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Periodic solitons and real algebraic curves.

S. M. NATANZON

Abstract

We describe a method of calculation of all physical algebraic-geometrical solutions of KP-equations.

1. In the middle of seventies in works of S. P. Novikov and others was discovered a new method of solution for some important differential equations of mathematical physics [4]. At first, I shortly remind this method.

Let

$$B = \{B_{ij}\} \in GL(g, C)$$

be a complex symmetric matrix such that the matrix ReB is negative definite. Then there exists a θ-function $\theta : C^g \to C$, where

$$\theta(z \mid B) = \sum_{N \in Z^g} \exp\{\frac{1}{2} \sum_{ij=1}^{g} B_{ij} N_i N_j + \sum_{i=1}^{g} N_i z_i\},$$

$$z = (z_1, \ldots, z_g), \quad N = (N_1, \ldots, N_g).$$

Let

$$e_k = (0, \ldots .1, 0, \ldots, 0), \quad k = 1, \ldots, g$$

be the standard basis of C^g and $f_k = Be_k$. Then

$$\theta(z + 2\pi i e_k) = \theta(z)$$

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and

$$\theta(z + f_k) = \exp(-\frac{1}{2}B_{kk} - z_k)\theta(z).$$

Let Γ be a group generated by

$$2\pi i e_1, \ldots, 2\pi i e_g, f_1, \ldots, f_g.$$

Then the quotient set $J = C^g/\Gamma$ is an Abelian variety. Let

$$\Phi : C^g \to J$$

be the natural projection.

In applications there often appear θ-functions connected which Riemann surfaces. Let P be a Riemann surface of genus g and

$$\{a_i, b_i, \quad i = 1, \ldots, g\} \in H_1(P, Z)$$

be a symplectic basis of $H_1(P, Z)$. This means that the intersection indexes are

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{ij}, \quad 1 \leq i, \quad j \leq g.$$

Let $\omega_1, \ldots, \omega_g$ be holomorphic differentials on P such that

$$\int_{a_k} \omega_j = 2\pi i \delta_{kj}.$$

Then the matrix

$$B_{ij} = \int_{b_i} \omega_j$$

is symmetric and ReB is negative definite. Thus it gives a θ-function $\theta = \theta(z \mid B)$.

2. Consider now Krichever's construction of τ-function. Let $p_0 \in P$, $p_0 \in V \subset P$, and $\epsilon : V \to C$ be a local map such that $\epsilon(p_0) = 0$. Then the differential ω_i has a representation

$$\omega_i = (w_1^i + w_2^i\epsilon + w_3^i\epsilon^2 + \ldots)d\epsilon \quad (i = 1, \ldots, g).$$

Consider vectors

$$w_n = (w_n^1, \ldots, w_n^g) \quad (n = 1, 2, \ldots).$$

Consider also a meromorphic differential Ω_i on P, which is holomorphic on $P - p_0$, has a representation

$$\Omega_i = \frac{i d\epsilon}{\epsilon^{i+1}} + (2\sum_{j=1}^{\infty} \alpha_{ij}\epsilon^j)d\epsilon$$

and such that

$$\oint_{a_k} \Omega_i = 0 \quad (k = 1, \ldots, g).$$

These conditions completely define $\alpha_{ij} \in C$. It is possibly to prove that

$$\alpha_{ij} = \alpha_{ji}.$$

An algebraic-geometrical τ-function is

$$\tau(z_1, z_2, \ldots) = \exp\{-\sum_{ij=1}^{\infty} \alpha_{ij} z_i z_j\}\theta(\sum_{i=1}^{\infty} z_i w_i + \Delta \mid B),$$

where $\Delta \in C^g$.

Solutions to a lot of important equations of mathematical physics can be expressed by $v = \ln \tau$. Consider for example the equations KP (Kadomtsev-Petviashvili), which describe waves in plasma.

$$\frac{3}{4}\partial_2^2 u = \partial_1[\partial_3 u - \frac{1}{4}(6u\partial_1 u + \partial_1^3 u)] \qquad KP1$$

$$-\frac{3}{4}\partial_2^2 \tilde{u} = \partial_1[\partial_3 \tilde{u} - \frac{1}{4}(6\tilde{u}\partial_1 \tilde{u} + \partial_1^3 \tilde{u})], \qquad KP2$$

where $\partial_i = \frac{\partial}{\partial z_i}$.

I. M. Krichever proved that

$$u(z_1, z_2, z_3) = -2\partial^2 v(z_1, z_2, z_3, 0, 0, \ldots)$$

is a solution of KP1 and

$$\tilde{u}(z_1, z_2, z_3) = u(z_1, iz_2, z_3)$$

is a solution of KP2. These functions u and $\tilde{u}$ are complex meromorphic functions. For physics applications, however, it is necessary that u and $\tilde{u}$ will be real and smooth functions on $(z_1, z_2, z_3) \in R^3$.

3. In 1988 B. A. Dubrovin and me proved [5] that the functions u and $\tilde{u}$ are real (on R^3) if and only if P is a real algebraic curve and ϵ, Δ satisfy some additional conditions.

