REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 10, número 2: 1997

Separable quotients of Banach spaces.

Jorge MUJICA

Abstract

In this survey we show that the separable quotient problem for Banach spaces in equivalent to several other problems from Banach space theory. We give also several partial solutions to the problem.

1 Introduction

The problem of whether every infinite dimensional Banach space has a separable infinite dimensional quotient seems to have been considered since the thirties, though the earliest explicit reference I know of is a paper of Rosenthal [30] of 1969.

In this survey we show that the separable quotient problem, as it is known, is equivalent to several other problems from Banach space theory. We give also several partial solutions to the problem.

In Section 1 we introduce the notion of Schauder basis of a Banach space. After stating the elementary properties of Schauder bases, we show that the separable quotient problem is equivalent to the problem of whether every infinite dimensional Banach space has an infinite dimensional quotient with a Schauder basis. The main results in this section are due to Bessaga and Pelczynski [3], and Johnson and Rosenthal [14].

In Section 2 we introduce the notion of quasi-complemented subspace of a Banach space. We show that every closed subspace of a separable

---

1991 Mathematics Subject Classification: 46B10, 46B15, 46B45, 46A08, 46G20.
Servicio Publicaciones Univ. Complutense. Madrid, 1997.

Banach space is quasi-complemented. We prove also that the separable quotient problem is equivalent to the problem of whether every infinite dimensional Banach space has a separable, infinite dimensional, quasi-complemented subspace. The main results in this section are due to Mackey [21] and Rosenthal [30].

In Section 3 we introduce the notion of barrelled space. We show that the separable quotient problem is equivalent to the problem of whether every infinite dimensional Banach space has a dense, nonbarrelled subspace. The main result in this section is due to Saxon and Wilansky [32].

Finally in Section 4 we show that a real Banach space has a separable, infinite dimensional quotient if its dual has an infinite dimensional subspace which is either reflexive or isomorphic to c_0 or ℓ^1. Thus the separable quotient problem is closely connected with the problem of whether every infinite dimensional Banach space has an infinite dimensional subspace which is either reflexive or isomorphic to c_0 or ℓ^1. The last mentioned problem remained open for a long time and was recently solved in the negative by Gowers [10]. The main result in this section is due to Hagler and Johnson [11].

This survey is based on the notes of a three-lecture minicourse delivered at the 43rd Brazilian Analysis Seminar, held at the University of São Paulo during May 23-25, 1996. I am indebted to the organizers for their kind invitation. I am also indebted to the referee for several suggestions which have helped to improve the presentation.

0 Notation and Terminology

Unless stated otherwise, the letters E and F always represent Banach spaces over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. E^* denotes the algebraic dual of E, whereas E' denotes the topological dual of E. $\mathcal{L}(E;F)$ denotes the Banach space of all continuous, linear operators from E into F. If $T \in \mathcal{L}(E;F)$, then $T' \in \mathcal{L}(F';E')$ denotes the dual operator. B_E denotes the closed, unit ball of E, whereas S_E denotes the unit sphere of E. For a set $A \subset E$, span A denotes the vector subspace of E spanned by A, whereas $[A]$ denotes the closure of span A in E.

1 Schauder Bases of Banach Spaces

A sequence (e_n) in a Banach space E is said to be a *Schauder basis* if for each $x \in E$ there is a unique sequence of scalars (λ_n) such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$, where the series converges in norm. The coordinate functionals

$$e'_n : \sum_{j=1}^{\infty} \lambda_j e_j \in E \to \lambda_n \in \mathbb{K}$$

and the projections

$$S_n : \sum_{j=1}^{\infty} \lambda_j e_j \in E \to \sum_{j=1}^{n} \lambda_j e_j \in E$$

are evidently linear, and the following result of Banach [2, p. 113] (see also [6, pp. 32-33] or [20, pp. 1-2]) shows that they are continuous.

1.1. Proposition. *Let (e_n) be a Schauder basis of E. Then there is a constant $c \geq 1$ such that $||S_n x|| \leq c||x||$ and $|e'_n(x)| \leq 2c||x||/||e_n||$ for every $x \in E$ and $n \in \mathbb{N}$.*

A sequence (e_n) in E is said to be a *basic sequence* if it is a Schauder basis of the closed subspace that it generates in E.

