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Separating ideals in dimension 2.

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Abstract

Experience shows that in geometric situations the separating ideal associated with two orderings of a ring measures the degree of tangency of the corresponding ultrafilters of semialgebraic sets. A related notion of separating ideals is introduced for pairs of valuations of a ring. The comparison of both types of separating ideals helps to understand how a point on a surface is approached by different half-branches of curves.

In this paper we study the geometry of separating ideals. Let A be a Noetherian ring with real spectrum $Sper(A)$ (cf. [2], Chapitre 7; [6], Kapitel III). For any two orderings $\alpha, \beta \in Sper(A)$, the *separating ideal* $\langle \alpha, \beta \rangle$ was defined by Madden in [8]. This is the ideal generated by the symmetric difference $(\alpha \backslash \beta) \cup (\beta \backslash \alpha)$. Alternatively, it can also be characterized as the smallest ideal $I \subseteq A$ that is convex with respect to both α and β and for which α and β induce the same total order on A/I ([8]). The separating ideal was first introduced as a tool for working at the Pierce-Birkhoff Conjecture ([8]; [10]). By its definition it is an algebraic artifact. But experience shows that in a ring arising from a geometric context, e.g. the coordinate ring of an algebraic variety over a real closed field, separating ideals carry geometric information. For example, suppose that $\alpha, \beta \in Sper(\mathbf{R}[X, Y])$ are defined by germs of half branches of curves at $0 \in \mathbf{R}^2$. Then $\langle \alpha, \beta \rangle$ measures the degree of

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tangency of the half branches. This leads us to study the relationship between tangent spaces and orderings.

The classical notion of valuations can be generalized from fields to rings (cf. [3], Chapitre VI, §3, No. 1, [4]; [9]). Suppose that $v : A \to \Gamma \cup \{\infty\}$ is a valuation. An ideal $I \subseteq A$ is called a *v–ideal* if

$$I = \{a \in A \mid \exists b \in I : v(a) \geq v(b)\}.$$

Every ordering $\alpha \in Sper(A)$ defines a valuation v_α of A. The v_α–ideals are the same as the ideals that are convex with respect to α. Similar to separating ideals of orderings, it is possible to introduce separating ideals of valuations. If v and w are valuations of A then $\langle v, w\rangle$ denotes the smallest ideal $I \subseteq A$ such that I is both a v–ideal and a w–ideal and the chains of v– and w–ideals containing I are identical. Basic properties of $\langle v, w\rangle$ are discussed in section 2. We are particularly interested in the relationship between the separating ideals $\langle \alpha, \beta\rangle$ and $\langle v_\alpha, v_\beta\rangle$ (where $\alpha, \beta \in Sper(A)$). It is obvious that always $\langle v_\alpha, v_\beta\rangle \subseteq \langle \alpha, \beta\rangle$. The question of when these ideals are equal is much more subtle.

The language needed to talk about the geometry of the different separating ideals is developed in section 1. Suppose that $M \subseteq A$ is a maximal ideal and that v is a valuation having M as a v–ideal. There is a largest v ideal M^v contained in M. Since $M^2 \subseteq M^v$, the factor module M/M^v can be considered as a factor space of the cotangent space M/M^2. If $v = v_\alpha$ for some $\alpha \in Sper(A)$ then M/M^v carries a total order. The translation of these structures into the Zariski tangent space provides a geometric way of looking at the situation. (Related concepts have also been introduced by Marshall – see [10], p. 1265.) Of particular interest is the question of what it means that a separating ideal is maximal. In section 2, Theorem 1 and Theorem 2 juxtapose the geometric meanings of this condition for separating ideals of valuations and for separating ideals of orderings.

In section 4 we consider our general concepts of separating ideals in the rather benign environment provided by a regular local ring (A, m, k) of dimension 2. The reason is that separating ideals are complete (or integrally closed) ideals, as are all v–ideals for any valuation v (cf. [14], Appendix 4). For two–dimensional regular local rings there exists a highly satisfactory theory of complete ideals, largely due to Zariski (cf. [13]; [14], Appendix 5). The most important tool in this theory are quadratic transformations. Therefore, in preparation for section 4, we discuss the behavior of separating ideals under quadratic transformations in section 3. For orderings and valuations there is a notion of strict transforms. Under hypotheses which are not too restrictive for geometric applications we show that the strict transform of the separating ideal of two valuations is the separating ideal of the strict transforms. A corresponding result for orderings is known from [1]. In section 4, after excluding a few trivial cases, we deal with orderings $\alpha, \beta \in Sper(A)$ such that $\langle\alpha,\beta\rangle$ is m-primary. Finitely many quadratic transformations transform $\langle\alpha,\beta\rangle$ into the maximal ideal of an iterated quadratic transform of (A, m, k). This means that the tangent directions of α and β become separated after finitely many transformations. This can come about in two different ways: Either the tangent spaces are distinct; or there is a common tangent space and only inside this space the directions differ. We show in Theorem 3 that these two cases can be distinguished in the ring (A, m, k), without doing any quadratic transformations, by comparing the separating ideals $\langle\alpha,\beta\rangle$ and $\langle v_\alpha, v_\beta\rangle$. From a conceptual point of view it is interesting that at least this piece of information can be obtained without leaving the ring A.

Notation and terminology. Throughout, all rings other than valuation rings are Noetherian. If A is a ring then $Sper(A)$ is its *real spectrum*. General references for the real spectrum are [2], Chapitre 7, and [6], Kapitel III. The points of the real spectrum are called *orderings*. If $\alpha \in Sper(A)$ then the prime ideal $supp(\alpha) = \alpha \cap -\alpha$ is its *support*. Similarly, if $v : A \to \Gamma \cup \{\infty\}$ is a valuation then $\{a \in A \mid v(a) = \infty\}$ is a prime ideal which is also called the *support* of v. General references for valuations of rings are [3], Chapitre VI, §3, No 1; [4]; [9]. In analogy to the usage for valuations of fields, we call $\{a \in A \mid v(a) \geq 0\}$ the *valuation ring* of v. But, of course, this subring of A does not have the properties one is used to from the valuation theory of fields. If $I \subsetneq A$ is an ideal then $v(I)$ denotes the smallest value of any element of I. This is well-defined since A is Noetherian. If $\alpha \in Sper(A)$ then $A/supp(\alpha)$ is a domain which is totally ordered with positive cone $\alpha/supp(\alpha)$; the totally ordered quotient field is $\kappa(\alpha)$. We denote the canonical image of $a \in A$ in $\kappa(\alpha)$ by $a(\alpha)$. An ideal $I \subsetneq A$ is an $\alpha - ideal$ (or is *convex* with respect to α) if $a \in I$ whenever $0 \leq a(\alpha) \leq b(\alpha)$ and $b \in I$.

With α we associate a valuation v_α: Let $V(\alpha) \subseteq \kappa(\alpha)$ be the convex subring of $\kappa(\alpha)$ generated by the image of A. This is a valuation ring. If $v'_\alpha : \kappa(\alpha) \to \Gamma \cup \{\infty\}$ is the corresponding valuation then v_α is defined to be

$$A \longrightarrow \kappa(\alpha) \xrightarrow{v'_\alpha} \Gamma \cup \{\infty\}.$$

Let $v : A \to \Gamma \cup \{\infty\}$ be any valuation of A. Then an ideal $I \subseteq A$ is a $v - ideal$ if there is some $\gamma \in \Gamma \cup \{\infty\}$ such that

$$I = \{a \in A \mid v(a) \geq \gamma\}.$$

The support of v is a v–ideal; it may be the only one. Given α or v there are a largest α–ideal and a largest *v–ideal* in A. They are called the *center* of α and the *center* of v. The center is a prime ideal, say p, and α (or v) extends uniquely to A_p. As A is Noetherian, the totally ordered set of α–ideals or v–ideals is anti–wellordered by inclusion. The immediate successor of I is denoted by I^α or I^v: it is the biggest α–ideal or v–ideal properly contained in I.

