

## Levelled O-minimal structures.

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### Abstract

We introduce the notion of “levelled structure” and show that every structure elementarily equivalent to the real exponential field expanded by all restricted analytic functions is levelled.

An expansion  $\mathfrak{R}$  of an ordered field  $(R, <, +, \cdot, 0, 1)$  is *o-minimal* if every subset of  $R$  (parametrically) definable in  $\mathfrak{R}$  is a finite union of points and open intervals; it is *exponential* if it defines an isomorphism of the ordered groups  $(R, <, +)$  and  $((0, \infty), <, \cdot)$ , where  $(0, \infty)$  denotes the positive elements of  $R$ .

**Example** The *ordered field of real numbers with restricted analytic functions* is the structure

$$\mathbb{R}_{\text{an}} := (\mathbb{R}, <, +, -, \cdot, 0, 1, (\tilde{f})_{f \in \mathbb{R}\{X, m\}, m \in \mathbb{N}}),$$

where  $\mathbb{R}\{X, m\}$  denotes the ring of all power series in  $X_1, \dots, X_m$  over  $\mathbb{R}$  that converge in a neighborhood of  $[-1, 1]^m$ , and where  $\tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}$  is defined for each  $f \in \mathbb{R}\{X, m\}$  by

$$\tilde{f}(x) := \begin{cases} f(x), & x \in [-1, 1]^m \\ 0, & \text{otherwise.} \end{cases}$$

We let  $\mathbb{R}_{\text{an,exp}}$  denote the o-minimal (see e. g. [2]) expansion of  $\mathbb{R}_{\text{an}}$  by the function  $x \mapsto e^x : \mathbb{R} \rightarrow \mathbb{R}$ .

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\*Partially supported by National Science Foundation grants DMS-9306159 and INT-922456, and an AMS Centennial Fellowship.

\*\*Supported by National Science Foundation Postdoctoral Fellowship DMS-9407549.

Mathematics Subject Classification: 03C80-11U09

Servicio Publicaciones Univ. Complutense. Madrid, 1997.

Given an exponential o-minimal expansion  $\mathfrak{R}$  of an ordered field  $(R, <, +, \cdot, 0, 1)$  there is a unique definable differentiable ordered group isomorphism

$$E : (R, <, +, 0) \rightarrow ((0, \infty), <, \cdot, 1)$$

satisfying  $E' = E$  on  $R$ . We denote this unique (hence 0-definable) function by  $\exp$ . The function  $\exp$  behaves (in  $\mathfrak{R}$ ) to a large extent as the real exponential function  $e^x$  behaves when working over the real numbers. (See [5] for details on the above.) The compositional inverse of  $\exp$  from  $(0, \infty)$  onto  $R$  is denoted by  $\log$ , and is called the *logarithm function* (of  $\mathfrak{R}$ ); we extend  $\log$  to be defined on  $R$  by setting  $\log(x) := 0$  for  $x \leq 0$ . For  $r \in R$  and  $a > 0$ , we put  $a^r := \exp(r \log a)$ .

Below, let  $\mathfrak{R}$  denote an o-minimal expansion of an ordered exponential field  $(R, <, +, -, \cdot, 0, 1, \exp)$ ; “definable” means “ $\mathfrak{R}$ -definable”—that is, “definable in  $\mathfrak{R}$  with parameters from  $R$ ”—unless stated otherwise. The reader is assumed to be familiar with the basic properties of o-minimal expansions of ordered exponential fields.

Whenever convenient, we regard any particular partial function as being totally defined by setting the function equal to 0 off its domain of definition.

Let  $e_0$  denote the identity on  $R$  and put  $e_{n+1}(t) := \exp(e_n(t))$  for  $n \in \mathbb{N}$  and  $t \in R$ . Similarly,  $\ell_0$  denotes the identity on  $R$  and  $\ell_{n+1}(t) := \log(\ell_n(t))$  for each  $n \in \mathbb{N}$  and  $t \in R$ . We may also write  $\ell_{-n}$  for  $e_n$ , depending on convenience; for example, ultimately we have  $\ell_{j+k}(t) = \ell_j(\ell_k(t))$  for all  $j, k \in \mathbb{Z}$ . (*Ultimately* abbreviates “for all sufficiently large positive arguments”.)

