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Morphisms of Klein surfaces.

F. J. CIRRE*

Abstract

We give an elementary proof of a theorem of Andreian Cazacu on the behaviour of morphisms of Klein surfaces under composition.

1 Introduction

Klein himself introduced in the past century the notion of Klein surface as a way to endow conformal structures on surfaces which may be non-orientable or with boundary. Of course, this notion agrees with the classical one of Riemann surface when dealing with orientable surfaces without boundary. In 1971, Alling and Greenleaf [A-G], founded the theory of Klein surfaces in modern terms. In addition to its own interest, this theory acquires more relevance since they proved that, in the *same way* as a compact Riemann surface is associated with a complex projective smooth algebraic curve, each compact Klein surface S can be associated with a real projective smooth algebraic curve whose field of rational functions is the field of meromorphic functions on S. Hence, the problem of classifying real algebraic curves up to birational transformations and that of determining the group of automorphisms of a real algebraic curve, are closely related to the study of isomorphisms between Klein surfaces.

Let us denote by ∂S the boundary of the Klein surface S. In this

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paper we prove the following

Theorem. *Let $f : S \to S'$ and $g : S' \to S''$ be two non-constant continuous maps between Klein surfaces such that $f(\partial S) \subset \partial S'$ and $g(\partial S') \subset \partial S''$. Consider the following statements:*

(1) f is a morphism

(2) g is a morphism

(3) $g \circ f$ is a morphism.

Then:

(i) (1) and (2) imply (3).

(ii) If f is surjective, (1) and (3) imply (2).

(iii) If f is open, (2) and (3) imply (1).

The basic part (i) of this theorem was proved in [A-G] while the statements of parts (ii) and (iii) are due to Andreian Cazacu [A]. Her proof of part (iii) is based on a powerful theorem of S. Stoilow [St, Ch. V, II.6], which was originally stated in the setting of interior transformations.

Our goal is to give a self-contained and easier proof of part (iii) by using only elementary and well-known results of complex analysis. For the sake of completeness we also include a proof of part (ii), which as far as we know does not appear in the literature.

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2 Preliminaries

2.1 Dianalyticity

Let U be an open set in $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is *antianalytic on U* if its complex conjugate, $\overline{f}$, is analytic on U, and *dianalytic on U* if its restriction to every connected component of U is either analytic or antianalytic. Easy computations show that

- If U is connected and f is simultaneously analytic and antianalytic, then f is constant.

- Let f and g be dianalytic functions on an open connected set U. If f and g are both either analytic or antianalytic, then $g \circ f$ is analytic. Otherwise, $g \circ f$ is antianalytic.

Let A be open in $\mathbb{C}^+ := \{z \in \mathbb{C} : Imz \geq 0\}$ but not in $\mathbb{C}$. A function $f : A \to \mathbb{C}$ is said to be *dianalytic on A* if it is the restriction of a dianalytic function $f_U : U \to \mathbb{C}$ where U is an open set in $\mathbb{C}$ containing A.

2.2 Klein surfaces

A *surface* is a Hausdorff connected topological space S together with a family $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ such that $\{U_i : i \in I\}$ is an open covering of S and each map $\varphi_i : U_i \to \varphi_i(U_i)$ is a homeomorphism onto an open set of $\mathbb{C}^+$. The family $\mathcal{A}$ is a *topological atlas* on S and its elements are *charts*. The *transition functions of S* are the homeomorphisms

$$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j).$$

The *orientability* of S is defined as for a real 2-manifold, under the identification of $\mathbb{C}$ with $\mathbb{R}^2$. The *boundary* of S is the set

$$\partial S = \{x \in S : \ \varphi_i(x) \in \mathbb{R} \text{ for all } i \in I \text{ with } x \in U_i\}.$$

The topological atlas $\mathcal{A}$ is said to be *dianalytic* if the transition functions are dianalytic. We say that two dianalytic atlases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{A} \cup \mathcal{B}$ is dianalytic. A *dianalytic structure on S* is the equivalence class of a dianalytic atlas on S.

A *Klein surface* is a surface S equipped with a dianalytic structure.

2.3 Morphisms of Klein surfaces

The *folding map* is the open continuous map

$$\Phi : \mathbb{C} \to \mathbb{C}^+ : x + \sqrt{-1}y \mapsto x + \sqrt{-1} \mid y \mid .$$

Obviously, $\Phi(z) = \Phi(\overline{z})$ and if A is a subset of $\mathbb{C}^+$ then $\Phi^{-1}(A) = A \cup \overline{A}$ where $\overline{A} := \{z \in \mathbb{C} : \overline{z} \in A\}$.