Every complete orthonormal sequence in a separable Hilbert space is a Schauder basis. If e_n is the scalar sequence that consists of zeros except for a one in position n, then the sequence (e_n) is a Schauder basis in each of the spaces c_0 or ℓ^p, where $1 \leq p < \infty$. All this is easy to prove.

It is much harder to find Schauder bases in spaces like $L^p[0,1]$ or $C[0,1]$. Schauder [33][34], who introduced the notion of Schauder basis, proved that the Haar system (h_n) is a Schauder basis of $L^p[0,1]$ whenever $1 \leq p < \infty$, whereas the Schauder system is a Schauder basis of $C[0,1]$. The *Haar system* (h_n) is defined by $h_1 = \chi_{[0,1]}$, $h_2 = \chi_{[0,\frac{1}{2}]} - \chi_{(\frac{1}{2},1]}$, $h_3 = \chi_{[0,\frac{1}{4}]} - \chi_{(\frac{1}{4},\frac{1}{2}]}$, $h_4 = \chi_{(\frac{1}{2},\frac{1}{4}]} - \chi_{(\frac{3}{4},1]}$, etc., whereas the *Schauder system* (s_n) is defined by $s_1 = \chi_{[0,1]}$ and $s_n(x) = \int_0^x h_{n-1}(t)dt$ for every $n \geq 2$. Schauder's results can be proved with the aid of the following useful criterion, which can be found in a paper of James [12] (see also [6, pp. 36-37] or [20, p. 2]). A nice, direct proof that the Schauder system is a

Schauder basis of $C[0,1]$ can be found in another article of James [13].

1.2. Proposition. *A sequence (e_n) of nonzero vectors in E is a basic sequence if and only if there is a constant $c \geq 1$ such that*

$$\|\sum_{j=1}^{m} \lambda_j e_j\| \leq c \|\sum_{j=1}^{n} \lambda_j e_j\|$$

for all $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$ *and* $m < n$ *in* $\mathbb{N}$.

1.3. Corollary. *(a) If (e_n) is a Schauder basis of a Banach space E, then (e'_n) is a basic sequence in E'.*

(b) If (e_n) is a Schauder basis of a reflexive Banach space E, then (e'_n) is a Schauder basis of E'.

Clearly every Banach space with a Schauder basis is necessarily separable, and the problem of whether every separable Banach space has a Schauder basis was posed by Banach [2, p. 111]. This problem, known as the basis problem, remained open for a long time, and was finally solved in the negative by Enflo [8]. We have however the following positive result.

1.4. Theorem. *Every infinite dimensional Banach space has a closed, infinite dimensional subspace with a Schauder basis.*

Theorem 1.4 was stated without proof by Banach [2, p. 238], and no proofs had been published before 1958, at which time several proofs appeared; see [3], [4] and [9]. A proof of Theorem 1.4, based on ideas of Mazur, made public by Pelczynski [27], can be found in [6, pp. 38-39] or [20, p. 4]. That proof rests on the following lemma, and will be the model for other proofs later on.

1.5. Lemma. *Let M be a finite dimensional subspace of an infinite dimensional Banach space E, and let $0 < \varepsilon < 1$. Then there exists $y \in S_E$ such that $\|x + \lambda y\| \geq (1 - \varepsilon)\|x\|$ for every $x \in M$ and $\lambda \in \mathbb{K}$.*

1.6. Corollary. *If E' has a reflexive, infinite dimensional subspace, then E has a reflexive, infinite dimensional quotient with a Schauder* *basis.*

Proof. By Theorem 1.4 E' has a reflexive, infinite dimensional subspace N with a Schauder basis. By Corollary 1.3 N' has a Schauder basis as well. Let $S : N \hookrightarrow E'$ be the inclusion mapping, let $J : E \hookrightarrow E''$ be the natural embedding, and let $T = S' \circ J : E \to N'$. Then one can readily verify that $T' = S$. Thus T' has a continuous inverse, and T is therefore surjective, by [36, p. 234, Th. 4.7-C].