A function  $f : R \rightarrow R$  is said to be *infinitely increasing* if  $f$  is ultimately strictly increasing and unbounded. Note that if  $f$  is definable, then  $f$  is infinitely increasing if and only if  $\lim_{t \rightarrow +\infty} f(t) = +\infty$ .

For functions  $f, g : R \rightarrow R$  with  $g$  ultimately nonzero, we write  $f(t) \sim g(t)$  if  $\lim_{t \rightarrow +\infty} f(t)/g(t) = 1$ .

Suppose that  $f : R \rightarrow R$  is a definable infinitely increasing function and there exists  $s \in \mathbb{Z}$  such that for some  $k \in \mathbb{Z}$  we have  $\ell_k(f(t)) \sim \ell_{k-s}(t)$ . Then  $s$  is unique and  $\ell_j(f(t)) \sim \ell_{j-s}(t)$  for all  $j \geq k$ . Following Rosenlicht [7], we then say that  $f$  has *level*  $s$  and we write  $\text{level}(f) = s$ . Equivalently, a definable infinitely increasing unary function  $f$  has level  $s$  if and only if there exists  $N \in \mathbb{N}$  such that  $\ell_{N+s}(f(t)) \sim \ell_N(t)$ .

**Definition.** *The structure  $\mathfrak{R}$  is levelled if every definable infinitely increasing unary function has level; its complete theory  $Th(\mathfrak{R})$  is levelled if every  $\mathfrak{A} \equiv \mathfrak{R}$  is levelled.*

We can now state the main result of this note.

**Theorem** ( $\mathbb{R}_{an,exp}$ ) *is levelled.*

We defer the proof until later.

Levelled structures have nice properties that can show up in unexpected ways. For example, it is shown in [6] that if  $\mathfrak{R}$  is levelled, and  $*$  :  $R^2 \rightarrow R$  is definable, continuous and  $(R, *)$  is a group, then  $(R, *)$  is definably homeomorphic to  $(R, +)$ . (It is not known whether this property holds for  $\mathfrak{R}$  without the assumption that  $\mathfrak{R}$  be levelled.)

We now list some basic properties of level; the proofs are easy and we omit them.

**Proposition.** *Let  $f, f_1, f_2$  be definable infinitely increasing unary functions with  $level(f) = s$ ,  $level(f_1) = s_1$  and  $level(f_2) = s_2$ .*

- (1) *For each  $k \in \mathbb{Z}$ ,  $l_k$  has level  $k$ .*
- (2) *If ultimately  $f_1(t) \leq f_2(t)$ , then  $s_1 \leq s_2$ .*
- (3) *If  $\alpha, \beta \in [1, \infty)$  are such that ultimately  $f_1(t) \leq f_2(t)^\alpha$  and  $f_2(t) \leq f_1(t)^\beta$ , then  $s_1 = s_2$ .*
- (4) *Both  $f_1 + f_2$  and  $f_1 \cdot f_2$  have level equal to  $\max(s_1, s_2)$ .*
- (5) *The (ultimately defined) composition  $f_1 \circ f_2$  has level  $s_1 + s_2$ .*

For  $A \subseteq R^{m+1}$  and  $x \in R^m$  put  $A_x := \{t \in R : (x, t) \in A\}$ , and for  $f : A \rightarrow R$  and  $x \in R^m$  define  $f_x : A_x \rightarrow R$  by  $f_x(t) := f(x, t)$ .

**Definition.** *The structure  $\mathfrak{R}$  is exponentially bounded, or  $e$ -bounded for short, if for each definable  $f : R \rightarrow R$  there exists  $n \in \mathbb{N}$  such that ultimately  $|f(t)| \leq e_n(t)$ .*

**Note.** Clearly, if  $\mathfrak{R}$  is levelled then  $\mathfrak{R}$  is  $e$ -bounded. On the other hand, if  $\mathfrak{R}$  is  $e$ -bounded, then for every  $m \in \mathbb{N}$  and definable function  $f : R^{m+1} \rightarrow R$  the set

$$\{level(f_x) : f_x \text{ has level}\}$$

is finite. This follows from (1) and (2) of the previous proposition, and the fact that for  $f$  as above there is some  $N \in \mathbb{N}$  such that for each  $x \in R^m$  ultimately we have  $|f_x(t)| \leq e_N(t)$ . (This fact is established over the reals using 4.18 of [4], but the proof given there goes through for  $\mathfrak{o}$ -minimal expansions of arbitrary ordered fields.)