A *morphism* between the Klein surfaces S and S' is a continuous map $f : S \to S'$ such that

i) $f(\partial S) \subset \partial S'$,

ii) given $s \in S$, there exist charts $(U, \varphi), (V, \psi)$ with $s \in U$ and $f(U) \subset V$ and an analytic function $F : \varphi(U) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccccc} U & & \xrightarrow{\quad f \quad} & & V \\ \varphi \big\downarrow & & & & \big\downarrow \psi \\ \varphi(U) & \xrightarrow{F} & \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C}^+ \end{array}$$

Since $\varphi(U)$ is contained in $\mathbb{C}^+$, F extends to an analytic function $\widehat{F} : \varphi(U) \cup \overline{\varphi(U)} \to \mathbb{C}$ by taking

$$\widehat{F}(z) = \begin{cases} F(z) & \text{if } z \in \varphi(U), \\ \overline{F(\overline{z})} & \text{if } z \in \overline{\varphi(U)}. \end{cases}$$

Indeed, if $\varphi(U)$ and $\overline{\varphi(U)}$ are disjoint, then $\widehat{F}$ is analytic on $\overline{\varphi(U)}$ since it is the composite of two antianalytic functions, namely, the complex conjugation and $\overline{F}$. In case of $\varphi(U)$ and $\overline{\varphi(U)}$ are not disjoint, $\varphi(U) \cap \mathbb{R}$ is not empty and the analyticity of $\widehat{F}$ on $\varphi(U) \cup \overline{\varphi(U)}$ is a consequence of Schwarz's Reflection Principle [S, Th. 16.4], provided that F maps the reals into the reals. But $F(\varphi(U) \cap \mathbb{R}) = F(\varphi(U \cap \partial S))$ and if $x \in U \cap \partial S$ then $f(x) \in V \cap \partial S'$; therefore $\Phi F \varphi(x) = \psi f(x) \in \mathbb{R}$ as required.

As to the derivative of $\widehat{F}$, straightforward computations show that it satisfies the same formula than $\widehat{F}$: *i.e.*, $\widehat{F}'(z) = \overline{\widehat{F}'(\overline{z})}$.

From condition i) in the definition, if S' has no boundary, then neither has S. In particular, when dealing with orientation preserving morphisms between Riemann surfaces, Φ can be omitted in the diagram. Hence this definition of morphism agrees with the classical one.

It is well-known that the image of an open set of $\mathbb{C}$ by a non-constant complex analytic function is also open. It follows that a non-constant morphism between Riemann surfaces is an open map. The same holds true for morphisms between Klein surfaces:

Claim 1. *If $f : S \to S'$ is a non-constant morphism between Klein surfaces, then f is open.*

Proof. It suffices to show that $f(U)$ is open for each U as in the definition. Since $f(U) = \psi^{-1}\Phi F\varphi(U)$ and Φ is an open map, the claim is obvious if $\varphi(U)$ is open in $\mathbb{C}$. If, on the contrary, $\varphi(U)$ is not open in $\mathbb{C}$ but in $\mathbb{C}^+$, then $F\varphi(U)$ may not be so in $\mathbb{C}$. However, it is easy to check that $\Phi F\varphi(U)$ equals $\Phi\widehat{F}(\varphi(U) \cup \overline{\varphi(U)})$ and since $\varphi(U) \cup \overline{\varphi(U)}$ is open in $\mathbb{C}$, we conclude as above that $f(U) = \psi^{-1}\Phi\widehat{F}(\varphi(U) \cup \overline{\varphi(U)})$ is open in S'.

This result points out that we cannot drop the assumption "*f is open*" in part (iii) of the theorem, as the following example, due to Andreian Cazacu [A], shows. Set $f : \mathbb{C} \to \mathbb{C} : x + \sqrt{-1}y \mapsto x + \sqrt{-1} \mid y \mid$ and $g = \Phi : \mathbb{C} \to \mathbb{C}^+$ (f is not the folding map since f has range $\mathbb{C}$). Clearly $g = g \circ f : \mathbb{C} \to \mathbb{C}^+$ is a morphism but not f since it is not open.

To finish this section, let us point out another property of morphisms between Klein surfaces: they are discrete, *i.e.*, they have discrete fibers.