1 Tangents

The (Zariski) tangent space of a maximal ideal m in a ring A is the dual $(m/m^2)^\vee$ of the k–vector space m/m^2 (with $k = A/m$). If v is a valuation of A with center m then it is obvious that $m^2 \subseteq m^v$. Therefore we can consider the subspace m^v/m^2 of the cotangent space at m. The set

$$T_v = \{p \in (m/m^2)^\vee \mid f(p) = 0 \text{ for all } f \in m^v/m^2\}$$

is a subspace of $(m/m^2)^\vee$ which is canonically isomorphic to $(m/m^v)^\vee$. We shall call T_v the *tangent space* of v. If $\alpha \in Sper(A)$ is centered in m we define the tangent space of α to be $T_\alpha = T_{v_\alpha}$. From the definitions it is clear that $m/m^{v_\alpha} = m/m^\alpha$ is a totally ordered vector space over the totally ordered field k and that m/m^α is archimedean relative k. To study the connections of T_α with this total order we consider the following more general situation:

Let k be a totally ordered field, V a finite dimensional totally ordered vector space over k. The positive cone of V is the union of a set $\mathcal{P}$ of closed convex polyhedral cones in V. With each $P \in \mathcal{P}$ one associates the dual cone $P^\vee$ in $V^\vee$ (cf. [12], Appendix 1). It is defined as

$$P^\vee = \{f \in V^\vee \mid \forall v \in P : f(v) \geq 0\}.$$

It is well-known that $P^\vee$ is a closed convex polyhedral cone in $V^\vee$. Such a cone in $V^\vee$ will be called a *$\mathcal{P}$-cone*, a cone $Q \subseteq V^\vee$ is an *$\mathcal{N}$-cone* if $-Q$ is a $\mathcal{P}$-cone. Every $\mathcal{P}$-cone can be represented in the form

$$\{f \in V^\vee \mid f(v_1) \geq 0 \,\&\ldots\&\, f(v_s) \geq 0\}$$

where $v_1, \ldots, v_s \geq 0$ in V. From this it is clear that every convex polyhedral cone in $V^\vee$ is the intersection of a $\mathcal{P}$-cone with an $\mathcal{N}$-cone. Finite intersections of $\mathcal{P}$-cones are $\mathcal{P}$-cones. Every $\mathcal{P}$-cone is full dimensional.

Lemma 1. *A finite union of convex polyhedral cones none of which is a $\mathcal{P}$-cone contains no $\mathcal{P}$-cone.*

Proof. Any inclusion

$$P \subseteq \bigcup_i (P_i \cap N_i)$$

(where P and the P_i are $\mathcal{P}$-cones, the N_i are $\mathcal{N}$-cones and $P_i \not\subseteq P_i \cap N_i$) yields the inclusion

$$P \cap (\bigcap_i -N_i) \subseteq \bigcup_i (N_i \cap -N_i).$$

For every i the intersection $N_i \cap -N_i$ is a proper linear subspace of $V^\vee$. A finite union of such subspaces cannot contain a full dimensional cone. However, $P \cap (\bigcap_i -N_i)$ is a $\mathcal{P}$-cone, hence is full dimensional, and is contained in $\bigcup_i (N_i \cap -N_i)$. This contradiction ends the proof.

■

Lemma 2. *Let $\mathcal{L}$ be the lattice of subsets of $V^\vee$ generated by the set of all closed convex polyhedral cones. In $\mathcal{L}$ the $\mathcal{P}$-cones are a basis for a prime filter.*

Proof. We must show that the set $\mathcal{L}'$ of elements of $\mathcal{L}$ containing some $\mathcal{P}$-cone has the following properties:

(1) $\{0\} \notin \mathcal{L}'$;

(2) if $Q \in \mathcal{L}'$ and $Q \subseteq P$ then $P \in \mathcal{L}'$;

(3) if $P, Q \in \mathcal{L}'$ then $P \cap Q \in \mathcal{L}'$;
(4) if $P \cup Q \in \mathcal{L}'$ then $P \in \mathcal{L}'$ or $Q \in \mathcal{L}'$.
Conditions (1), (2), (3) are evident, (4) is immediate from Lemma 1.

■

We have shown that a total order on V defines a prime filter in $\mathcal{L}$ which has a basis consisting of full dimensional cones. Conversely, let $\mathcal{L}'$ be a prime filter in $\mathcal{L}$ having a basis $\mathcal{B}'$ which consists of full dimensional closed convex polyhedral cones. Then $\mathcal{L}'$ defines a total order for V. To prove this pick any cone $P^\vee \in \mathcal{B}'$ and consider its dual cone $P \subseteq V$. It is claimed that the union T of these dual cones is the positive cone of a total order of V. First we check that $T \cap -T = \{0\}$. So, suppose that $v \in T \cap -T$. Then there are $P^\vee, Q^\vee \in \mathcal{B}'$ such that $v \in P$ and $-v \in Q$, i.e.,

$$P^\vee \cap Q^\vee \subseteq \{f \in V^\vee \mid f(v) = 0\}.$$

Since the basis $\mathcal{B}'$ consists of full dimensional cones this implies that $v = 0$. Next we prove $T \cup -T = V$: If $v \in V$ then

$$H^+ = \{f \in V^\vee \mid f(v) \geq 0\},\ H^- = \{f \in V^\vee \mid f(v) \leq 0\}$$

are both closed convex polyhedral in $V^\vee$. Their union is $V^\vee$, hence belongs to $\mathcal{L}'$. By primality of $\mathcal{L}'$ one of H^+ and H^- belongs to $\mathcal{L}'$, hence contains some $P^\vee \in \mathcal{B}'$, say $P^\vee \subseteq H^+$. Then $f(v) \geq 0$ for every $f \in P^\vee$, i.e., $v \in P$. Finally we have to check the algebraic properties of the prospective positive cone T: Pick $v, w \in T$, say $v \in P$, $w \in Q$ with $P^\vee, Q^\vee \in \mathcal{B}'$, and $0 < \lambda, \mu \in k$. There is some $R^\vee \in \mathcal{B}'$ contained in $P^\vee \cap Q^\vee$. But then $v, w \in R$ and

$$f(\lambda v + \mu w) = \lambda f(v) + \mu f(w) \geq 0$$

for every $f \in R^\vee$. This proves $\lambda v + \mu w \in R \subseteq T$.