**Proposition.** *The following are equivalent:*

- (1)  $\text{Th}(\mathfrak{R})$  is levelled.
- (2) For every  $m \in \mathbb{N}$  and definable function  $f : R^{m+1} \rightarrow R$  there exist integers  $N, s(1), \dots, s(k)$  with  $N \geq 0, s(1), \dots, s(k)$  such that for every  $x \in R^m$ , if  $f_x$  is infinitely increasing, then  $\ell_N(f_x(t)) \sim \ell_{N-s(i)}(t)$  for some  $i \in \{1, \dots, k\}$ .
- (3)  $\mathfrak{R}$  is  $e$ -bounded, and for every  $m \in \mathbb{N}$  and definable function  $f : R^{m+1} \rightarrow R$  there exists  $N \in \mathbb{N}$  such that for every  $x \in R^m$ , if  $f_x$  is infinitely increasing, then  $\ell_{N+s}(f_x(t)) \sim \ell_N(t)$  for some integer  $s (= s(x))$ .

**Proof.** (1)  $\Rightarrow$  (2). We may assume that  $f$  is 0-definable, say by an  $(m + 2)$ -ary formula  $\varphi$  in the language of  $\mathfrak{R}$ . Let  $v = (v_1, \dots, v_m)$ , and for each pair of integers  $(j, s)$  let  $\psi_{j,s}(v)$  be the  $m$ -ary formula expressing: "If  $\varphi(v, t, y)$  defines an infinitely increasing function  $y = F_v(t)$ , then  $\ell_j(F_v(t)) \sim \ell_{j-s}(t)$ ." Since  $\text{Th}(\mathfrak{R})$  is levelled, for every  $\mathfrak{A} \equiv \mathfrak{R}$  and every  $a \in A^m$  (where  $A$  is the underlying set of  $\mathfrak{A}$ ) there exist  $j, s \in \mathbb{Z}$  such that  $\mathfrak{A} \models \psi_{j,s}(a)$ . By compactness, there exist integers  $j(1), \dots, j(k), s(1), \dots, s(k)$  such that

$$\mathfrak{R} \models \forall v \left[ \psi_{j(1),s(1)}(v) \vee \dots \vee \psi_{j(k),s(k)}(v) \right].$$

Put  $N := \max\{0, j(1), \dots, j(k), s(1), \dots, s(k)\}$ .

(2)  $\Rightarrow$  (1). Let  $\mathfrak{A} \equiv \mathfrak{R}$  and  $g$  be an  $\mathfrak{A}$ -definable infinitely increasing unary function; say that  $g$  is defined by  $\varphi(a, t, y)$  with  $a \in A^m$  for some  $m \in \mathbb{N}$  and  $\varphi$  an  $(m + 2)$ -ary formula in the language of  $\mathfrak{R}$ . Let  $X$  be the 0-definable set consisting of all  $x \in A^m$  such that  $\varphi(x, t, y)$  defines an infinitely increasing unary function  $y = f_x(t)$ . Now define  $f : A^{m+1} \rightarrow A$  by

$$f(x, t) := \begin{cases} f_x(t), & x \in X \\ 0, & \text{otherwise.} \end{cases}$$

Then  $f$  is 0-definable, and by elementary equivalence there exist  $N \in \mathbb{N}$  and  $s \in \mathbb{Z}$  with  $\ell_N(f_a(t)) \sim \ell_{N-s}(t)$ ; that is,  $g$  has level  $s$ .

That (2)  $\Rightarrow$  (3) is clear, and (3)  $\Rightarrow$  (2) follows from the note preceding the statement of this proposition. ■

**Note.** If  $\text{Th}(\mathfrak{R})$  is levelled and  $\mathfrak{R}' := (R, <, +, \cdot, 0, 1, \exp, \dots)$  is a reduct of  $\mathfrak{R}$ , then  $\text{Th}(\mathfrak{R}')$  is levelled.

(This is immediate from the preceding proposition, but this fact can also be established directly by a basic model-theoretic argument.)