Claim 2. *If $f : S \to S'$ is a non-constant morphism between Klein surfaces, then f is discrete.*

Proof. It is enough to show that for each $s \in S$ the fiber $f^{-1}(f(s))$ is discrete in U, the neighbourhood of s given in the definition of morphism. Since the fibers of Φ are finite, the proof reduces to verify that the preimage of a finite set by a non-constant complex analytic function is a discrete set. This is an easy exercise in complex analysis for which only the Identity Principle [S, Th 10.8] is needed.

3 Proof of the theorem

In this proof all the neighbourhoods considered will be open, and the open and connected subsets of $\mathbb{C}$ will be called domains. When restricting a map h, the expression $h|_X$ will be written $h|$ if no confusion may arise.

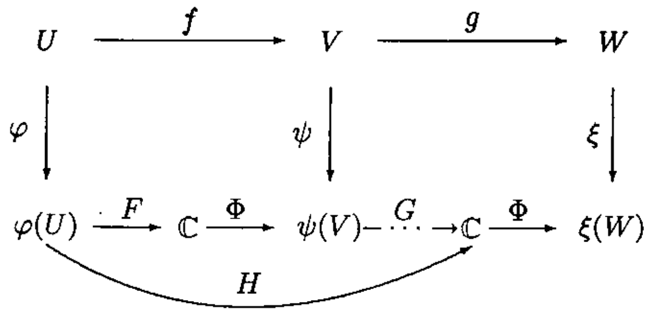
(i) (1)+(2) imply (3).

This was proved by Alling and Greenleaf [A-G, Theorem 1.4.3]. The proof is based on the Schwarz's Reflection Principle and the fact that the composition of analytic maps is also analytic.

(ii) If f is surjective, (1)+(3) imply (2).

Given $s' \in S'$ let $s \in S$ be such that $f(s) = s'$. Since f and $g \circ f$ are morphisms there exist charts $(U, \varphi), (V, \psi)$ and (W, ξ) with $s \in U$,

$f(U) = V, g(V) = W$ and there exist analytic maps F and H such that $\Phi F = \psi f \varphi^{-1}$ and $\Phi H = \xi g f \varphi^{-1}$.



We look for an analytic map $G : \psi(V) \to \xi(W) \cup \overline{\xi(W)}$ such that $\Phi G = \xi g \psi^{-1}$. Let $\widehat{F}$ and $\widehat{H}$ be the analytic extensions of F and H, respectively, to $A := \varphi(U) \cup \overline{\varphi(U)}$ as defined in section 2. The diagram suggests how to find G: roughly speaking it will be the composite of local inverses of $\widehat{F}$ with $\widehat{H}$.

1. The analytic function $\widehat{F} : A \to \mathbb{C}$ has analytic local inverses except in the discrete set $D := \{a \in A : \widehat{F}'(a) = 0\}$. That is, for each $p \in A \setminus D$ there exist neighbourhoods of p and $\widehat{F}(p)$ in $A \setminus D$ and $\widehat{F}(A \setminus D)$, that we shall denote by A_p and B_p, respectively, and an analytic map $L_p : B_p \to A_p$ such that $\widehat{F}(A_p) = B_p, \widehat{F}|_{A_p} \circ L_p = id_{B_p}$ and $L_p \circ \widehat{F}|_{A_p} = id_{A_p}$.

Restricting charts if necessary, we will suppose that D is a finite set.

2. For each $p \in A \setminus D$ we define the non-constant analytic map

$$\widehat{G_p} := \widehat{H} \circ L_p : B_p \to \mathbb{C}.$$

We claim that $\widehat{G_p} = \widehat{G_q}$ in $B_p \cap B_q$ if this intersection is nonempty. To see this we use the following lemma.

Lemma. *Let B be a domain in $\mathbb{C}$ and let $G_1, G_2 : B \to \mathbb{C}$ be two non-constant analytic maps such that $\Phi G_1 = \Phi G_2$. Then $G_1 = G_2$.*

Proof. Choose a nonempty domain Y of the preimage of $\mathbb{C} \setminus \mathbb{R}$ under G_1. Then the sets $M_1 = Y \cap \{G_1 = G_2\}$ and $M_2 = Y \cap \{G_1 = \overline{G_2}\}$ are disjoint and closed on Y. Further, $Y = M_1 \cup M_2$ since $\Phi G_1 = \Phi G_2$ and so, either $Y = M_1$ or $Y = M_2$. In the latter case G_1 should be both analytic and antianalytic on Y, *i.e.* $G_1|_Y$ should be constant, which is impossible because G_1 is an open map. Hence $G_1 = G_2$ on Y and by the Identity Principle $G_1 = G_2$ on B.

