

A failure of quantifier elimination.

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Abstract

We show that \log is needed to eliminate quantifiers in the theory of the real numbers with restricted analytic functions and exponentiation.

We let \mathcal{L}_{an} be the first order language of ordered rings augmented by function symbols \widehat{f} where f is an analytic function defined on an open $U \supset [0, 1]^n$ for some n . We interpret \widehat{f} as a function on \mathbf{R}^n by

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in [0, 1]^n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ be the language obtained by adding to \mathcal{L}_{an} unary function symbols f_r for each $r \in \mathbf{R}$. We interpret f_r as the function

$$f_r(x) = \begin{cases} x^r & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and denote $f_r(x)$ by x^r . Finally we let $\mathcal{L}_{\text{an,exp}}$ be the language $\mathcal{L}_{\text{an}} \cup \{\text{exp}\}$ and $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}} = \mathcal{L}_{\text{an}}^{\mathbf{R}} \cup \{\text{exp}\}$.

In [2] we showed that the $\mathcal{L}_{\text{an,exp}}$ -theory of \mathbf{R} admits quantifier elimination in the language $\mathcal{L}_{\text{an,exp}} \cup \{\log\}$. Indeed, we remark there that exp is unnecessary as we could actually eliminate quantifiers in the language

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$\mathcal{L}_{\text{an}} \cup \{\log\} \cup \{x^q : q \in \mathbf{Q}\}$. Here we show that although \exp and \log are interdefinable, \log is essential for quantifier elimination.

Theorem. *Let $\phi(x, y)$ be the formula*

$$\exists z (\exp(\exp z) = x \wedge y = z \exp z).$$

Then $\phi(x, y)$ is not equivalent to a quantifier free $\mathcal{L}_{\text{an}, \exp}^{\mathbf{R}}$ -formula.

Of course $\phi(x, y)$ is equivalent to the quantifier free $\mathcal{L}_{\text{an}} \cup \{\log\}$ -formula

$$x > 1 \wedge y = (\log x)(\log \log x).$$

There are several previous “failure of quantifier elimination” theorems for the reals with exponentiation. Osgood’s example

$$y > 0 \wedge \exists w (wy = x \wedge z = ye^w)$$

is not equivalent to a quantifier free formula in the language $\{+, -, \cdot, <, 0, 1, \exp\}$ (or any expansion by total real analytic functions (see for example [1])), while, in unpublished work, van den Dries and Macintyre showed that

$$\exists z (z^2 = x \wedge y = e^z)$$

is not equivalent to a quantifier free formula in the language $\{+, -, \cdot, \frac{1}{x}, \exp, <, 0, 1\}$. Both of these formulas are equivalent to a quantifier free $\mathcal{L}_{\text{an}, \exp}^{\mathbf{R}}$ -formulas.

In [4] Gabrielov gives several “failure of quantifier elimination” results of a different spirit.

The most interesting open question of this kind is whether the theory of $(\mathbf{R}, +, \cdot, \exp)$ admits quantifier elimination in either the language $\mathcal{L} = \{+, \cdot, -, <, 0, 1\} \cup \{\exp, \log\}$ or \mathcal{L} augmented by all semialgebraic functions. It seems that to eliminate quantifiers one needs to add some implicitly defined restricted analytic functions, so we expect both of these questions to have a negative answer.

Let $f(x) = (\log x)(\log \log x)$ and let Γ be the graph of f . We say that an open set $U \subseteq \mathbf{R}^2$ contains a *tail* of Γ if $(x, f(x)) \in U$ for all sufficiently large x .

Let $\phi(x, y)$ be the above formula. Suppose for purposes of contradiction that ϕ is equivalent to a quantifier free $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -formula, say

$$\phi(x, y) \Leftrightarrow \bigvee_{i=1}^r (F_i(x, y) = 0 \wedge \bigwedge_{j=1}^s G_{i,j}(x, y) > 0)$$

for some $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -terms $F_i, G_{i,j}$. Let

$$Y_i = \{(x, y) : F_i(x, y) = 0 \wedge \bigwedge_{j=1}^s G_{i,j}(x, y) > 0\}.$$

By o-minimality there is an i such that $(x, y) \in Y_i$ if and only if $y = f(x)$ for sufficiently large x . Fix such an i .

Let $W_0 = \{(x, y) : F_i(x, y) = 0\}$ and let $W_j = \{(x, y) : G_{i,j}(x, y) > 0\}$ for $j = 1, \dots, s$. Each W_j contains a tail of Γ . Suppose that for each i there is an M_i such that $\{(x, y) \in \Gamma : x > M_i\}$ is in the interior of W_i . Then $\{(x, y) \in \Gamma : x > \max M_i\}$ is in the interior of Y_i , a contradiction. Thus a tail of Γ must be in the boundary of at least one of the W_j .

