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Complete decomposability of quotients by complex conjugation for real complete intersection surfaces.

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Abstract

Suppose X is a non-singular complex surface defined over $\mathbb{R}$, which is a complete intersection constructed by the method of a small perturbation, and $Y = X/\text{conj}$ is its quotient by the complex conjugation conj $X \to X$. Assume that conj has a fixed point. It is proved that Y is diffeomorphic to a connected sum $\#_n \mathbb{C}P^2 \#_m \overline{\mathbb{C}P}^2$ if $w_2(Y) \neq 0$, or $\#_n(S^2 \times S^2)$ if $w_2(Y) = 0$.

1 Introduction

We mean by a real variety (real curve, real surface etc.) a pair (X, conj), where X is a complex variety (reduced and irreducible) and conj $X \to X$ an anti-holomorphic involution called *the real structure* or *the complex conjugation.* Given an algebraic variety over $\mathbb{R}$ we consider the set of its complex points with the natural complex conjugation (the Galois transformation) as the corresponding real variety. The fixed point set of conj will be denoted by $X_{\mathbb{R}}$ and called *the real part of* X. We put $Y = X/\text{conj}$ and identify in the notation $X_{\mathbb{R}}$ with its image $q(X_{\mathbb{R}})$ under the quotient map $q\, X \to Y$.

If (X, conj) is a nonsingular real curve then the topological type of Y depends only on the genus of X, the number of components in $X_{\mathbb{R}}$ and orientability of Y; the latter depends on vanishing of the

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fundamental class $[X_{\mathbb{R}}] \in H_1(X;\mathbb{Z}/2)$. On the other hand, any connected compact surface with nonempty boundary can appear as Y for some (X, conj).

The subject of the author's interest is the topology of Y in the case of nonsingular real surfaces. In this case Y is a closed 4-manifold with the map q being a 2-fold covering branched along $X_{\mathbb{R}}$. Moreover, Y inherits from X an orientation and a smooth structure making q smooth and orientation preserving (cf. [2]). The natural question is to describe the diffeomorphism types of 4-manifolds which can arise as the quotients Y for real surfaces X. Another interesting question is if the topological types of X and $X_{\mathbb{R}}$ together with some information about the fundamental class $[X_{\mathbb{R}}] \in H_2(X;\mathbb{Z}/2)$ determine the topology of Y like in the case of curves.

In many examples when Y is simply connected (it is simply connected for example if X is simply connected and $X_{\mathbb{R}} \neq \emptyset$) one can prove that it splits into a connected sum of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ or $S^2 \times S^2$. Let us call such decomposability property CDQ-property (complete decomposability for quotients) and call a real surface CDQ-surface if it is satisfied. Note that CDQ-surfaces X must have $X_{\mathbb{R}} \neq \emptyset$ and that surfaces X with $Y \cong S^4$ are CDQ, because the empty set of copies in a connected sum is allowed. The latter diffeomorphism holds for $X = \mathbb{C}P^2$, which follows easily from Cerf's theorem (different versions of the proof can be found in [2, 7, 12]).

For quadric and cubic real surfaces X, with $X_{\mathbb{R}} \neq \emptyset$, CDQ-property was set up by M. Letizia [9]. For real $K3$ surfaces (X, conj) with $X_{\mathbb{R}} \neq \emptyset$, and therefore for quartic real surfaces, CDQ-property was noticed by S. Donaldson [3]. Further, S. Akbulut [1] analyzed the diffeomorphism type of Y for double planes branched along real curves of arbitrary even multi-degree, $2n$, whose real scheme consists of n ovals in $\mathbb{R}P^2$ ordered by inclusion; this gave examples of CDQ surfaces of general type. The method of Akbulut was further developed by the author [4], which brought new families of real CDQ double planes (in particular it gave a new proof that $K3$ surfaces are CDQ). In [5] CDQ property is set up for all real rational and Enriques surfaces with simply connected Y. However, not all real surfaces with simply connected quotient Y are CDQ. In [6] it was shown that Y can have non vanishing signature when it is Spin and simply connected; similar examples in symplectic category were also constructed by R. Gompf (personal communication).