Back to our claim, it is enough to prove that $\Phi\widehat{G_p} = \Phi\widehat{G_q}$. In fact we shall show that both are equal to $\xi g\psi^{-1}\Phi$. Let $y \in B_p \cap B_q$.

If $L_p(y) \in \varphi(U)$,

$$\Phi\widehat{G_p}(y) = \Phi\widehat{H}L_p(y) = \Phi H L_p(y) = \xi g\psi^{-1}\Phi F L_p(y) = \xi g\psi^{-1}\Phi(y)$$

and also if $L_p(y) \in \overline{\varphi(U)}$,

$$\Phi\widehat{G_p}(y) = \Phi\overline{H}\left(\overline{L_p(y)}\right) = \Phi H\left(\overline{L_p(y)}\right) = \xi g\psi^{-1}\Phi F\left(\overline{L_p(y)}\right) =$$

$$= \xi g\psi^{-1}\Phi\left(\overline{\widehat{F}\left(L_p(y)\right)}\right) = \xi g\psi^{-1}\Phi\widehat{F}L_p(y) = \xi g\psi^{-1}\Phi(y).$$

Analogously, for $\widehat{G_q}$ we obtain $\Phi\widehat{G_q} = \xi g\psi^{-1}\Phi$ as desired.

3. This allows us to glue together the functions $\widehat{G_p}$ and define a global analytic function $\widehat{G}$ on $\widehat{F}(A \setminus D) = \cup_{p\in A\setminus D}B_p$ by

$$\widehat{G} : \widehat{F}(A \setminus D) \to \mathbb{C} : z \mapsto \widehat{G_p}(z) \quad \text{if } z \in B_p,$$

which verifies $\Phi\widehat{G} = \xi g\psi^{-1}\Phi|_{\widehat{F}(A\setminus D)}$.

Since $\widehat{F}(A\setminus D) \supset \widehat{F}(A)\setminus\widehat{F}(D)$ and $\widehat{F}(D)$ is a finite set, we may extend $\widehat{G}$ analytically to $\widehat{F}(A)$ by Riemann's Removable Singularities Theorem [S, Th. 11.4], provided that $\widehat{G}$ is locally bounded in $\widehat{F}(D)$. But this is clear because $\Phi\widehat{G}$ coincides with $\xi g\psi^{-1}\Phi|$. This analytic extension, which we also denote by $\widehat{G} : \widehat{F}(A) \to \mathbb{C}$, is in particular defined on $\psi(V)$ since

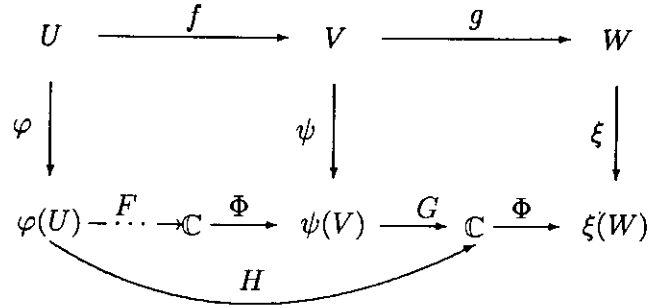
$$\psi(V) = \Phi F\varphi(U) \subset F\varphi(U) \cup \overline{F\varphi(U)} = \widehat{F}(A).$$

So, the restriction of $\widehat{G}$ to $\psi(V)$ is an analytic function $G : \psi(V) \to \mathbb{C}$ which verifies $\Phi G = \xi g\psi^{-1}$. Hence g is a morphism.

Notice that if S is compact, then the assumption "*f is surjective*" may be dropped because it is a consequence of the hypothesis. Indeed, since f is open and continuous, $f(S)$ has to be both open and compact. Since S' is Hausdorff and connected, $f(S) = S'$.

(iii) If f is open, (2)+(3) imply (1).

Given $s \in S$ there exist charts $(U, \varphi), (V, \psi)$ and (W, ξ) with $s \in U$, $f(U) = V, g(V) = W$ and there exist analytic functions G and H such that $\xi g = \Phi G \psi$ and $\xi g f = \Phi H \varphi$.



We look for an analytic map $F : \varphi(U) \to \psi(V) \cup \overline{\psi(V)}$ such that $\Phi F = \psi f \varphi^{-1}$. The diagram suggests that F must be the composite of H with local inverses of the analytic extension $\widehat{G}$ of G defined on $A := \psi(V) \cup \overline{\psi(V)}$ (see section 2).