In the case of a reflexive Banach space E, Corollary 1.6 is due to Pelczynski [28]. Corollary 1.6 provides a partial, positive solution to the following problem, posed by Pelczynski [28], and which seems to remain open.

1.7. Problem. *Does every infinite dimensional Banach space have an infinite dimensional quotient with a Schauder basis?*

The following two theorems also provide partial, positive solutions to Problem 1.7.

1.8. Theorem. *If E' has a subspace isomorphic to c_0, then E has a complemented subspace isomorphic to ℓ^1.*

1.9. Theorem. *Every separable, infinite dimensional Banach space has an infinite dimensional quotient with a Schauder basis.*

Theorem 1.8 is due to Bessaga and Pelczynski [3], whereas Theorem 1.9 is due to Johnson and Rosenthal [14]. We follow the proof of Johnson and Rosenthal [14] to prove both theorems. We first prove Theorem 1.8 in detail, and then indicate the necessary modifications to prove Theorem 1.9.

Proof of Theorem 1.8. Let $R : c_0 \hookrightarrow E'$ be an isomorphic embedding, and let (e_n) be the canonical Schauder basis of c_0. Then there are constants $b \geq a > 0$ such that

$$a \cdot \sup |\lambda_n| \leq \| \sum_{n=1}^{\infty} \lambda_n R e_n \| \leq b \cdot \sup |\lambda_n|$$

for every $(\lambda_n) \in c_0$. If we set $\psi_n = Re_n/||Re_n||$ for every n, then

$$\frac{a}{b}\sup|\lambda_n| \le ||\sum_{n=1}^{\infty}\lambda_n\psi_n|| \le \frac{b}{a}\sup|\lambda_n| \tag{1.1}$$

for every $(\lambda_n) \in c_0$. Since $\sum_{n=1}^{\infty}|\varphi(e_n)| < \infty$ for every $\varphi \in \ell^1$, it follows that

$$\sum_{n=1}^{\infty}|\psi_n(x)| < \infty \quad \text{for every} \quad x \in E. \tag{1.2}$$

Let $\varepsilon_1 > 0$ and consider the quotient mapping

$$Q_1 : E \to E/^{\perp}[\psi_1].$$

Since the closed unit ball of $E/^{\perp}[\psi_1] = [\psi_1]'$ is compact, there is a finite set $A_1 \subset B_E$ such that for each $u \in [\psi_1]'$ with $||u|| \le 1$ there is $x \in A_1$ such that

$$|u(\psi) - \psi(x)| \le \frac{\varepsilon_1}{3}||\psi|| \quad \text{for every} \quad \psi \in [\psi_1].$$

Then by (1.2) we can find $p_2 > 1$ such that

$$|\psi_{p_2}(x)| \le \frac{\varepsilon_1}{3} \quad \text{for every} \quad x \in A_1.$$

Now let (ε_n) be a sequence such that $0 < \varepsilon_n < 1$ and $\sum_{n=1}^{\infty}\varepsilon_n < \infty$. Then proceeding inductively we can find a strictly increasing sequence (p_n) in $\mathbb{N}$, and an increasing sequence of finite sets $A_n \subset B_E$ such that

(i) For each $u \in [\psi_{p_1}, \ldots, \psi_{p_n}]'$ with $||u|| \le 1$ there is $x \in A_n$ such that

$$|u(\psi) - \psi(x)| \le \frac{\varepsilon_n}{3}||\psi|| \quad \text{for every} \quad \psi \in [\psi_{p_1}, \ldots, \psi_{p_n}].$$

(ii) $|\psi_{p_{n+1}}(x)| \le \frac{\varepsilon_n}{3}$ for every $x \in A_n$.

To simplify notation we set $\varphi_n = \psi_{p_n}$ for every n.