In the present paper we show that CDQ-surfaces can be found among complete intersections in $\mathbb{C}P^{n+2}$ of arbitrary multi-degrees $(d_1, \dots, d_n)$. Note that existence of such examples was not known already for real surfaces of degree 5. More precisely, it appears that the construction of real complete intersections by the method of small perturbation produces CDQ-surfaces. This gives some arguments for the following relatively moderate conjecture: *any deformation type of simply connected complex algebraic surfaces contains a CDQ-surface.*

The proof is inspired by [11] and is based on the Deformation theorem proved there, which we apply in equivariant version (since the arguments in the proof of [11, Theorem 2.4, Corollary 2.6] are applicable without essential changes in equivariant setting, I have only indicated in the present paper the changes to be done). Then we use Laudenbach–Poenaru theorem [8] to set up CDQ-property, which may remind the usage of Cerf's theorem in the proof of $\mathbb{C}P^2/\operatorname{conj} \cong S^4$.

2 Main results

Let (V, conj) be a real variety. A holomorphic line bundle $p\ L \to V$ will be called *a real bundle* if it is supplied with an anti-linear involution $\operatorname{conj}_L\ L \to L$, which commutes with conj and p:

$$\begin{array}{ccc} L & \xrightarrow{\operatorname{conj}_L} & L \\ p\downarrow & & \downarrow p \\ V & \xrightarrow{\operatorname{conj}} & V \end{array}$$

A section $f\ V \to L$ is called real if $\operatorname{conj}_L \circ f = f \circ \operatorname{conj}$. The zero divisor of a real section f is a real subvariety of V (unless it is non-reduced or reducible, which is ruled out by the further conditions).

Assume now that (V, conj) is a real nonsingular 3–fold, $L_i \to V$, $i = 1, 2$, are real linear bundles and $f_i\ V \to L_i$ are real sections whose zero divisors $X_0^{(i)}$ are nonsingular and intersect each other transversally. Assume further that $f\ V \to L$ is a real section of $L = L_1 \otimes L_2$ with the zero divisor X intersecting transversally surfaces $X_0^{(i)}$, $i = 1, 2$, and curve $A = X_0^{(1)} \cap X_0^{(2)}$. Consider the section $f_\varepsilon : V \to L_0$, $f_\varepsilon = f_1 \otimes f_2 + \varepsilon f$,

$\varepsilon \in \mathbb{R}$, and denote by X_ε its zero divisor, which is nonsingular for a sufficiently small $\varepsilon \neq 0$, as it can be easily seen.

Theorem 1. *Suppose $X_0^{(1)}$ and $X_0^{(2)}$ are CDQ-surfaces, A is connected, $A_{\mathbb{R}} \neq \emptyset$ and $\varepsilon > 0$. Then X_ε is CDQ-surface provided that ε is small enough.*

We discuss in the rest of this section some of the corollaries of this theorem and prove it in §3.

Let (V, conj) be as above. A real linear bundle L will be called CDQ-*bundle* if it is very ample and admits a real section with a nonsingular CDQ zero divisor.

Lemma 2. *If L is a CDQ-bundle then its multiples $L^{\otimes d}$, $d \geq 1$, are CDQ-bundles as well.*

Proof. Let $X_0^{(1)}$ be a CDQ-divisor of L. We prove by induction on d that there exists a CDQ-divisor, $X_0^{(d)}$, of $L^{\otimes d}$ which intersects $X_0^{(1)}$ transversally along a curve having nonempty real part. This claim is trivial for $d = 1$, since we can perturb $X_0^{(1)}$ so that the result will intersect $X_0^{(1)}$ transversally and contain a given real point of it.

Suppose that $X_0^{(d)}$ satisfies the induction assumption. By Lefschetz Theorem A is connected. A generic real section of $L^{\otimes(d+1)}$ has zero divisor X transversal to $X_0^{(1)}$, $X_0^{(d)}$ and to $X_0^{(1)} \cap X_0^{(d)}$, hence, we can apply Theorem 1 and get a CDQ-divisor X_ε by a perturbation of $X_0^{(1)} \cup X_0^{(d)}$ via X. We can also choose X containing a real point of $X_0^{(1)}$, since $L^{\otimes(d+1)}$ is very ample. Then, for a sufficiently small $\varepsilon > 0$, X_ε intersects $X_0^{(1)}$ transversally and $X_\varepsilon \cap X_0^{(1)} = X \cap X_0^{(1)}$ has nonempty real part.

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Theorem 3. *For arbitrary integers $n, d_1, \ldots, d_n \geq 1$ there exists a CDQ-surface $X \subset \mathbb{C}P^{n+2}$ which is a complete intersection of multi-degree $(d_1, \ldots, d_n)$.*

Proof is carried by induction on n. The diffeomorphism $\mathbb{C}P^2/\mathrm{conj} \cong S^4$ was mentioned in the introduction, therefore, $\mathcal{O}_{\mathbb{C}P^3}(1)$ is a CDQ-bundle. By Lemma 2, $\mathcal{O}_{\mathbb{C}P^3}(d)$, for any $d \geq 1$, is a CDQ-bundle as well.