It is obvious that the two constructions of assigning a prime filter of cones to a total order and of assigning a total order to a prime filter of cones are inverse to each other. We summarize the results obtained so far in

Proposition 1. *Let V be a finite dimensional vector space over the totally ordered field k. Let $\mathcal{L}$ be the lattice of subsets of $V^\vee$ generated* *by the set of closed convex polyhedral cones. Then there is a bijective correspondence between the total orders of V and the prime filters in $\mathcal{L}$ generated by full dimensional closed convex polyhedral cones.*

∎

The situation which is most interesting for the discussion of m/m^α is the one in which the total order of V is archimedean relative k. This case can be characterized in the following way:

Proposition 2. *The order of V is archimedean relative k if and only if $\{0\}$ is the only proper subspace W of $V^\vee$ such that $\langle W \cap P^\vee \rangle = W$ for every $P^\vee \in \mathcal{B}'$. (Notation: $\langle X \rangle$ is the linear subspace generated by X; $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{B}'$ are as in the proof of Proposition 1.)*

Proof. First suppose that V is archimedean and assume (by way of contradiction) that $W = \langle W \cap P^\vee \rangle$ for every $P^\vee \in \mathcal{B}'$, where $\{0\} \subset W \subset V^\vee$. Let

$$V_1 = \{v \in V \mid f(v) = 0 \ \ \forall f \in W\}.$$

Then $\{0\} \subset V_1 \subset V$. It suffices to show that V_1 is convex. So, pick $0 \leq v \leq v_1$, $v \in V$, $v_1 \in V_1$. Now pick $0 \leq v_2, \ldots, v_s$ in V such that $v, v_1 - v, v_1, \ldots, v_s$ generate a full dimensional cone in V and let $P^\vee \in V^\vee$ be its dual. There is a basis, say $f_1, \ldots, f_t$, of W which is contained in $P^\vee$. In particular, $f_i(v_1) = 0, f_i(v) \geq 0, f_i(v_1 - v) \geq 0$ for all $i = 1, \ldots, t$. But this implies $f_i(v) = 0$ for all i, hence $v \in V_1$. This is a contradiction, and the first part of the proof is complete. – Now suppose that V is nonarchimedean. Let $\{0\} \subset V_1 \subset V$ be the largest convex subspace. In $V^\vee$ consider the subspace

$$W = \{f \in V^\vee \mid f(v_1) = 0 \ \forall v_1 \in V_1\}.$$

It is obvious that $\{0\} \subset W \subset V^\vee$. Pick $v_1, \ldots, v_s \geq 0$ in V generating a cone P and let $P^\vee$ be its dual. Let $\bar{v}_1, \ldots, \bar{v}_s \geq 0$ be the images in V/V_1, let $\bar{P}$ be the cone in V/V_1 generated by these elements. As $\bar{P}$ contains no nontrivial linear subspace the dual cone $\bar{P}^\vee \subset (V/V_1)^\vee$ is full dimensional. The subspace $W \subset V^\vee$ may be identified with $(V/V_1)^\vee$. Then $\bar{P}^\vee$ contains a basis of W and is obviously contained in $P^\vee$. This shows that $\langle W \cap P^\vee \rangle = W$ for each such cone P.

∎

We return to the original setup stemming from an ordering $\alpha \in Sper(A)$ centered at m:

Definition. *The tangent direction D_α of α is the prime filter of polyhedral cones in T_α defined by the archimedean (over k) total order of m/m^α.*

■

Example. Let $A = \mathbf{R}[X, Y]$ and consider the irreducible polynomial $Y^2 - X^2(X+1)$. There are four orderings $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of A having support $(Y^2 - X^2(X+1))$ which are centered at (X, Y). For example, α_1 can be described as

$$\alpha_1 = \{P \in A \mid \exists\, 0 < \varepsilon \in \mathbf{R} : \left.\begin{matrix} (0 < x, y < \varepsilon \;\&\; y^2 - x^2(x+1) = 0 \\ \rightarrow P(x,y) \geq 0) \end{matrix}\right\}.$$

α_2, α_3 and α_4 are given in the same way, only with "$0 < x, y < \varepsilon$" replaced by "$0 < -x, y < \varepsilon$", "$0 < -x, -y < \varepsilon$" and "$0 < x, -y < \varepsilon$", respectively. One finds that $T_{\alpha_1} = T_{\alpha_3}$ is the line $X = Y$ and $T_{\alpha_2} = T_{\alpha_4}$ is the line $X = -Y$. On these lines the only full dimensional polyhedral cones are half lines. Thus, every tangent direction must be a half line on the appropriate tangent line. For $\alpha_1, \alpha_2, \alpha_3$ and α_4 these are the half lines lying in the first, second, third and fourth quadrant, respectively.

■

2 Separating ideals

Separating ideals of orderings have been treated extensively in [8] and [1]. Here we will first discuss some basic properties of separating ideals of valuations.

Lemma 3. *If v is a valuation of A then every proper v–ideal has a prime ideal as its radical. In particular, $\sqrt{\langle v, w\rangle}$ is always a prime ideal.*

Proof. Let Γ be the value group of v and suppose that $I = \{a \in A \mid v(a) \geq \gamma\}$ is an ideal, where $\gamma \in \Gamma \cup \{\infty\}$. Then I is proper if and only if

$\gamma > 0$. Let $\Delta \in \Gamma$ be the largest convex subgroup that does not contain γ. One checks that

$$\sqrt{I} = \{a \in A \mid \forall \delta \in \Delta : \delta < v(a)\}.$$

Now it is obvious that $\sqrt{I}$ is a prime ideal.

■

From the definition of the support and the center of a valuation it is clear that we always have $supp(v) \subseteq center(v)$.

Let $\Delta \subset \Gamma$ be the convex subgroup of the value group of v generated by the set

$$\{\gamma \in \Gamma \mid \gamma < 0 \ \& \ \exists a \in A : v(a) = \gamma\}.$$

Then a coarsening $\bar{v}$ of v is defined by composing v with the canonical map $\Gamma \to \Gamma/\Delta$. The valuation ring of $\bar{v}$ in A is A itself. Both v and $\bar{v}$ define exactly the same valuation ideals, and

$$center(\bar{v}) = center(v) = \{a \in A \mid \forall \delta \in \Delta : \delta < v(a)\}$$

(recall that A is Noetherian). Denote the center of v and $\bar{v}$ by p. Next we study the behavior of the $v-$ and $\bar{v}$–ideals under localization at p. Note that both v and $\bar{v}$ extend uniquely to valuations v_p and $\bar{v}_p$ of A_p and that $\bar{v}_p$ defines the trivial valuation on the residue field. If $\alpha \in Sper(A)$ then $v_\alpha = \bar{v}_\alpha$ by definition of v_α.

Lemma 4. *Mutually inverse maps between the sets of v–ideals and v_p–ideals are defined by*

$$I \longrightarrow IA_p, \ J \longrightarrow J \cap A.$$

Proof. Suppose that I is a v–ideal and that $\frac{a}{s} \in IA_p$ with $a \in I, s \notin p$. Pick $\frac{b}{t} \in A_p$ such that $v_p\left(\frac{b}{t}\right) \geq v_p\left(\frac{a}{s}\right)$. This implies $v(bs) \geq v(at)$. Since $s \in A \backslash p$ there is some $r \in A \backslash p$ such that $v(r) \leq -v(s)$. Then

$$v(b) \geq v(brs) \geq v(atr).$$

Since $atr \in I$ we see that $b \in I$ and $\frac{b}{t} \in IA_p$. This proves that IA_p is indeed a v_p–ideal. On the other hand, it is trivial that $J \cap A$ is a

v–ideal for any v_p–ideal $J \subseteq A_p$. To prove that $IA_p \cap A = I$ we pick $b = \frac{a}{s} \in IA_p \cap A$ (with $a \in I, s \in A \backslash p$). Again, choosing $r \in A \backslash p$ with $v(r) \leq -v(s)$ we have

$$v(b) \geq v(brs) = v(ar)$$

with $ar \in I$. It follows that $b \in I$. Finally, it is a general property of localizations that $(J \cap A)A_p = J$.