1. Set

$$D_1 = \{a \in A : \widehat{G}'(a) = 0\}, \quad D_2 = \{a \in A : \widehat{G}(a) \in \mathbb{R}\}.$$

We construct local inverses of $\widehat{G}$ on $Y := A \setminus (D_1 \cup D_2)$.

First, for any $p \in Y \cap \mathbb{C}^+$ there exist neighbourhoods of p and $\widehat{G}(p)$ in $Y \cap \mathbb{C}^+$ and $\widehat{G}(Y \cap \mathbb{C}^+)$ that we shall denote by A_p and B_p, respectively, and an analytic function $L_p : B_p \to A_p$ such that $\widehat{G}(A_p) = B_p$, $\widehat{G}|_{A_p} \circ L_p = id_{B_p}$ and $L_p \circ \widehat{G}|_{A_p} = id_{A_p}$.

Moreover, if $p \in Y \cap \mathbb{C}^+$ it turns out that $\overline{p} \in Y \cap \mathbb{C}^-$, where $\mathbb{C}^- := \overline{\mathbb{C}^+}$, because $\widehat{G}'(\overline{p}) = \overline{\widehat{G}'(p)} \neq 0$ and $\widehat{G}(\overline{p}) = \overline{\widehat{G}(p)} \notin \mathbb{R}$. Consequently, there exist neighbourhoods $A_{\overline{p}} = \overline{A_p}$ and $B_{\overline{p}} = \overline{B_p}$ of $\overline{p}$ and $\widehat{G}(\overline{p})$ on $Y \cap \mathbb{C}^-$ and $\widehat{G}(Y \cap \mathbb{C}^-)$ respectively, such that the analytic function $L_{\overline{p}} : B_{\overline{p}} \to A_{\overline{p}}$ defined by $L_{\overline{p}}(z) = \overline{L_p(\overline{z})}$ verifies $\widehat{G}|_{A_{\overline{p}}} \circ L_{\overline{p}} = id_{B_{\overline{p}}}$ and $L_{\overline{p}} \circ \widehat{G}|_{A_{\overline{p}}} = id_{A_{\overline{p}}}$.

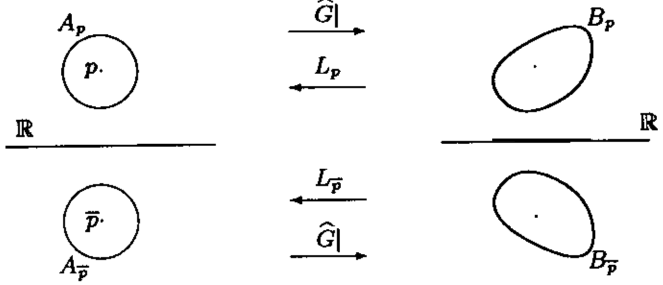


Figure 1.

Note that B_p does not intersect $\mathbb{R}$ since A_p does not intersect D_2. In particular, B_p and $B_{\overline{p}}$ are disjoint. (Figure 1 represents only the case $B_p \subset \mathbb{C}^+$).

2. For each $p \in Y \cap \mathbb{C}^+$ we define the analytic function

$$\begin{array}{rcl} F_p : N_p := (\psi f \varphi^{-1})^{-1}(A_p) & \to & A_p \cup A_{\overline{p}} \\ x & \mapsto & \begin{cases} L_p \circ H(x) & \text{if} \quad H(x) \in B_p \\ L_{\overline{p}} \circ H(x) & \text{if} \quad H(x) \in B_{\overline{p}}. \end{cases} \end{array}$$

The function F_p is well defined: if $x \in (\psi f \varphi^{-1})^{-1}(A_p)$, then $\Phi H(x) = \xi g f \varphi^{-1}(x) = \Phi G \psi f \varphi^{-1}(x) \in \Phi G(A_p) = \Phi(B_p)$ and so, $H(x) \in B_p \cup B_{\overline{p}}$.