Assume now that we are given a complete intersection of real hypersurfaces, $X = H_1 \cap \ldots \cap H_n \subset \mathbb{C}P^{n+2}$, of multi-degree $(d_1, \ldots, d_n)$ and that X is CDQ-surface. Choose hypersurfaces $H_i' \subset \mathbb{C}P^{n+3}$, $i = 1, \ldots, n$, so that $H_i = H_i' \cap \mathbb{C}P^{n+2}$, and the intersection $V = H_1' \cap \ldots \cap H_n'$ is transversal. Then the bundle $L \to V$ induced from $\mathcal{O}_{\mathbb{C}P^{n+3}}(1)$ is a CDQ-bundle, since X is its zero divisor. By Lemma 2, $L^{\otimes d}$ is also CDQ-bundle, hence, there exists a CDQ complete intersection of multi-degree $(d_1, \ldots, d_n, d)$.

■

Remark. The method used for Theorem 3 can be applied similarly if instead of $\mathbb{C}P^{n+2}$ we consider for instance products of projective spaces or the weighted projective space $P(1,1,1,n)$, $n \geq 1$ (in the later case we apply Lemma 2 in a version with V having an isolated singularity and note that generic hypersurfaces of degree n in $P(1,1,1,n)$ are isomorphic to $\mathbb{C}P^2$).

3 Proof of Theorem 1

Denote by $N^{(i)}$ a conj-invariant compact tubular neighborhood of A in $X_0^{(i)}$ and put $M^{(i)} = N^{(i)}/\operatorname{conj}$, $B = A/\operatorname{conj}$, $Y^{(i)} = X_0^{(i)}/\operatorname{conj}$ and $Y_\varepsilon = X_\varepsilon/\operatorname{conj}$. It can be easily seen that $M^{(i)}$ is a regular neighborhood of B in $Y^{(i)}$. Let $2k$ denote the number of imaginary points in $A \cap X$.

Proposition 4. *There exists a diffeomorphism $\varphi\, \partial M^{(1)} \to \partial M^{(2)}$, such that $Y_\varepsilon \cong (Cl(Y^{(1)} - M^{(1)}) \cup_\varphi Cl(Y^{(2)} - M^{(2)}))\, \#k\,\overline{\mathbb{C}P}^2$.*

Let us derive first Theorem 1 from the above proposition.

Proof of Theorem 1. Since A is connected and has nonempty real part, B is a compact connected surface with a nonempty boundary, hence, $M^{(i)}$ are handlebodies with one 0-handle and several 1-handles embedded into $Y^{(i)}$. It is well known that if we glue a pair of simply connected 4-manifolds, $Y^{(i)}, i = 1, 2$, along the boundary of such handlebodies the result is diffeomorphic to $Y^{(1)} \# Y^{(2)} \# g\, Z$, where $g = b_1(B)$ and $Z = S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ (it is a corollary of [8], cf. [10]). This implies complete decomposability of Y_ε if $Y^{(i)}$ are completely decomposable. ■

Proof of Proposition 4. Follows closely the scheme of [11]: by blowing up $\widehat{V} \to V$ we lift the pencil $X_t = X_0 + tX$ to get a real fibered surface $\widehat{V} \to \mathbb{C}P^1$; then we apply the deformation theorem in the equivariant version. More precisely, assume that $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$ is sufficiently small. Then X_ε intersects $X_0^{(i)}$, $i = 1, 2$, transversally along the curve $C_i = X_0^{(i)} \cap X$. Consider first the blow-up, $\widetilde{V} \to V$, along C_1 and denote by $\widetilde{C}_i$, $\widetilde{X}_0^{(i)}$, and $\widetilde{X}_t$ the proper images of C_i, $X_0^{(i)}$ and X_t. The pencil $\widetilde{X}_t$ has the base-curve $\widetilde{C}_2$, therefore, the next blow-up $\widehat{V} \to \widetilde{V}$ along $\widetilde{C}_2$ gives a fibering over $\mathbb{C}P^1$ with fibers $\widehat{X}_t$ (here and below we mark by a hat the proper image in $\widehat{V}$).