We have no example at present of an  $o$ -minimal expansion of an ordered exponential field whose complete theory is known to be not levelled. However, Boshernitzan [1] has shown that there are real analytic functions  $f : (a, \infty) \rightarrow \mathbb{R}$  satisfying  $f(t + 1) = e^{f(t)}$  for  $t > a$  whose germs at  $+\infty$  belong to Hardy fields; such a function clearly cannot be ultimately bounded by any fixed compositional iterate of  $e^x$ , hence does not have level. Also established in [1] is the existence of ultimately real analytic solutions to the functional equation  $g(g(x)) = e^x$  (a so-called “half-iterate” of  $e^x$ ) whose germs belong to Hardy fields. No such function could have level (otherwise,  $1 = \text{level}(g \circ g) = 2\text{level}(g)$ ). It seems plausible that  $(\mathbb{R}, <, +, \cdot, \exp)$  could be expanded by some such functions to an  $o$ -minimal structure.

### Proof of the Theorem

We now fix some  $\mathfrak{R} \equiv \mathbb{R}_{\text{an}, \text{exp}}$ , with underlying set  $R$ . We must show that  $\mathfrak{R}$  is levelled.

We let  $\mathcal{L}_{\text{an}}$  and  $T_{\text{an}}$  denote respectively the language and the theory of  $\mathbb{R}_{\text{an}}$ , and  $\mathcal{L}_{\text{an}, \text{exp}}$  and  $T_{\text{an}, \text{exp}}$  denote respectively the language and the theory of  $\mathbb{R}_{\text{an}, \text{exp}}$ .

We assume familiarity with the main results from [2,3]; we must first modify some of the constructions from those papers.

If  $G$  is a divisible ordered abelian group, then  $R((t^G))$  denotes the field of formal power series of the form  $f = \sum a_g t^g$ , where  $g$  ranges over  $G$ , each  $a_g \in R$  and  $\text{supp } f := \{g : a_g \neq 0\}$  is well ordered. Since the reduct of  $\mathfrak{R}$  to  $\mathcal{L}_{\text{an}}$  is a model of  $T_{\text{an}}$ , we can naturally equip  $R((t^G))$  with an  $\mathcal{L}_{\text{an}}$ -structure so that  $R((t^G)) \models T_{\text{an}}$ .

There is a natural valuation  $v : R((t^G))^{\times} \rightarrow G$  given by  $v(f) := \min \text{supp } f$ . We extend this valuation to  $R((t^G))$  by putting  $v(0) := \infty$ , with  $v(f) < \infty$  for all  $f \in R((t^G))^{\times}$ .

In the following, we say that a map  $F$  from an ordered ring  $D$  into an ordered ring  $D'$  is a *partial exponential* if  $F$  is an order-preserving homomorphism from the additive group of  $D$  into the multiplicative group of positive elements of  $D'$ .

**Construction of  $R((t))^E$**

We construct a chain of divisible ordered abelian groups

$$\{0\} := \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \dots$$

such that  $\Gamma_{n-1}$  is a convex subgroup of  $\Gamma_n$  for each  $n \in \mathbb{N}$ . Putting  $K_n := R((t^{\Gamma_n}))$  for each  $n \in \mathbb{Z}$  with  $n \geq -1$ , we will obtain an  $\mathcal{L}_{\text{an}}$ -elementary chain

$$K_{-1} \prec K_0 \prec K_1 \prec \dots$$

where  $\Gamma_{n-1}$  is an ordered  $R$ -subspace of  $K_n$  for each  $n \in \mathbb{N}$ . We identify  $K_{-1} = R((t^{(0)}))$  with  $R$ . We will define partial exponential maps  $E_{n-1} : K_{n-1} \rightarrow K_n$  such that  $E_{n-1} \subset E_n$  for each  $n \in \mathbb{N}$ .

Let  $\Gamma_0 := R$ . Let  $E_{-1} : R \rightarrow R((t^R))$  be given by  $E_{-1}(r) := \exp(r)$ . Suppose now that  $n > 0$  and that  $\Gamma_m$  and  $E_{m-1}$  have been constructed for  $m < n$ . Put

$$O_n := \{x \in K_n : v(x) \geq \gamma \text{ for some } \gamma \in \Gamma_{n-1}\}$$

and

$$m_n := \{x \in K_n : v(x) > \Gamma_{n-1}\}.$$

Note that  $O_n = K_{n-1} \oplus m_n$ . We extend  $E_{n-1}$  to a partial exponential  $\widehat{E}_n : O_n \rightarrow K_n$  by setting  $\widehat{E}_n(x) := E_{n-1}(r) \sum_{i \in \mathbb{N}} (\alpha^i / i!)$  for  $x = r + \alpha$  with  $r \in K_{n-1}$  and  $\alpha \in m_n$ . (Note that  $\sum_{i \in \mathbb{N}} (\alpha^i / i!)$  is well-defined since  $v(\alpha) > 0$ .) Let  $J_n := \{x \in K_n : \text{supp } x < \Gamma_{n-1}\}$ ; so  $K_n = J_n \oplus O_n$  as  $K_{n-1}$ -linear spaces. Then we put  $\Gamma_{n+1} := J_n \oplus \Gamma_n \subseteq K_n$ , ordered as an  $R$ -linear subspace of  $K_n$ , so  $\Gamma_n$  is convex in  $\Gamma_{n+1}$ .