Thus we have shown that there is an $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -term F such that a tail of Γ is in either the boundary of $\{(x, y) : F(x, y) = 0\}$ or the boundary of $\{(x, y) : F(x, y) > 0\}$. Unfortunately, since our terms need not be continuous, we must consider both possibilities. The next lemma shows that we can in fact choose F such that the first possibility holds and F is analytic on a neighborhood of a tail of Γ .

Lemma 1. *Let $f(x) = (\log x)(\log \log x)$. There is an $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -term $F(x, y)$ which is analytic on an open $U \subseteq \mathbf{R}^2$ containing a tail of Γ such that $F(x, f(x)) = 0$ for sufficiently large x , and for all x there are at most finitely many y such that $(x, y) \in U$ and $F(x, y) = 0$. Moreover, we can choose F such that all of its subterms are analytic on U .*

Proof. We know there is an $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -term $F(x, y)$ with the following property:

(*) There is an open $U \subseteq \mathbf{R}^2$ containing a tail of Γ such that Γ is in the boundary of either

- a) $\{(x, y) \in U : F(x, y) = 0\}$ or
- b) $\{(x, y) \in U : F(x, y) > 0\}$.

We may, by induction on terms, assume that if any nonconstant subterm of F is replaced by the constant term 0 or 1, then the resulting term does not have property (*).

We next try to find an open $V \subseteq U$ containing a tail of Γ such that F and all of its subterms are analytic on V . We try to prove this by induction on subterms of F . We will see that the only obstructions to this induction will lead to a new term F_1 with property (*) such that F_1 and all of its subterms are analytic on an open set containing a tail of Γ .

- If a subterm t of F is a constant or variable, it is analytic on all of U .

- Suppose t_0 and t_1 are subterms of F and t_i is analytic on V_i where V_i is an open subset of U containing a tail of Γ . Then $V = V_0 \cap V_1$ contains a tail of Γ and $t_0 \pm t_1$, $t_0 t_1$ and $\exp(t_i)$ are analytic on V .

- Suppose t_1, \dots, t_n and $h = \widehat{g}(t_1, \dots, t_n)$ are subterms of F , where \widehat{g} is the function symbol for a restricted analytic function and t_1, \dots, t_n are analytic on an open set U_i containing a tail of Γ . Using the \mathfrak{o} -minimality of $\mathbf{R}_{\text{an,exp}}$ one of the following holds for each i .

Case 1. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, y) \in (-\infty, 0] \cup (1, +\infty)$ for all $(x, y) \in V_i$.

Case 2. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, y) = 1$ for all $(x, y) \in V_i$.

Case 3. There is an open $V_i \subseteq U_i$ containing a tail of Γ such that $0 < t_i(x, y) < 1$ for all $(x, y) \in V_i$.

If we are not in cases 1)-3) then $t_i(x, y)$ must be equal to 0 or 1 on a tail of Γ . Since $t_i(x, y)$ is analytic on an open neighborhood of a tail of Γ , we must be in one of the following two cases.

Case 4. There is an open set $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, f(x)) = 0$ but $\{y : (x, y) \in V_i \wedge t_i(x, y) = 0\}$ is finite for sufficiently large x .

Case 5. There is an open set $V_i \subseteq U_i$ containing a tail of Γ such that $t_i(x, f(x)) = 1$ but $\{y : (x, y) \in V_i \wedge t_i(x, y) = 1\}$ is finite for sufficiently large x .

Cases 4) or 5) are the cases where our induction breaks down. In case 4) we replace F by $t_i(x, y)$. Then $t_i(x, y)$ satisfies (*) and t_i and all of its subterms are analytic on V_i . In case 5) we replace F by $t_i(x, y) - 1$. In either case the new term has the desired property.

In case 1)

$$\widehat{g}(t_1, \dots, t_n) = \widehat{g}(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$$

for all $(x, y) \in V_i$. Thus we could replace this occurrence of t_i by 0 to obtain a new term F^* such that $F^* = F$ on an open set containing a tail of Γ . This contradicts our assumptions on F . Similarly in case 2) we can replace this occurrence of t_i by 1 contradicting our assumptions on F .

Thus we may assume we are in case iii). Let $V = \bigcap_{i=1}^n V_i$. Then $(t_1(x, y), \dots, t_n(x, y)) \in (0, 1)^n$ for all $(x, y) \in V$ and h is analytic on V .