We claim that $F_p = F_q$ in $N_p \cap N_q$ if this intersection is nonempty. Indeed, if $x \in N_p \cap N_q$ then $y := \psi f \varphi^{-1}(x)$ belongs to $A_p \cap A_q$ and so $\Phi H(x) = \Phi G(y)$ belongs to $B_p \cap B_q$. This yields two possibilities:

If $H(x) = G(y)$, then it belongs to $B_p \cap B_q$; thus

$$F_p(x) = L_p H(x) = L_p G(y) = y = L_q G(y) = L_q H(x) = F_q(x).$$

Analogously, if $H(x) = \overline{G(y)} = \widehat{G}(\overline{y})$, then it belongs to $B_{\overline{p}} \cap B_{\overline{q}}$; thus

$$F_p(x) = L_{\overline{p}} H(x) = L_{\overline{p}} \widehat{G}(\overline{y}) = \overline{y} = L_{\overline{q}} \widehat{G}(\overline{y}) = L_{\overline{q}} H(x) = F_q(x).$$

This proves the claim. Further, in both cases we have obtained the identity

$$\Phi F_p(x) = \psi f \varphi^{-1}(x).$$

As a consequence, if we set

$$N := \bigcup_{p \in Y \cap \mathbb{C}^+} N_p \subset \varphi(U),$$

we get a well defined analytic function $F^* : N \to \mathbb{C}$ given by $F^*(x) = F_p(x)$ if $x \in N_p$, which verifies $\Phi F^* = \psi f \varphi^{-1}|_N$. Moreover, it also verifies $\widehat{G}F^* = H|_N$.

In order to extend F^* to $\varphi(U)$ we have to assure that $\varphi(U) \setminus N$ is *thin enough.*

3. For short, we denote

$$l := \psi f \varphi^{-1} : \varphi(U) \to \psi(V),$$

which is a continuous and open map because f is so. Since $Y \cap \mathbb{C}^+ = \cup_p A_p$, where p runs over $Y \cap \mathbb{C}^+$, we have $l^{-1}(Y \cap \mathbb{C}^+) = \cup_p l^{-1}(A_p) = N$. Therefore, since $Y = A \setminus (D_1 \cup D_2)$ and $\varphi(U) = l^{-1}(A)$ one gets

$$\varphi(U) \setminus N = l^{-1}(A) \setminus l^{-1}(Y \cap \mathbb{C}^+) = l^{-1}(D_1) \cup l^{-1}(D_2).$$

3.1. *$l^{-1}(D_1)$ is discrete in $\varphi(U)$.*
First, let us note that for any x in $l^{-1}(D_1)$, $\Phi H(x) = \Phi G l(x)$ belongs to $\Phi G(D_1)$, *i.e.*, $H(x) \in G(D_1) \cup \overline{G(D_1)} = \widehat{G}(D_1)$ and so, x belongs to $H^{-1}(\widehat{G}(D_1))$. Thus $l^{-1}(D_1) \subset H^{-1}(\widehat{G}(D_1))$ and we only have to prove the discreteness of $H^{-1}(\widehat{G}(D_1))$. But this follows from the proof of claim 2 of section 2 since we may suppose from the beginning that D_1 is a finite set.

Note that this proves that l, and so f, is a discrete map.

3.2. *$l^{-1}(D_2)$ is a proper (global) real analytic set of $\varphi(U)$.*
Indeed, from the equality $\Phi \widehat{G} l = \Phi H$ it follows readily that $l^{-1}(D_2)$ equals $H^{-1}(\mathbb{R})$, *i.e.*, it is the zero set in $\varphi(U)$ of the imaginary part of H.

Summarizing, the complement of N in $\varphi(U)$ is *thin* : it is the union of a discrete set and a proper real analytic set.

4. Continuous extension of F^*.

We have defined an analytic function

$$F^* : \varphi(U) \setminus \left(l^{-1}(D_1) \cup H^{-1}(\mathbb{R})\right) \to \mathbb{C}$$

which verifies $\Phi F^* = l|$. Evidently, l extends continuously ΦF^* to $\varphi(U)$ which in particular implies that F^* is locally bounded in $\varphi(U)$. Thus, F^* may be extended analytically to the discrete set $l^{-1}(D_1) \setminus H^{-1}(\mathbb{R})$. We also call $F^* : \varphi(U) \setminus H^{-1}(\mathbb{R}) \to \mathbb{C}$ this extension.

Suppose we have found a continuous extension $F : \varphi(U) \to \mathbb{C}$. Then ΦF has to coincide with l, that is, for each $x \in \varphi(U)$, $F(x)$ has to be either $l(x)$ or $\overline{l(x)}$. We shall show that the behaviour of F^* near x gives the right choice that makes F continuous.