Finally, extend  $\widehat{E}_n$  to the partial exponential  $E_n : K_n \rightarrow K_{n+1}$  given by

$$E_n(x) := t^{-a} \widehat{E}_n(b) \text{ for } x = a + b \text{ with } a \in J_n \text{ and } b \in O_n.$$

Put  $R((t))^E := \bigcup K_n$ ,  $\Gamma := \bigcup \Gamma_n$  and  $E := \bigcup E_n$ . Then  $R((t))^E \models T_{\text{an}}$  and  $E : R((t))^E \rightarrow R((t))^E$  is a partial exponential that

agrees with the restricted exponential function on  $[-1, 1]$  and ultimately dominates all polynomials. Note that  $R((t))^E$  is a subfield of  $R((t^\Gamma))$ .

**Construction of  $R((t))^{LE}$**

Similarly as in §2 of [3] we obtain an  $\mathcal{L}_{\text{an,exp}}$ -embedding  $\Phi : R((t))^E \rightarrow R((t))^E$  such that  $\Phi(t^{-1}) = E(t^{-1})$ . Let  $x$  denote  $t^{-1}$ . Put  $L_0 := R((t))^E$ . We can find an  $\mathcal{L}_{\text{an,exp}}$ -extension  $L_1$  of  $L_0$  and an isomorphism  $\eta_1 : L_1 \rightarrow R((t))^E$  such that  $\eta_1$  maps  $R((t))^E$  onto  $\Phi(R((t))^E)$ . Then  $E(\eta_1^{-1}(x)) = x$ . Indeed, every positive element  $g$  of  $R((t))^E$  has a logarithm in  $L_1$  (that is, there exists  $h \in L_1$  such that  $E(h) \doteq g$ ). We continue by constructing for each  $n \in \mathbb{N}$  an  $\mathcal{L}_{\text{an,exp}}$ -extension  $L_{n+1}$  of  $L_n$  and an isomorphism  $\eta_{n+1} : L_{n+1} \rightarrow R((t))^E$  such that  $\eta_{n+1}$  maps  $L_n$  onto  $\Phi(R((t))^E)$ . Every element of  $L_n$  has a logarithm in  $L_{n+1}$ . Finally, put  $R((t))^{LE} := \bigcup L_n$ .

Every positive element of  $R((t))^{LE}$  has a logarithm in  $R((t))^{LE}$ . Thus, from the axiomatization of  $T_{\text{an,exp}}$  from [2], we see that  $R((t))^{LE} \models T_{\text{an,exp}}$ . By §5 of [2], we may identify the field  $\mathcal{H}$  of germs at  $+\infty$  of definable unary functions with the smallest elementary substructure of  $R((t))^{LE}$  containing  $R$  and the element  $x = t^{-1} \in R((t))^{LE}$ . Therefore, in what follows we routinely identify any given definable unary function  $f$  with its germ  $f \in \mathcal{H}$ , which in turn is identified with the element  $f \in R((t))^{LE}$ . In particular, note that for every definable unary function  $f$  we have  $E(f) = \exp(f)$ , and if  $f$  is ultimately positive then  $E(\ell(f)) = f$ . Thus, there is no harm in denoting the logarithm function for  $R((t))^{LE}$  by  $\ell$ , and using the notation  $\ell_k$  for  $k \in \mathbb{Z}$  in the obvious fashion. Note in particular that  $\eta_n^{-1}(x) = \ell_n(x)$  for all  $n \geq 1$ .

Given definable unary functions  $f$  and  $g$  with  $g$  ultimately nonzero, we have  $f(x) \sim g(x)$  if and only if  $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$  if and only if  $v(f-g) > v(g)$ . Thus, given nonzero  $f, g \in R((t))^{LE}$ , we write  $f \sim g$  for  $v(f-g) > v(g)$ , that is,  $f = g(1+\epsilon)$  for some  $\epsilon \in R((t))^{LE}$  with  $v(\epsilon) > 0$ . Note also that  $v(f) = v(g)$  if and only if  $f \sim cg$  for some nonzero  $c \in R$ . It is easy to see that  $\sim$  is a congruence relation on the multiplicative group of nonzero elements of  $R((t))^{LE}$ .