■

The Lemma also applies to $\bar{v}$, of course. So the v–ideals, $\bar{v}$–ideals, v_p–ideals and $\bar{v}_p$–ideals are essentially the same thing. This makes it possible to replace v by $\bar{v}$ in many places and to use localization techniques. In particular, for any v and w we have $\langle v, w \rangle = \langle \bar{v}, \bar{w} \rangle$.

The separating ideal $\langle v, w \rangle$ of any two valuations is the entire ring if $center(v) \neq center(w)$. Since this is a rather unexciting separating ideal we will restrict our attention entirely to the case that $center(v) = center(w)$. Denoting this center by p again it is a consequence of the foregoing considerations that the separating ideal is well–behaved under localization:

Proposition 3. *If v_p and w_p are the unique extensions of v and w to A_p then $\langle v_p, w_p \rangle = \langle v, w \rangle A_p$ and $\langle v, w \rangle = \langle v_p, w_p \rangle \cap A$.*

■

The next result gives us a set of generators for $\langle v, w \rangle$:

Proposition 4. *Let v, w be valuations of A having a common center, and assume that $v = \bar{v}$, $w = \bar{w}$. Then either v and w are both trivial and $\langle v, w \rangle = supp(v) = supp(w)$ or $\langle v, w \rangle$ is generated by the set*

$$M = \{f \in A \mid \exists g \in A : \; v(f) < v(g) \;\&\; w(f) \geq w(g) \\ \text{or } \exists g \in A : v(f) \geq v(g) \;\&\; w(f) < w(g)\}.$$

Proof. Since the case of trivial valuations is clear we suppose that v is non–trivial. Suppose that $f \in M$ and $v(f) < v(g), w(f) \geq w(g)$. Then

$$I = \{a \in A \mid v(a) > v(f)\}$$

is a v-ideal which contains g, but not f. Hence, I is not a w-ideal. This implies that $I \subset \langle v, w\rangle$, i.e., $v(f) \geq v(\langle v, w\rangle)$. This proves that $M \subseteq \langle v, w\rangle$. Conversely, $\langle v, w\rangle$ is obviously generated by the set

$$N = \{f \in A \mid v(f) = v(\langle v, w\rangle)\}.$$

Now pick $f \in N$ and suppose that also

$$w(f) = w(\langle v, w\rangle).$$

By definition of $\langle v, w\rangle$ the ideals

$$I = \{a \in A \mid v(a) > v(f)\}$$

and

$$J = \{a \in A \mid w(a) > w(f)\}$$

do not coincide. Thus, there is some $a \in I \backslash J$ or some $a \in J \backslash I$. In either case this element a shows that the condition defining M holds for f, i.e., $f \in M$. Next suppose that

$$w(f) > w(g) = w(\langle v, w\rangle)$$

with some suitable $g \in \langle v, w\rangle$. Then there exist $a \in A$ such that

$$v(f + ag) = v(\langle v, w\rangle),\ w(f + ag) = w(\langle v, w\rangle).$$

By the first case discussed above we see that $f + ag \in M$. Also, since $g \in \langle v, w\rangle$ it is clear that $v(g) \geq v(f)$. The definition of M shows that $g \in M$. But then $f \in (M)$. Altogether this proves that $N \subseteq (M)$, and we conclude that $(M) = \langle v, w\rangle$.

■

The description of $\langle v, w\rangle$ in Proposition 4 is very much reminiscent of the definition of $\langle \alpha,\ \beta\rangle\,(\alpha, \beta \in Sper(A))$ in [8]). Instead of using functions changing sign between α and β, now we consider functions which are of different order of magnitude with respect to v and w. If $v = v_\alpha$ and $w = v_\beta$ then, obviously, this is a much coarser approach than Madden's.

To show the geometric significance of separating ideals we will now establish a connection with the tangent spaces and tangent directions of §1.

Theorem 1. *Let (A, m, k) be a local ring and suppose that the valuations v and w are both centered at m and have support properly contained in m. Then $\langle v, w\rangle = m$ if and only it $T_v \neq T_w$.*

Theorem 2. *Let (A, m, k) be a local ring and suppose that $\alpha, \beta \in Sper(A)$ are both centered at m, induce the same total order on k and have support properly contained in m. Then $\langle \alpha, \beta\rangle = m$ if and only if the tangent directions of α and β are different.*

Proof of Theorem 1. From the definitions it is obvious that $m^v/m^2 \neq m^w/m^2$ in m/m^2 if and only if $T_v \neq T_w$ in $(m/m^2)^\vee$. It is also clear that $\langle v, w\rangle = m$ if and only if $m^v \neq m^w$.

■

Proof of Theorem 2. First suppose that $\langle \alpha, \beta\rangle = m$. If $\langle \alpha, \beta\rangle = \langle v_\alpha, v_\beta\rangle$ then Theorem 1 implies that $T_\alpha \neq T_\beta$. By the definition of $\mathcal{P}$-cones, T_α is spanned by every cone belonging to the tangent direction of α and T_β is spanned by every cone belonging to the tangent direction of β. Thus, the tangent directions must be different. If $\langle \alpha, \beta\rangle \supset \langle v_\alpha, v_\beta\rangle$ then $m^\alpha = m^\beta$, i.e., $T_\alpha = T_\beta$, but α and β define different total orders on $m/m^\alpha = m/m^\beta$. The tangent directions of α and β must be different since Proposition 1 shows that they determine the total orders of $m/m^\alpha = m/m^\beta$. Finally suppose that $\langle \alpha, \beta\rangle \subset m$. Then $m^\alpha = m^\beta$ and α and β define the same total order on $m/m^\alpha = m/m^\beta$. Therefore the tangent directions agree.

■

Returning to the example at the end of §1 we see that all separating ideals between any two of the orderings $\alpha_1, \alpha_2, \alpha_3$, and α_4 are equal to the maximal ideal (X, Y). But the separating ideals $\langle v_{\alpha_i}, v_{\alpha_j}\rangle$ (with $i \neq j$) are equal to m if $i + j \equiv 1 (mod\ 2)$ or different from m if $i + j \equiv 0 (mod\ 2)$ as the tangents are different or equal in these two cases.

From [14], Appendix 4, it is clear that valuation ideals and convex ideals in a ring A are always complete (= integrally closed). Among the complete ideals the simple complete ideals play a particular role. These are the ideals which cannot be represented nontrivially as a product of other ideals. They are of great importance for the theory of complete ideals in 2-dimensional rings (cf. [13]; [14], Appendix 5; [7]). From Noetherianity it is clear that every complete ideal in A can be factored into a product of simple complete ideals. For v-ideals it is immediately clear that they can even be factored into simple v-ideals (cf. the discussion following Proposition 2.1 in [1]). It is shown in [1], Proposition 2.2, that under a suitable hypothesis the separating ideal $\langle\alpha,\beta\rangle$ is simple. Note that $I^\alpha \supseteq Im$ if (A,m,k) is a local ring and $I \subsetneq A$ is an α-ideal, hence $I/^\alpha$ is a k-vector space. In this situation the hypothesis is that $\dim_k I/I^\alpha = 1$. Using an obvious variation of this hypothesis it can also be shown that separating ideals of valuations are simple:

