the general case.

With respect to the Wijsman convergence, the following two examples prove that there is not a general relation between Wijsman and Mosco convergence.

Example 1. Let E be a separable reflexive space such that the norm of E^* does not fulfil Kadec property (see [B-F]). Norm and weak topology does not agree on S_{E^*}, and therefore there exists a sequence, $\{x_n^*\}_n \subset S_{E^*}$, w-convergent (equivalently w^*-convergent) to a $x^* \in S_{E^*}$, which does not converges for the norm. Let's define:

$$P_n(x) = (x_n^*(x))^k \quad P(x) = (x^*(x))^k$$

$\{P_n\}_n$ is Wijsman convergent to P because $\overline{P_n(B(a,r))} = [(x_n^*(a) - r)^k, (x_n^*(a)+r)^k]$ converges to $\overline{P(B(a,r))} = [(x^*(a)-r)^k, (x^*(a)+r)^k]$ if k is odd (if k is even is similar). But $\{P_n\}_n$ does not converge uniformly on bounded sets to P, and hence neither in the Mosco sense because of reflexivity of E.

Example 2. Let us consider the space l_1, and define:

$$P_n(x) = e_1^*(x)^k - 3^k e_n^*(x)^k \quad P(x) = e_1^*(x)^k$$

Kuratowski convergence holds, and being l_1 a Schur space, Mosco convergence too. But on the othe hand

$$\overline{P(B(e_1, \frac{1}{2}))} = [(\frac{1}{2})^k, (\frac{3}{2})^k] \quad 0 \in \overline{P_n(B(e_1, \frac{1}{2}))}$$

and consequently we do not have Wijsman convergence. Let us remember that Wijsman convergence is a metric property and for reflexive spaces, throughout a renorming, it may agree with Mosco convergence for convex sets (see [T]).

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