The projections $\widehat{X}_\varepsilon \to X_\varepsilon$, $\widehat{X}_0^{(1)} \to X_0^{(1)}$ are biregular, as well as $\widehat{X}_0^{(2)} \to \widetilde{X}_0^{(2)}$, whereas $\widetilde{X}_0^{(2)} \to X_0^{(2)}$ is the blow-up at $C_1 \cap X_0^{(2)} = A \cap X$.

The real structure on V can be obviously lifted to the real structure, $\operatorname{conj}_{\widehat{V}}\ \widehat{V} \to \widehat{V}$, and we have $\widehat{Y}^{(1)} \cong Y^{(1)}$, $\widehat{Y}_\varepsilon \cong Y_\varepsilon$ and $\widehat{Y}^{(2)} \cong Y^{(2)} \# k(\overline{\mathbb{C}P}^2)$, where $\widehat{Y}^{(1)}$, $\widehat{Y}_\varepsilon$, $\widehat{Y}^{(2)}$ denote the quotients by $\operatorname{conj}_{\widehat{V}}$ of $\widehat{X}_0^{(1)}$, $\widehat{X}_\varepsilon$ and $\widehat{X}_0^{(2)}$. The latter diffeomorphism follows because blows-up at real points do not change the diffeomorphism type of the quotient, since $\mathbb{C}P^2/\operatorname{conj} \cong S^4$, whereas a pair of blow-ups at conjugated imaginary points descends to a blow-up in the quotient. Restrictions give diffeomorphisms between $M^{(i)}$ and regular neighborhoods of $\widehat{A}/\operatorname{conj}_{\widehat{V}}$ in $\widehat{X}_0^{(i)}/\operatorname{conj}_{\widehat{V}}$ for $\widehat{A} = \widehat{X}_0^{(1)} \cap \widehat{X}_0^{(2)}$.

To complete the proof we use the following equivariant version of the deformation theorem [11, Theorem 2.4, Corollary 2.6]. Assume that W is a complex analytic 3-fold supplied with a real structure $\operatorname{conj}_W\ W \to W$ and $f\ W \to \Delta$ a nonconstant proper holomorphic mapping of W into a disc, $\Delta \subset \mathbb{C}$, around zero, such that $f \circ \operatorname{conj}_W = \operatorname{conj} \circ f$, where $\operatorname{conj}\ \Delta \to \Delta$ is the complex conjugation on $\mathbb{C}$. Assume further that f has a critical value only at zero and the zero divisor X_0 of f splits in two nonsingular irreducible conj_W-invariant components $X_0^{(i)}$, $i = 1, 2$, of multiplicity 1 crossing transversally along a nonsingular irreducible curve A. Suppose that $U \subset W$ is a sufficiently small conj_W-invariant tubular neighborhood of A, so that $N^{(i)} = U \cap X_0^{(i)}$ is a conj_W-invariant tubular neighborhood of A in $X_0^{(i)}$, $i = 1, 2$.

Deformation Theorem. *There exists a* conj_W*-equivariant bundle isomorphism* $\overline{\varphi}\ \partial N^{(1)} \to \partial N^{(2)}$ *reversing orientations of fibers, such that*

$X_t = f^{-1}(t)$ is conjugation-equivariantly diffeomorphic to $Cl(X_0^{(1)} - N^{(1)}) \cup_{\overline{\varphi}} Cl(X_0^{(2)} - N^{(2)})$ for a non-critical value $t \in \Delta$.

Proof. This theorem in non-equivariant version is proved in [11] (in more general form). The proof in our equivariant setting follows the same scheme with some not essential modifications: we need to choose a conj_W-invariant metric on W and instead of the fibering $U \cap X_t \to A$ considered in [11] deal with its quotient, $(U \cap X_t)/\mathrm{conj}_W \to B$, and then apply similarly the arguments on the reduction of the structure group. More precisely, these arguments can be applied in *exactly the same way* for the restriction of this fibering over the complement $B - V$ of a regular neighborhood V of ∂B. To extend reduction of structure group to the whole B we note that the product of the complementary fibering over $\mathrm{Cl}V$ with the D^1-fibering $\mathrm{Cl}V \to \partial B$ is a $D^2 \times I$-fibering over ∂B. For the associated $S^1 \times I$-fibering the structure group is already reduced, so we need to use relative reduction of $\mathrm{Diff}(D^2 \times I)$. This is possible due to Cerf's theorem, since the connected components of ∂B are just circles. (Note that we can prove equivariant version of the deformation theorem in higher dimensions following the same scheme, but instead of Cerf's theorem we need to make use of Hatcher's theorem on $\mathrm{Diff}(S^3)$.)

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