Proposition 5. *Let (A,m,k) be a local ring, let v,w be valuations centered at m. If $\dim\ I/I^v = 1$ for all v-ideals properly containing $\langle v,w\rangle$ then $\langle v,w\rangle$ is simple.*

Proof. Without loss of generality one may assume that v and w induce the trivial valuation on k. If $\langle v,w\rangle$ is not simple then it can be written as $\langle v,w\rangle = IJ$ where $I,J \subsetneq m$ are both v-ideals and w-ideals. Note that the hypothesis applies to I and J. It is clear that

$$v(I)+v(J)=v(\langle v,w\rangle),\quad w(I)+w(J)=w(\langle v,w\rangle).$$

Since $\langle v,w\rangle^v$ is not a w-ideal or $\langle v,w\rangle^w$ is not a v-ideal the symmetric difference of the two ideals is not empty. Without loss of generality, assume that there exists $c\in\langle v,w\rangle^v\setminus\langle v,w\rangle^w$. Then $c\in\langle v,w\rangle = IJ$ and $w(c)=w(\langle v,w\rangle)$. There are $x_1,\dots,x_k\in I$, $y_1,\dots,y_k\in J$ such that $c=x_1y_1+\dots+x_ky_k$. Suppose that the enumeration is such that

$$\begin{aligned} w(x_1) &= \dots = w(x_l) = w(I),\\ w(y_1) &= \dots = w(y_l) = w(J) \end{aligned}$$

and, for every $i=l+1,\dots,k$, $w(x_i)>w(I)$ or $w(y_i)>w(J)$. The same relations hold if w is replaced by v. Since $w(c)=w(IJ)$ we must have $l\geq 1$. By hypothesis, $\dim\ I/I^v = 1 = \dim\ J/J^v$. Hence there are $p_2,\dots,p_l,q_2,\dots,q_l\in A^*$ such that

$$\begin{aligned} x_2 &\equiv p_2x_1,\quad \dots,\quad x_l\equiv p_lx_1 \ mod\ I^v=I^w,\\ y_2 &\equiv q_2y_1,\quad \dots,\quad y_1\equiv q_ly_1 \ mod\ J^v=J^w. \end{aligned}$$

This yields

$$c = (1 + p_2q_2 + \ldots + p_lq_l)x_1y_1 + z$$

with $v(z) > v(\langle v, w\rangle)$ and $w(z) > w(\langle v, w\rangle)$. Now $w(1 + p_2q_2 + \ldots + p_lq_l) = 0$ implies that also $v(1 + p_2q_2 + \ldots + p_lq_l) = 0$. But then

$$v(c) = v(x_1) + v(y_1) = v(\langle v, w\rangle),$$

a contradiction.

∎

It was mentioned before that for $\alpha, \beta \in Sper(A)$ the separating ideal of the corresponding valuations is contained in the separating ideal of the orderings. Assuming the hypotheses of Proposition 5 we take a closer look at this relationship. To do so define

$$\begin{aligned} A^+ &= \{a \in A \mid sign\ a(\alpha) = sign\ a(\beta)\}, \\ A^- &= \{a \in A \mid sign\ a(\alpha) \neq sign\ a(\beta)\}. \end{aligned}$$

Thus, $\langle\alpha, \beta\rangle$ is the ideal of A generated by A^-. If Γ_α is the value group of v_α (including ∞) then we set

$$\Gamma'_\alpha = v_\alpha(A),\ \Gamma^+_\alpha = v_\alpha(A^+),\ \Gamma^-_\alpha = v_\alpha(A^-).$$

If ∞ is deleted then Γ'_α is a submonoid of Γ_α. By the definition of v_α, Γ'_α is contained in the positive cone of Γ_α. Now we associate a sign with each element of Γ'_α. For $\gamma \in \Gamma'_\alpha$ set

$$\sigma(\gamma) = \begin{cases} -1 & \text{if } \gamma \in \Gamma^-_\alpha \setminus \Gamma^+_\alpha \\ +1 & \text{if } \gamma \in \Gamma^+_\alpha \setminus \Gamma^-_\alpha \\ 0 & \text{if } \gamma \in \Gamma^-_\alpha \cap \Gamma^+_\alpha \text{ or } \gamma = \infty. \end{cases}$$

Lemma 5. *If $\langle\alpha, \beta\rangle \subseteq m$ then $\gamma = v_\alpha(\langle v_\alpha, v_\beta\rangle)$ is the smallest element of Γ'_α with sign 0.*

Proof. Suppose that $\sigma(\delta) = 0$ and that $\delta < \gamma$. Then there are $a \in A^+, b \in A^-$ (say, $a(\alpha) > 0, b(\alpha) > 0, a(\beta) > 0, b(\beta) < 0$) with $v_\alpha(a) = v_\alpha(b) = \delta$. The α–ideal

$$I = \{x \in A \mid v_\alpha(x) \geq \delta\}$$

properly contains $\langle v_\alpha, v_\beta \rangle$, hence dim $I/I^\alpha = 1$ and there is some $c \in A^*$ such that $v_\alpha(a+bc) > \delta$. This implies $c(\alpha) < 0$ and (since α and β define the same total order on k) $c(\beta) < 0$. We conclude that $v_\beta(a) \geq v_\beta(a+bc)$ and hence, by Proposition 4, $a \in \langle v_\alpha, v_\beta \rangle$, a contradiction. It remains to show that $\sigma(\gamma) = 0$. If $\gamma = \infty$ the claim is clearly true. So assume that $\gamma < \infty$. Pick $a \in M$ (the generating set of $\langle v_\alpha, v_\beta \rangle$ discussed in Proposition 4) with $v_\alpha(a) = \gamma$. First suppose that there is some $b \in A$ with

$$v_\alpha(a) < v_\alpha(b), \ v_\beta(a) \geq v_\beta(b).$$

We may assume that $a(\alpha) > 0$. A β–ideal is defined by

$$I = \{x \in A \mid v_\beta(x) \geq v_\beta(b)\}.$$

As the total order induced by β on I/I^β is archimedean over k there is some $c \in A^*$ such that $(a + cb)(\beta) < 0$. Since $(a + cb)(\alpha) > 0$ it follows that $a + cb \in A^-$. Similarly, there is some $d \in A^*$ such that $(a + db)(\beta) > 0$. Since $(a + db)(\alpha) > 0$ we see that $a + db \in A^+$. Altogether this proves $\sigma(\gamma) = 0$. Finally suppose that there is some $b \in A$ such that

$$v_\alpha(a) \geq v_\alpha(b), v_\beta(a) < v_\beta(b).$$

Interchanging the roles of α and β in the foregoing discussion one proves $\sigma(\gamma) = 0$.

∎

The well–ordered monoid Γ'_α has a unique minimal set Γ''_α of generators. To determine the sign of every $\gamma < v_\alpha(\langle v_\alpha, v_\beta \rangle)$ it is only necessary to know the signs of the elements of

$$\Gamma^{(3)}_\alpha = \{\delta \in \Gamma''_\alpha \mid \delta < v_\alpha(\langle v_\alpha, v_\beta \rangle)\}.$$

In particular, if $\langle v_\alpha, v_\beta \rangle \subset \langle \alpha, \beta \rangle$ then $v_\alpha(\langle \alpha, \beta \rangle) \in \Gamma^{(3)}_\alpha$. (This strengthens the result of [1], that $\langle \alpha, \beta \rangle$ has to be a simple ideal.) So, whenever $\Gamma^{(3)}_\alpha$ is finite (this is the case, for example, if v_α is discrete of finite rank) and dim $I/I^\alpha = 1$ for every α–ideal and β ranges in $Sper(A)$ then there are only finitely many possiblities for $\langle \alpha, \beta \rangle$ with $\langle \alpha, \beta \rangle \supset \langle v_\alpha, v_\beta \rangle$.