First, it is obvious that F^* extends continuously to any point x in $l^{-1}(\mathbb{R})$, which is a subset of $H^{-1}(\mathbb{R})$, by defining $F(x) = l(x) = \overline{l(x)}$.

Let M be *the subset of* $H^{-1}(\mathbb{R}) \setminus l^{-1}(\mathbb{R})$ *consisting of those points* x *such that for any neighbourhood* U^x *of* x, $U^x \setminus H^{-1}(\mathbb{R})$ *has more than two connected components.*

The set M is discrete in $\varphi(U)$ since H' vanishes on it. Indeed, given $x \in M$, if $H'(x) \neq 0$ then $H|^{-1}$ is a homeomorphism between a small open disc $U^{H(x)}$ centered at $H(x)$ and a neighbourhood U^x of x. In particular, the number of connected components of $U^{H(x)} \setminus \mathbb{R}$ and that of $U^x \setminus H^{-1}(\mathbb{R})$ should coincide. This is impossible if $x \in M$ since $U^{H(x)} \setminus \mathbb{R}$ has exactly two connected components.

Now we extend F^* continuously to $H^{-1}(\mathbb{R}) \setminus M$.

Since the extension is obvious for points in $l^{-1}(\mathbb{R})$, we just have to deal with points in $(H^{-1}(\mathbb{R}) \setminus l^{-1}(\mathbb{R})) \setminus M$. Given such a point x, let U^x be an open connected neighbourhood of x not intersecting $l^{-1}(\mathbb{R})$ such that $U^x \setminus H^{-1}(\mathbb{R})$ has two connected components. One of these components is mapped by H onto a domain in $\mathbb{C}^+ \setminus \mathbb{R}$ and we denote it by U^x_+, and the other is mapped by H onto a domain in $\mathbb{C}^- \setminus \mathbb{R}$ and we denote it by U^x_-. The reason is clear: $H(U^x)$ is a domain in $\mathbb{C}$ intersecting $\mathbb{R}$ but $H(U^x_+ \cup U^x_-)$ does not intersect $\mathbb{R}$. Restricting U^x we may suppose that $H(U^x_+) = \overline{H(U^x_-)}$.

Let us denote by δ the arc $U^x \cap H^{-1}(\mathbb{R})$. Our purpose is to extend F^* continuously to δ and for this it is enough to prove that "$F^*(U^x_+)$ *is contained in* $\mathbb{C}^+$ *if and only if* $F^*(U^x_-)$ *is contained in* $\mathbb{C}^+$". Indeed, since $l(U^x)$ does not intersect $\mathbb{R}$ the equality $\Phi F^* = l$ gives that for $\varepsilon \in \{+,-\}$, $F^*|_{U^x_\varepsilon}$ equals either $l|_{U^x_\varepsilon}$, or $\bar{l}|_{U^x_\varepsilon}$. Now the image of $l|_{U^x_\varepsilon}$ (respectively, $\bar{l}|_{U^x_\varepsilon}$) is contained in $\mathbb{C}^+$ (respectively, $\mathbb{C}^-$). Thus if the claim is true, then either $F^*|_{U^x_+ \cup U^x_-} = l|_{U^x_+ \cup U^x_-}$ or $F^*|_{U^x_+ \cup U^x_-} = \bar{l}|_{U^x_+ \cup U^x_-}$. Therefore the continuity of l and $\bar{l}$ on U^x ensures the existence of a continuous extension of $F^*|_{U^x_+ \cup U^x_-}$ to δ.

In order to prove the above claim, we first observe that $Gl(U^x)$ is open in $\mathbb{C}$. Indeed, since l is an open map and $l(U^x)$ does not intersect $\mathbb{R}$, $l(U^x)$ is open in $\mathbb{C}$ and hence so is $Gl(U^x)$. Further, it contains a real interval, $Gl(\delta)$, since $l(\delta)$ is in $D_2 = \widehat{G}^{-1}(\mathbb{R})$.

Let us prove then our claim, *i.e.*, that

$$F^*(U_+^x) \subset \mathbb{C}^+ \Leftrightarrow F^*(U_-^x) \subset \mathbb{C}^+.$$

If $F^*(U_+^x) \subset \mathbb{C}^+$, then $Gl(U_+^x) = G\Phi F^*(U_+^x) = GF^*(U_+^x) = H(U_+^x)$ which is in $\mathbb{C}^+$. (Figure 2 illustrates this case).