• Suppose h and t are subterms of F , $h = t^r$ and t is analytic on an open set U containing a tail of Γ . As above, we can find an open set $V \subseteq U$ containing a tail of Γ such that one of the following holds:

Case 1. $t(x, y) \leq 0$ for all $(x, y) \in V$,

Case 2. $t(x, f(x)) = 0$ and $\{y : (x, y) \in V \wedge t(x, y) = 0\}$ for sufficiently large x , or

Case 3. $t(x, y) > 0$ for $(x, y) \in V$.

As above case 1) can not happen as we could simplify F by replacing h by 0. In case 2) we can use t instead of F and we are done. Thus we may assume that we are in case 3) and note that h is analytic on V .

This completes the induction. Either we will find a simpler term satisfying the conditions of the theorem or we will eventually thin U to an open V containing a tail of Γ such that F is analytic on V . In the later case, since F is analytic on V , $\{(x, y) \in V : F(x, y) > 0\}$ is open. Thus we must be in case a) of (*) and F is the desired term.

Let $F(x, y)$ be the term guaranteed by lemma 1. Note that since F and all of its subterms are analytic on U , one can show by induction that all of the partial derivatives of F are equal to $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -terms on U .

Let $p \in \Gamma \cap U$. By repeated application of the Weierstrass division theorem we can find an open neighborhood V of p , $n \in \mathbf{N}$ and an analytic function g on V such that on V

$$F(x, y) = (y - f(x))^n g(x, y)$$

and there is no point $(x, y) \in V \setminus \{p\}$ such that $y = f(x)$ and $g(x, y) = 0$. Note that for each $m \leq n$ there is an analytic h_m on V such that

$$\frac{\partial^m F}{\partial y^m}(x, y) = \frac{n!}{(n - m)!} (y - f(x))^{n-m} (g(x, y) + (y - f(x))h_m(x, y)).$$

Let G be an $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -term such that $G = \frac{\partial^{n-1}F}{\partial y^{n-1}}$ on U . Then G vanishes identically on $\Gamma \cap V$ and $\frac{\partial G}{\partial y}$ does not vanish on $\Gamma \cap V \setminus \{p\}$. By analytic continuation and o-minimality

$$G(x, f(x)) = 0$$

and

$$\frac{\partial G}{\partial y}(x, f(x)) \neq 0$$

for sufficiently large x .

Since (e^{e^z}, ze^z) parameterizes the curve $y = f(x)$, $G(e^{e^z}, ze^z) = 0$ for sufficiently large z . Differentiating with respect to z we see that

$$0 = e^z e^{e^z} \frac{\partial G}{\partial x}(e^{e^z}, ze^z) + (z+1)e^z \frac{\partial G}{\partial y}(e^{e^z}, ze^z)$$

and

$$z = -e^{e^z} \frac{\frac{\partial G}{\partial x}(e^{e^z}, ze^z)}{\frac{\partial G}{\partial y}(e^{e^z}, ze^z)} - 1 \tag{1}$$

for sufficiently large z .

Suppose \mathcal{M} is a nonstandard model of the $\mathcal{L}_{\text{an,exp}}$ -theory of \mathbf{R} , $x \in \mathcal{M}$, and $x > \mathbf{R}$. Let N be the smallest $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -substructure of \mathcal{M} containing $\mathbf{R}(e^{e^x}, xe^x)$, i.e. N is the smallest subset of \mathcal{M} containing $\mathbf{R}(e^{e^x}, xe^x)$ and closed under $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -terms and exponentiation. In fact N is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of \mathcal{M} containing $\mathbf{R}(e^{e^x}, xe^x)$ and closed under exp . Since G and $\frac{\partial G}{\partial y}$ are $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -terms, $x \in N$. We will obtain a contradiction by showing this fails when \mathcal{M} is the logarithmic-exponential series field $\mathbf{R}((t))^{\text{LE}}$ constructed in [3].

For the remainder of the proof we assume familiarity with the notation and results from [3].

Lemma 2. *Let $x = t^{-1} \in \mathbf{R}((t))^{\text{LE}}$. Let $N \subset \mathbf{R}((t))^{\text{LE}}$ be the smallest $\mathcal{L}_{\text{an,exp}}^{\mathbf{R}}$ -substructure of $\mathbf{R}((t))^{\text{LE}}$ containing $\mathbf{R}(e^{e^x}, xe^x)$. Then $x \notin N$.*

Proof. We first note that in fact $N \subset \mathbf{R}((t))^{\text{E}}$. We build a chain $(F_\alpha : \alpha < \lambda)$ of truncation closed $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary substructures of N such that:

- i) $F_\alpha = \bigcup_{\beta < \alpha} F_\beta$ if α is a limit ordinal,

- ii) there is $y_\alpha \in F_\alpha$ such that $F_{\alpha+1}$ is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$ containing $F(e^{y_\alpha})$ for all $\alpha < \lambda$, and
- iii) $N = \bigcup_{\alpha < \lambda} F_\alpha$.