3 Quadratic transformations

One of the most important tools in the investigation of complete ideals in regular local domains of dimension 2 are quadratic transformations (cf. [13]; [14], Appendix 5; [5]). Since we also wish to employ this technique we will have to clarify the behavior of separating ideals under quadratic transformations. The setup is as follows: Let (A, m, k) be a regular local ring of dimension n. Let K be the quotient field of A. A quadratic transformation of A is a regular local subring A' of K dominating A which is obtained in the following way: A' is the localization of an extension $A[x^{-1}m]$ of A (where $x \in m\backslash m^2$) at a prime ideal restricting to m in A. The properties of the morphism $Spec(A') \to Spec(A)$ of schemes corresponding to the canonical inclusion $A \to A'$ are well-known. The properties of the functorial map $Sper(A') \to Sper(A)$ are discussed in [1]. Working in the same vein we will look at the extension of valuations from A to A' now.

Let v be valuation of A with center m and support p. First suppose that $p = m$, i.e., v is essentially the trivial valuation of the residue field k. For every $x \in m\backslash m^2$ the prime ideal $mA[x^{-1}m] = xA[x^{-1}m] \subsetneq A[x^{-1}m]$ has dimension $n - 1$. Any valuation of $A[x^{-1}m]$ whose support contains $xA[x^{-1}m]$ is an extension of v. So, v is always extendible to a valuation of a quadratic transformation $A' = A[x^{-1}m]_q$, where $q \subsetneq A[x^{-1}m]$ is a prime ideal containing x. In general there are many different such extensions. Now suppose that $supp(v) \subset m$. If $x \in supp(v)$ then for any extension w of v to a valuation of $A[x^{-1}m]$ the support contains $xA[x^{-1}m] = mA[x^{-1}m]$, i.e., $supp(w) \cap A = m$. But this is impossible. So v is extendible only if $x \notin supp(v)$. If this condition holds then $A' = A[x^{-1}m]_q$ (with $q \cap A = m$) is contained in $A_{supp(v)}$. Since v extends uniquely to a valuation of $A_{supp(v)}$ it is clear that v extends uniquely to A'. This extension of v is called the *transform* of v and is denoted by v'. Similarly, if $\alpha \in Sper(A)$ is centered at m and if $x \notin supp(\alpha)$ then the unique extension of α to an ordering of A' is denoted by α' and is called the *transform* of α. Finally, for an ideal I the strict transform is denoted by I'. In [1], §3, the notion of a quadratic transformation of A along a valuation or along an ordering is explained. It is shown there (loc. cit., Lemma 3.1) that $\langle \alpha', \beta' \rangle \supseteq \langle \alpha, \beta \rangle'$ if α is centered at m, $supp(\alpha) \subset m$ and $\langle \alpha, \beta \rangle \subset m$. We shall deal exclusively with quadratic transformations along a valuation or an ordering. If $\alpha \in Sper(A)$ and

A' is the quadratic transformation of A along α then it is clear that $v_{\alpha'} = (v_\alpha)'$. Similar to [1], Lemma 3.1, we prove

Proposition 6. *Let v, w be valuations of A, centered at m, $\langle v, w\rangle \subset m$. Let A' be the quadratic transform of A along v. Then the transforms v' and w' are both defined and $\langle v', w'\rangle \supseteq \langle v, w\rangle'$.*

Proof. The ring A' is obtained as a localization $A[x^{-1}m]_q$ where $v(x) = v(m)$, i.e., $x \in m\backslash m^v$. Since we are assuming $\langle v, w\rangle \subset m$ it follows that $\langle v, w\rangle \subseteq m^w = m^v$. Therefore $supp(w) \neq m$ and $x \notin supp(w)$, and this shows that both v' and w' are defined. Let r be the order of $\langle v, w\rangle$ (cf. [14], p. 362). Then

$$\langle v, w\rangle' = x^{-r}\langle v, w\rangle A',$$

and this ideal is generated by the set $\{\frac{a}{x^r} \mid a \in M\}$ (with M as in Proposition 4). It suffices to show that every $\frac{a}{x^r}$, $a \in M$, belongs to $\langle v', w'\rangle$. For example, suppose that there is some $b \in A$ with $v(a) < v(b), w(a) \geq w(b)$. Since $A' \subseteq A_{supp(v)}$ and $A' \subseteq A_{supp(w)}$ it follows immediately that

$$v'\left(\frac{a}{x^r}\right) < v'\left(\frac{b}{x^r}\right), \quad w'\left(\frac{a}{x^r}\right) \geq w'\left(\frac{b}{x^r}\right),$$

i.e., $\frac{a}{x^r} \in \langle v', w'\rangle$.

■

In [1], Example 3d, it is shown that in the case of orderings the containment $\langle \alpha', \beta'\rangle \supseteq \langle \alpha', \beta'\rangle$ can be proper for 3–dimensional rings. Similar examples can be constructed to show the same phenomenon for separating ideals of valuations.

4 Two-dimensional regular local rings

Now we leave the general discussion of separating ideals and turn to the case that (A, m, k) is a regular local ring of dimension 2. Since we are aiming particularly at a geometric understanding of separating ideals of orderings on a smooth real algebraic surface defined over a real closed field we also assume that the residue field k is real closed. The theory of complete ideals is very well developed in this situation; this allows us to obtain more specific results about separating ideals than in the preceding sections.

Usually the investigation of complete ideals in a regular local ring is concerned mostly with m–primary ideals. We will first see that these are also the most interesting ideals from the point of view of separating ideals. So, pick $\alpha, \beta \in Sper(A)$, $\alpha \neq \beta$, having a common center $p \subset A$. If $height(p) \leq 1$ then we can localize at p to obtain a regular local ring of dimension at most 1. Since separating ideals are easy to handle in such a setting we will not concern ourselves with this situation here. So we assume that the center of α and β is m. If $\langle \alpha, \beta \rangle = (0)$ then $\alpha = \beta$ contrary to the choice of α and β. If $height(\langle \alpha, \beta \rangle) = 1$ then $q = \sqrt{\langle \alpha, \beta \rangle}$ is a prime ideal of height one. Again we can localize at q to get into a one–dimensional situation. Therefore we shall now assume that $\langle \alpha, \beta \rangle$ is an m–primary ideal. The additional assumption at $supp(\alpha) \subset m, supp(\beta) \subset m$ serves to exclude trivial cases.

The radical r of $\langle v_\alpha, v_\beta \rangle$ is a prime ideal (Lemma 3). Whenever there is a prime v_α–.or v_β–ideal q of height 1 containing r then $\alpha + q$ and $\beta + q$ are specializations of α and β and (since $\langle \alpha, \beta \rangle$ is m–primary) $\langle \alpha, \beta \rangle = \langle \alpha + q, \beta + q \rangle$. Therefore, to study the separating ideal $\langle \alpha, \beta \rangle$ is essentially the same problem as to study the separating ideal of the orderings $\bar{\alpha}$ and $\bar{\beta}$ induced in A/q by α and β. With A/q we are in a 1–dimensional situation again, although in this case the ring A/q is not necessarily regular. However, the investigation of 1–dimensional rings is not our main concern here. So we only mention that related situations were considered in [10] and that the separating ideal in A/q can be analyzed by looking at the integral closure $\overline{A/q}$ in $qf(A/q)$ which is a Dedekind domain.