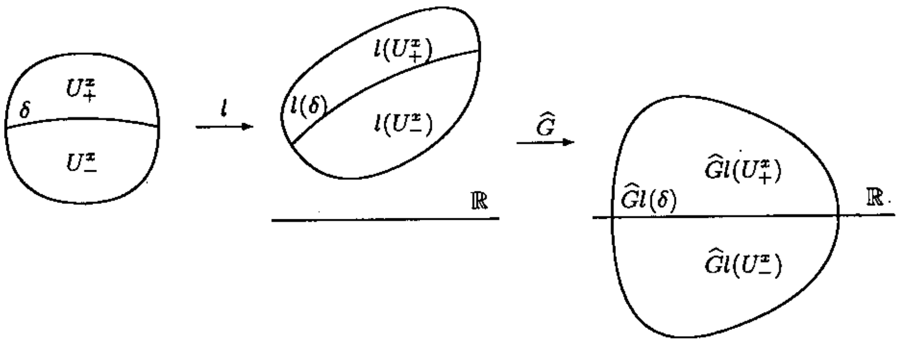


Figure 2.

Suppose that $F^*(U_-^x) \subset \mathbb{C}^-$. Then $Gl(U_-^x)$ coincides with $Gl(U_+^x)$ since $Gl(U_-^x) = G\Phi F^*(U_-^x) = G\overline{F^*(U_-^x)} = \overline{\widehat{G}F^*(U_-^x)} = \overline{H(U_-^x)} = H(U_+^x)$. So, $Gl(U^x) = Gl(U_+^x \cup U_-^x \cup \delta) = H(U_+^x) \cup Gl(\delta)$ which clearly contradicts the opennes of $Gl(U^x)$ in $\mathbb{C}$. Thus, $F^*(U_+^x) \subset \mathbb{C}^+$ implies $F^*(U_-^x) \subset \mathbb{C}^+$ and the converse follows by symmetry.

This completes the proof of the existence of a continuous extension F of F^* to $\varphi(U) \setminus M$, which also verifies $\Phi F = l|_{\varphi(U)\setminus M}$.

The last step is to show that F is, in fact, analytic on $\varphi(U)$.

5. *F is analytic on* $\varphi(U)$.

Let x be a point of $H^{-1}(\mathbb{R}) \setminus M$ (recall that F is analytic outside $H^{-1}(\mathbb{R})$). For such a point x there exists an open connected neighbourhood U^x in $\varphi(U) \setminus M$ such that $U^x \setminus H^{-1}(\mathbb{R})$ has two connected components U_+^x and U_-^x. Moreover, the boundaries of U_+^x and U_-^x share the open Jordan arc $\delta := U^x \cap H^{-1}(\mathbb{R})$ which is rectifiable and accesible (accesible means that each $a \in \delta$ can be joined to any point of U_+^x, respectively U_-^x, by a continuous curve $\alpha : [0,1] \to U_+^x$, respectively U_-^x, with $\alpha(0) = a$).

Hence, the analyticity of F in U^x is a consequence of the following

Theorem. ([S, Th. 16.3]) *Let* $\{G_1, f_1\}$ *and* $\{G_2, f_2\}$ *be two elements* (that is, G_i is a domain and f_i is an analytic function on G_i) *whose domains are disjoint but share an accesible Jordan boundary arc* δ, *where* δ *is open and rectifiable. Suppose* f_i *is continuous in* $G_i \cup \delta$ *for* $i = 1, 2$ *and moreover suppose that* f_1 *and* f_2 *coincide on* δ. *Then the function* Θ *defined by*

$$\Theta(z) = \begin{cases} f_1(z) & \text{if } z \in G_1 \\ f_1(z) = f_2(z) & \text{if } z \in \delta \\ f_2(z) & \text{if } z \in G_2 \end{cases}$$

is analytic on $G_1 \cup \delta \cup G_2$.

Applying this theorem to the elements $\{U_+^x, F|_{U_+^x}\}$ and $\{U_-^x, F|_{U_-^x}\}$ we conclude that F is analytic on $U^x \subset \varphi(U) \setminus M$ and therefore on $\varphi(U) \setminus M$.

Finally, the equality $\Phi F = \psi f \varphi^{-1}|_{\varphi(U)\setminus M}$ shows that F is locally bounded in M. This, together with the discreteness of M ensures the analytic continuation of F to the whole $\varphi(U)$, where the equality $\Phi F = \psi f \varphi^{-1}$ also holds. Hence, f is a morphism.

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Departamento de Geometría y Topología
Facultad C.C. Matemáticas, U.C.M.
28040 Madrid, Spain