Let us describe these conditions. Let (P, β) be a real algebraic curve. This means that $\beta : P \to P$ is an antiholomorphic involution. The fix points of β form a set $\mathrm{Re}(P, \beta)$ of real points of (P, β). It disintegrates on $k \leq g + 1$ simple contours $a_0, \ldots, a_{k-1}$. We suppose that $p_0 \in a_0$. A local map $\epsilon : V \to C$ is called real if $\epsilon\beta = \bar{\epsilon}$. A differential ω is called positive on a if $\omega = f(\epsilon)d\epsilon$, where ϵ is real and $f(V \cap a) \subset R$, $f(V \cap a) \geq 0$.

The involution β gives an antiholomorphic involution $\beta_J : J \to J$. The fixed points of the involution β_J form $m = 2^{k-1}$ tori

$$T_{\epsilon_1, \ldots, \epsilon_{k-1}} \quad (\epsilon_i = 0, 1).$$

The Abelian map gives a one-to-one correspondence between points of

$$T_{\epsilon_1, \ldots, \epsilon_{k-1}}$$

and a set of divisors $D \in P^g$ such that a set $a_i \cap D$ contains

$$n_i \equiv \epsilon_i \quad (mod 2)$$

points.

Theorem [5]. *The function $u(z_1, z_2, z_3)$ is real on R^3 if and only if the ϵ is a real local map and $\beta_J \Delta = \Delta$. It is smooth if and only if $k = g + 1$ and $\Delta \in \Phi(T_{1,\ldots,1})$.*

A local map ϵ is called imaginary if $\epsilon\tau = -\bar{\epsilon}$. The fixed points of the involution $-\beta_J$ form $m = 2^{k-1}$ tori $I_{\delta_1, \ldots, \delta_{k-1}}$ $(\delta_i = \pm 1)$. The Abelian map gives a one-to-one correspondence between points of $I_{\delta_1, \ldots, \delta_{k-1}}$ and a set of divisors $D \in P^g$ such that $D + \tau D$ is the divisor of zeros of a holomorphic differential ω, which is positive on a_0 and has a sign δ_i on a_i.

Theorem [5]. *The function $\tilde{u}(z_1, z_2, z_3)$ is real on R^3 if and only if the ϵ is an imaginary local map and $\tau_J \Delta = -\Delta$. It is smooth if and only if $P \backslash Re(P, \tau)$ is non connected and $\Delta \in \Phi(I_{1,\ldots,1})$.*

4. For calculations by these theorems it is necessary to find the matrix B, the vectors w_n and the α_{11}. For arbitrary Riemann surfaces

this is Schottky's problem, which has not now effective solution. But in 1987 A. I. Bobenko [2] found a method of calculation B, W_n, α_{11} for real algebraic curves, which was based on classical results of W. Burnside [3] and H. F. Baker [1] and modern results of the theory of Fuchsian groups [6].

Let G be a Schottky group on $\Omega \subset C \cup \infty$ with generators $\sigma_1, \dots, \sigma_g$, where

$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n}$$

such that $P = \Omega/G$. Put us

$$G/G_n = \{\sigma = \sigma_{i_1}^{j_1} \dots \sigma_{i_k}^{j_k} \mid i_k \neq n\}$$

and

$$G_m \setminus G/G_n = \{\sigma = G/G_n \mid i_1 \neq m\}.$$

Consider the series

$$\sum_{nm} = \sum_{\sigma \in G_m \setminus G/G_n} | \ln\{B_m . A_m, \sigma B_n, \sigma A_n\} |,$$

where

$$\{z_1, z_2, z_3, z_4\} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

One can prove [1,2,3], that if $A_n, B_n, \mu_n \in R$, then $\sum_{nm} < \infty$ and

$$B_{nm} = \frac{1}{2\pi i} \left(\sum_{\sigma \in G_n \setminus G/G_m} \ln\{B_m, A_m, \sigma B_n, \sigma A_n\} + \delta_{nm} \ln \mu_n \right),$$

$$w_n^i = \sum_{\sigma \in G/G_n} ((\sigma A_i)^n - (\sigma B_i)^n),$$

$$\alpha_{11} = \sum_{\sigma \in G \setminus 1} c^{-2},$$

where

$$\sigma z = \frac{az + b}{cz + d}.$$

The involution $z \mapsto \bar{z}$ gives an involution $\beta : P \to P$. One can prove [2] that this construction gives all real algebraic curves (P, τ) for $GH = H$ where $H = \{z \in C \mid Im z > 0\}$.

Thus $\Gamma = G\ |_H$ is a Fuchsian group with a standard system of generators $\{\sigma_1, \ldots, \sigma_g\}$. Sets of numbers $\{A_n, B_n, \mu_n \quad n = 1, \ldots, g\}$, which correspond to such a system of generators, were found in [6].

Thus we describe a scheme of calculation of all algebraic-geometrical physical solutions of KP-equations. I. M. Krichever proved that these solutions approximate all quasi-periodical solutions of KP. In [7,8] an analogous method has been used for an integration of two-dimensional Schroedinger operators.

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Independent Moscow University, Russia
and
University of Strasbourg, France