We are left with two cases now: Either $r = (0)$ and (0) and m are the only prime v_α–and v_β–ideals, or $r = m$. In the first case we have $v_\alpha = v_\beta$, and α and β are total orders of A. Moreover, by [1], Lemma 4.5, dim $I/I^\alpha = 1$ for every α–ideal $I \supset \langle \alpha, \beta \rangle$, hence $\langle \alpha, \beta \rangle$ is simple. In the second case we show that both $\langle \alpha, \beta \rangle$ and $\langle v_\alpha, v_\beta \rangle$ are simple. First note that [1], Lemma 4.5, is applicable, hence $\langle \alpha, \beta \rangle$ is simple. For $\langle v_\alpha, v_\beta \rangle$, consider the sequences $p_1 = m \supset p_2 \supset \ldots$ of simple v_α–ideals and $q_1 = m \supset q_2 \supset \ldots$ of simple v_β–ideals. Since the sequences of v_α–ideals and v_β–ideals diverge after finitely many steps it follows from Zariski's theory of complete ideals (cf. [14], Appendix 5) that there must be some n such that p_n is not a v_β–ideal or q_n is not a v_α–ideal. (Strictly speaking, to be able to apply Zariski's theory we need to work with valuations of the field $qf(A)$ and not only with valuations of A. However, if, for example, v_α is not a valuation of $qf(A)$, i.e., if $supp(v_\alpha) \neq (0)$ then one can replace v_α by the valuation obtained as the composition v of the valuation belonging to the discrete valuation ring $A_{supp(v_\alpha)}$ and the valuation of $qf(A/supp(v_\alpha))$ defining v_α. The separating ideal of v and v_α is $supp(v_\alpha)$. If we are only interested in a finite initial part of the sequence of v_α we can therefore work equally well with v as with v_α.) So $\langle v_\alpha, v_\beta \rangle$ properly contains a simple v_α–ideal or a simple v_β–ideal. But if I is any v_α–ideal (or v_β–ideal) properly containing a simple v_α–ideal (or v_β–ideal) then $dim\ I/I^\alpha = 1$ (or $dim\ I/I^\beta = 1$).

We have shown so far that $\langle \alpha, \beta \rangle$ is always simple and that $\langle v_\alpha, v_\beta \rangle = 0$ or $\langle v_\alpha, v_\beta \rangle$ is simple. For the further analysis of the separating ideals we will use (iterated) quadratic transformations. As we saw in §3, we always have

$$\begin{array}{rcl} \langle \alpha, \beta \rangle' & \subseteq & \langle \alpha', \beta' \rangle, \\ \langle v_\alpha, v_\beta \rangle' & \subseteq & \langle v'_\alpha, v'_\beta \rangle. \end{array}$$

In the present setting this can be improved. It is shown in [1], Proposition 4.7, that $\langle \alpha, \beta \rangle' = \langle \alpha', \beta' \rangle$ if $\langle \alpha, \beta \rangle \subset m$. For valuations the corresponding result is almost immediate from [14], p. 390 ff. For, if $\langle v_\alpha, v_\beta \rangle = (0)$ then it is clear that $\langle v'_\alpha, v'_\beta \rangle = (0) = \langle v_\alpha, v_\beta \rangle'$, and if $\langle v_\alpha, v_\beta \rangle$ is m–primary then $\langle v_\alpha, v_\beta \rangle'$ is a simple v'_α– and v'_β–ideal, hence the sequence of v'_α–ideals preceding $\langle v_\alpha, v_\beta \rangle'$ and the sequence of v'_β–ideals preceding $\langle v_\alpha, v_\beta \rangle'$ both agree. But then $\langle v'_\alpha, v'_\beta \rangle \subseteq \langle v_\alpha, v_\beta \rangle'$ which proves the transformation formula for separating ideals of valuations. (It is clear that the same argument works for any two valuations v, w of A such that $\langle v, w \rangle$ is simple and m–primary.)

If $A \subsetneq A' \subsetneq A'' \subsetneq \ldots$ is the sequence of iterated quadratic transformations of A along α then we denote the r–th iterated transforms of $\alpha, \beta, \ldots$ by $\alpha^{(r)}, \beta^{(r)}, \ldots$. Suppose that r is minimal such that

$$\langle \alpha, \beta \rangle^{(r)} = m^{(r)} \subsetneq A^{(r)}.$$

Using the terminology of tangent directions introduced in §1, this means that $A^{(r)}$ is the first quadratic transform in which the tangent directions of $\alpha^{(r)}$ and $\beta^{(r)}$ are separated (Theorem 2). Now there are two possible cases: Either $\langle \alpha, \beta \rangle^{(r)} = \langle v_\alpha, v_\beta \rangle^{(r)}$ (which means that even the tangents have been separated – cf. Theorem 1) or $\langle\alpha,\beta\rangle^{(r)} \supset \langle v_\alpha, v_\beta\rangle^{(r)}$ (which means that the tangents of $\alpha^{(r)}$ and $\beta^{(r)}$ agree, but $\alpha^{(r)}$ and $\beta^{(r)}$ approach the common center from different directions along the tangent). This proves

Theorem 3. *If $A^{(r)}$ is the first quadratic transform of A along v_α with $\langle\alpha^{(r)},\beta^{(r)}\rangle = m^{(r)}$ then $\alpha^{(r)}$ and $\beta^{(r)}$ have different tangents or different tangent directions along the same tangent according as $\langle\alpha,\beta\rangle = \langle v_\alpha, v_\beta\rangle$ or $\langle\alpha,\beta\rangle \supset \langle v_\alpha, v_\beta\rangle$.*

■

The paper concludes with a couple of remarks.

Remark 1. Suppose that $\alpha \in Sper(A)$ is given and that Γ''_α is defined as at the end of section 2. Let I be any simple m–primary α–ideal and let J be any simple m–primary α–ideal of the form

$$J = \{a \in A \mid v_\alpha(a) \geq \gamma\}$$

where $0 < \gamma \in \Gamma''_\alpha$. Then using the technique of quadratic transformations it is possible to show that there exist some $\eta, \vartheta \in Sper(A)$ such that $I = \langle\alpha,\eta\rangle = \langle v_\alpha, v_\eta\rangle$ and $J = \langle\alpha,\vartheta\rangle \supset \langle v_\alpha, v_\vartheta\rangle$.