Claim. Suppose F is a truncation closed \mathcal{L}_{an} -elementary substructure of $\mathbf{R}((t))^{\mathbf{E}}$ and the value group of F is an \mathbf{R} -vector space. Then F is an $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary substructure.

If $y \in F$ and $y > 0$, then $y = at^g(1 + \epsilon)$ where $a \in \mathbf{R}$, $a > 0$, $t^g, \epsilon \in F$ and $v(\epsilon) > 0$. Then $y^r = a^r t^{rg}(1 + \epsilon)^r$. Since $z \mapsto (1 + z)^r$ is analytic near zero, $(1 + \epsilon)^r \in F$. Since the value group of F is an \mathbf{R} -vector space, $t^{rg} \in F$. Thus $y^r \in F$. By the quantifier elimination from [5], F is an $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$.

The above claim, the truncation results of §3 of [3] and the valuation theoretic results from §3 of [2] guarantee that if F is a truncation closed $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$, $y \in \mathbf{R}((t))^{\mathbf{E}}$, $v(y) \notin v(F)$ and F^* is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$ containing $F(y)$, then F^* is truncation closed and the value group of F^* is $v(F) \oplus \mathbf{R}v(y)$.

Let F_0 be the the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$ containing $\mathbf{R}(e^{e^x}, xe^x)$. By the above remarks F_0 is truncation closed. We can then build $(F_\alpha : \alpha < \lambda)$ satisfying i)-iii) above. Since

$$e^x = t^{-x} \text{ and } e^{e^x} = t^{-e^x},$$

the value group of F_0 is $\mathbf{R}(1 + x) \oplus \mathbf{R}e^x$. Clearly $\mathbf{R}(1 + x)$ is a convex subgroup of the value group of F_0 . We argue that $\mathbf{R}(1 + x)$ is a convex subgroup of the value group of F_α for all $\alpha < \lambda$. Thus $\mathbf{R}(1 + x)$ is a convex subgroup of the value group of N . In particular $x \notin N$.

In fact the value group of F_0 is of the form $\mathbf{R}(1 + x) \oplus H$ where $\text{supp } h < \mathbf{R}$ for all $h \in H$. The next claim allows us to inductively show that this is true for the value group of F_α for all α .

Claim. Let $F \subset \mathbf{R}((t))^{\mathbf{E}}$ be a truncation closed $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel with value group $G = \mathbf{R}(1 + x) \oplus H$ where $\text{supp } h < \mathbf{R}$ for all $h \in H$. Suppose $y \in F$, $e^y \notin F$ and F_1 is the smallest $\mathcal{L}_{\text{an}}^{\mathbf{R}}$ -elementary submodel of $\mathbf{R}((t))^{\mathbf{E}}$ containing $F(e^y)$. Then F_1 is truncation closed and G_1 , the value group of F_1 , is $\mathbf{R}(1 + x) \oplus H_1$ where $\text{supp } h_1 < \mathbf{R}$ for all $h_1 \in H_1$.

Let $y = \alpha + \beta$ where $\text{supp } \alpha < 0$ and $v(\beta) \geq 0$. By our assumptions on G , $\text{supp } \alpha < \mathbf{R}$. Since $e^\beta \in F$, $F(e^y) = F(e^\alpha)$ and $e^\alpha = t^{-\alpha}$. Thus the value group of F_1 is $G \oplus \mathbf{R}\alpha$. Thus $\text{supp } (r\alpha + h) < 0$ for all $h \in H$. Since $H_1 = \mathbf{R}\alpha \oplus H$, this proves the claim.

References

- [1] L. van den Dries, *Remarks on Tarski's problem concerning $(\mathbf{R}, +, \cdot, \exp)$* , *Logic Colloquium '82*, G. Lolli, G. Longo and A. Marcja eds., North-Holland 1984.
- [2] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, *Ann. Math.* 140 (1994), 183-205.
- [3] L. van den Dries, A. Macintyre, and D. Marker, *Logarithmic-exponential power series*, *J. London Math. Soc.*, to appear.
- [4] A. Gabrielov, *Counter-examples to quantifier elimination for fewnomials and exponential expressions*, preprint 1995.
- [5] C. Miller, *Expansions of the real field with power functions*, *Ann. Pure Appl. Logic* 68 (1994), 79-94.

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