**Lemma.** *Let  $f, g \in R((t))^{LE}$  with  $f, g > 0$ , and  $v(g) < 0$ .*

- (1) *If  $f = gh$  with  $g > h^r$  for all  $r \in R$ , then  $\ell(f) \sim \ell(g)$ .*
- (2) *If  $v(f) = v(g)$ , then  $\ell_k(f) \sim \ell_k(g)$  for all  $k > 0$ .*

**Proof.** For (1), note that for all positive  $r \in R$  we have

$$r(\ell(f) - \ell(g)) = r\ell(h) = \ell(h^r) < \ell(g),$$

that is,  $v(\ell(f) - \ell(g)) > v(\ell(g))$ .

An easy induction on  $k$  yields (2). ■

**Claim.** Let  $g \in R((t))^E$  with  $v(g) < 0$  and  $g > 0$ , and let  $k$  be the least positive integer such that  $v(g) \in \Gamma_k$  (as in the construction of  $R((t))^E$ ). Then  $\ell_{2+k}(g) \sim \ell_2(x)$ .

**Proof.** We prove this by induction on  $k$ . First, suppose  $k = 0$ . Then  $v(g) = v(x^r)$  for some positive  $r \in R$ , and by (2) of the Lemma we have

$$\ell_2(g) \sim \ell_2(x^r) = \ell(r) + \ell_2(x) \sim \ell_2(x).$$

Suppose now that the result holds for a certain  $k > 0$  and let  $v(g) \in \Gamma_{k+1} \setminus \Gamma_k$ . Then  $v(g) = \delta + \gamma$  where  $\delta \in J_k$ ,  $\delta < 0$  and  $\gamma \in \Gamma_k$ . Hence,  $v(\delta) \in \Gamma_k \setminus \Gamma_{k-1}$  and  $v(\delta) < \Gamma_{k-1}$ , so  $\ell_{2+k}(-\delta) \sim \ell_2(x)$  by the inductive assumption. Also, we have  $g = t^\delta(at^\gamma + \mu)$ , with  $a \in R$  and  $v(\mu) > \gamma$ . Now  $\delta < \Gamma_k$ , so  $\ell(g) \sim \ell(t^\delta) = -\delta$  and  $\ell_{2+k+1}(g) \sim \ell_{2+k}(-\delta)$  by (1) and (2) of the Lemma, respectively. Thus,  $\ell_{2+k+1}(g) \sim \ell_2(x)$ . ■

**Claim.** Let  $g \in L_n$  (as in the construction of  $R((t))^{LE}$ ),  $g > 0$  and  $v(g) < 0$ . Then there exists  $s \in \mathbb{Z}$  such that  $\ell_{2+n+s}(g) \sim \ell_{2+n}(x)$ .

**Proof.** Let  $f_n : L_n \rightarrow R((t))^E$  be as in the construction of  $R((t))^{LE}$ . By the previous claim, there is some  $k \in \mathbb{N}$  such that  $\ell_{2+k}(f_n(g)) \sim \ell_2(x)$ . Since  $f_n$  is an  $\mathcal{L}_{\text{an,exp}}$ -isomorphism, we have

$$\ell_{2+k}(g) \sim \ell_2(f_n^{-1}(x)) = \ell_2(\ell_n(x)) = \ell_{2+n}(x).$$

Hence  $\ell_{2+n+s}(g) \sim \ell_{2+n}(x)$ , where  $s = k - n$ . ■

**Definition.** An element  $f \in R((t))^{LE}$  has level  $s$  for  $s \in \mathbb{Z}$  if  $f > 0$ ,  $v(f) < 0$  and there is an  $N \in \mathbb{N}$  such that  $\ell_{N+s}(f) \sim \ell_N(x)$ .

It is immediate from the preceding claim and the construction of  $R((t))^{LE}$  that every  $g \in R((t))^{LE}$  with  $g > 0$  and  $v(g) < 0$  has level  $s$  for some  $s \in \mathbb{Z}$ . Hence,  $\mathfrak{R}$  is levelled. ■

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