■

Remark 2. There is yet another way to distinguish the two ways how the iterated transforms $\alpha^{(r)}$ and $\beta^{(r)}$ are separated. It involves the notion of minimal reductions (cf. [11]). Every m–primary ideal $I \subsetneq A$ has many minimal reductions, each of which is generated by a system of parameters (i.e., by two elements which generate an m–primary ideal). I is the integral closure of each such minimal reduction. First suppose that $\langle\alpha,\beta\rangle \supset \langle v_\alpha, v_\beta\rangle$ and pick a minimal reduction (a, b) of $\langle\alpha,\beta\rangle$. As $\langle\alpha,\beta\rangle^\alpha = \langle\alpha,\beta\rangle^\beta$ and $dim\langle\alpha,\beta\rangle/\langle\alpha,\beta\rangle^\alpha = 1$ we see that one generator, say a, of the minimal reduction has value $v_\alpha(\langle\alpha,\beta\rangle)$ and, after replacing b by a linear combination $ca + b$ with $c \in A^*$ if necessary, $v_\alpha(b) > v_\alpha(\langle\alpha,\beta\rangle)$. Since $\langle v_\alpha, v_\beta\rangle \subseteq \langle\alpha,\beta\rangle^\alpha = \langle\alpha,\beta\rangle^\beta$ we also have $v_\beta(a) = v_\beta(\langle\alpha,\beta\rangle)$, $v_\beta(b) > v_\beta(\langle\alpha,\beta\rangle)$. Now suppose that $\langle\alpha,\beta\rangle = \langle v_\alpha, v_\beta\rangle$ and that (a, b) is a minimal reduction of $\langle\alpha,\beta\rangle$ again. We will show that either $v_\alpha(a) = v_\alpha(\langle\alpha,\beta\rangle)$ and $v_\beta(b) = v_\beta(\langle\alpha,\beta\rangle)$ or $v_\alpha(b) = v_\alpha(\langle\alpha,\beta\rangle)$ and $v_\beta(a) = v_\beta(\langle\alpha,\beta\rangle)$.

To prove this assume that $v_\alpha(b) > v_\alpha(\langle\alpha,\beta\rangle)$ and $v_\beta(b) > v_\beta(\langle\alpha,\beta\rangle)$. It is easy to show that if $s = ord(\langle\alpha,\beta\rangle)$ the transform $\langle\alpha,\beta\rangle'$ under a quadratic transformation $A \subsetneqq A' = A[x^{-1}m]_q$ has $\left(\frac{a}{x^s}, \frac{b}{x^s}\right)$ as a minimal reduction. In particular $v_\alpha\left(\frac{a}{x^s}\right) < v_\alpha\left(\frac{b}{x^s}\right)$ and $v_\beta\left(\frac{a}{x^s}\right) < v_\beta\left(\frac{b}{x^s}\right)$. Now apply quadratic transformations along α to transform $\langle\alpha,\beta\rangle$ into $m^{(r)}$. Then (a,b) is transformed into a minimal reduction $(a^{(r)}, b^{(r)})$ of $m^{(r)}$ with $v_\alpha(a^{(r)}) < v_\alpha(b^{(r)}), v_\beta(b^{(r)}) < v_\beta(b^{(r)})$. But $m^{(r)}$ is basic (i.e., it has no proper reductions), hence $m^{(r)} = (a^{(r)}, b^{(r)})$. It is obvious that $m^{(r)2} \subsetneqq m^{(r)\alpha} \cap m^{(r)\beta}$. Moreover, $b^{(r)} \in m^{(r)\alpha} \cap m^{(r)\beta}$. But then

$$\left(m^{(r)2}, b^{(r)}\right) \subseteq m^{(r)\alpha} \cap m^{(m)\beta}.$$

Since $dim\ m^{(r)}/(m^{(r)2}, b^{(r)}) = 1$ this implies that

$$m^{(r)\alpha} = \left(m^{(r)2}, b^{(r)}\right) = m^{(r)\beta}.$$

Since the order of this ideal is 1 it is a simple complete ideal. Also it is a v_α–ideal and a v_β–ideal. Therefore it is the transform of some simple v_α– and v_β–ideal $I \subsetneqq A$. Since $I^{(r)} \subset \langle\alpha,\beta\rangle^{(r)} = \langle v_\alpha, v_\beta\rangle^{(r)}$ we see that $I \subset \langle v_\alpha, v_\beta\rangle$. Thus, there is a simple v_α– and v_β–ideal in A which is properly contained in $\langle v_\alpha, v_\beta\rangle$. But then the sequences of v_α– and v_β–ideals agree up to I. This contradicts the definition of $\langle v_\alpha, v_\beta\rangle$, and the proof of the claim is complete.

■

Remark 3. Suppose that k is a totally ordered field. Let $A = k[X,Y]$ be the polynomial ring in two variables, K the quotient field of A. Let α be a total order of A having center $p = (X,Y)$. In $Sper(A)$ the closed specialization of α is denoted by β. Refering to α, suppose that $0 < Y < X$. Set $Y' = \frac{Y}{X}$ and consider $A' = A[Y']$. In $Sper(A')$ there is a unique point α' restricting to α in $Sper(A)$. Let β' be the closed specialization of α' in $Sper(A')$. Under suitable hypotheses a directional form was associated with the valuation v_α in [14], p. 364, Definition 1. The purpose of this remark is to point out a connection between this directional form and our tangent directions introduced in section 1.

The exceptional divisor of the quadratic transformation $A \to A'$ is represented by the factor ring A'/XA'. This ring is isomorphic to the polynomial ring $B = k[Y']$. In K let V_α be the valuation ring of v_α, let M_α be the maximal ideal. There is a canonical homomorphism $B \to V_\alpha/M_\alpha$. If this is injective then V_α is the valuation ring of the order valuation of A belonging to the maximal ideal p. In this case there is no directional form, so there is nothing to do.

Now suppose that $B \to V_\alpha/M_\alpha$ is not injective. The kernel is generated by some irreducible polynomial $\overline{f}$ in the variable Y'. This is essentially the directional form of [14]. In A let L be the k-vector space of linear forms. The total order α of A restricts to a total order of L. The Zariski cotangent space p/p^2 can be identified canonically with L, the Zariski tangent space $(p/p^2)^\vee$ can be identified with k^2. Given $0 \leq l_1, \ldots, l_r \in L$, let

$$P(l_1, \ldots, l_r) = \{x \in k^2 \mid \forall i : l_i(x) \geq 0\}.$$

The set of all these cones in k^2 defines a tangent direction according to section 1. Each one of these cones corresponds to a closed interval on the exceptional divisor of the quadratic transformation. The set of these intervals is the basis of a filter on k. The following conditions are equivalent:

(1) The directional form $\overline{f}$ is linear.

(2) The totally ordered vector space L is not archimedean over k.

(3) β' defines a k-rational point on the exceptional divisor.

(4) The intersection of the intervals contains a k-rational point.

If these conditions hold then the unique nontrivial convex subspace of L is generated by a linear form f with proper transform $f' \in A'$ such that the residue of f' in B is the directional form. The k-rational point of (4) is unique and is the same as the k-rational point of (3) and the same as the point defined by the linear directional form in (1).

For the discussion of the general case let R be the real closure of k. The total order of V_α/M_α restricts to a total order of $B/(\overline{f})$. Since $B/(\overline{f})$ is an algebraic extension of k there is a unique embedding into R. Let $z \in R$ be the image of $Y' + (\overline{f})$. Then, identifying the R-rational points on the exceptional divisor with R, z belongs to the intersection of the system of intervals determined above. Clearly, z is one of the roots of the directional form. The total order of $B/(\overline{f})$ gives enough information to distinguish z among the different roots of $\overline{f}$. Thus, it is legitimate to consider z as the R–rational point on the exceptional divisor determining the tangent for the total order α. Of course, the total order of L contains less information than the total order of $B/(\overline{f})$. Therefore, the intersection of the intervals will, in general, contain more roots of $\overline{f}$ than just z. So the directional form offers a choice of tangents for α. With the information coming from the total order of L the field of candidates is narrowed down, but not necessarily down to 1. If there is only one candidate left then, of course, this is the true tangent.

Finally, to mention one particularly important special case, suppose that k is real closed, i.e., $k = R$. Then z is a k–rational point and the equivalent conditions (1) – (4) apply.

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