We next claim that

$$||\varphi + \lambda\varphi_{n+1}|| \ge (1 - \varepsilon_n)||\varphi|| \quad \text{for all} \quad \varphi \in [\varphi_1, \ldots, \varphi_n], \ \lambda \in \mathbb{K}. \tag{1.3}$$

It suffices to prove (1.3) when $||\varphi|| = 1$. Since $||\varphi_{n+1}|| = 1$, (1.3) is obvious when $|\lambda| \ge 2$, so we assume $|\lambda| < 2$. Given $\varphi \in [\varphi_1, \ldots, \varphi_n]$

with $\|\varphi\| = 1$, there is $u \in [\varphi_1, \dots, \varphi_n]'$ such that $u(\varphi) = \|u\| = 1$. By (i) there is $x \in A_n \subset B_E$ such that $|u(\varphi) - \varphi(x)| \le \varepsilon_n/3$. By using this and (ii) we get that

$$\begin{aligned} \|\varphi + \lambda\varphi_{n+1}\| &\ge |(\varphi + \lambda\varphi_{n+1})(x)| \ge |\varphi(x)| - |\lambda\varphi_{n+1}(x)| \\ &\ge |u(\varphi)| - |u(\varphi) - \varphi(x)| - |\lambda\varphi_{n+1}(x)| \\ &\ge 1 - \frac{\varepsilon_n}{3} - 2\frac{\varepsilon_n}{3} = 1 - \varepsilon_n. \end{aligned}$$

We now use (1.3) to prove that (φ_n) is a basic sequence in E' in the same way Lemma 1.5 is used to prove Theorem 1.4. Indeed for $\lambda_1, \dots, \lambda_n$ in $\mathbb{K}$ and $m < n$ in $\mathbb{N}$ we have that

$$\|\sum_{j=1}^{n} \lambda_j\varphi_j\| \ge \prod_{k=m}^{n-1} (1 - \varepsilon_k)\|\sum_{j=1}^{m} \lambda_j\varphi_j\| \ge \prod_{k=m}^{\infty} (1 - \varepsilon_k)\|\sum_{j=1}^{m} \lambda_j\varphi_j\|.$$

Thus (φ_n) is a basic sequence in E', by Proposition 1.2. Let (φ_j') denote the sequence of coordinate functionals, and let (S_j) denote the sequence of canonical projections. Thus

$$S_m\varphi = \sum_{j=1}^{m} \varphi_j'(\varphi)\varphi_j \quad \text{for every} \quad \varphi \in [\varphi_j]$$

and $\|S_m\| \le \prod_{k=m}^{\infty}(1 - \varepsilon_k)^{-1}$. In particular $\lim \|S_m\| = 1$.

Now let $T : E \to [\varphi_j]'$ be defined by $Tx(\varphi) = \varphi(x)$ for every $x \in E$ and $\varphi \in [\varphi_j]$.

We claim that $T(E) \subset [\varphi_j']$. Indeed if $x \in E$ and $\varphi = \sum_{j=1}^{\infty}\varphi_j'(\varphi)\varphi_j \in [\varphi_j]$, then

$$Tx(\varphi) = \varphi(x) = \sum_{j=1}^{\infty} \varphi_j'(\varphi)\varphi_j(x).$$

But since $\sum_{j=1}^{\infty}|\varphi_j(x)| < \infty$, by (1.2), we conclude that $Tx = \sum_{j=1}^{\infty}\varphi_j(x)\varphi_j' \in [\varphi_j']$, as asserted.

We next show that $[\varphi_j'] \subset T(E)$. In certainly suffices to show that given $u \in span(\varphi_j')$ with $\|u\| \le 1$, and given $\varepsilon > 0$, we can find $x \in B_E$ such that $\|Tx - u\| \le 2\varepsilon$. Indeed given u and ε, choose $n \in \mathbb{N}$ such that $u \in [\varphi_1', \dots, \varphi_n']$, $\|S_j\| \le 2$ for every $j \ge n$ and $\sum_{j=n}^{\infty}\varepsilon_j \le \varepsilon$. Then

$$\|\varphi_j'\| \le \|S_j - S_{j-1}\| \le 4 \quad \text{for every} \quad j \ge n.$$

By (i) there is $x \in A_n \subset B_E$ such that

$$|u(\varphi) - \varphi(x)| \leq \frac{\varepsilon_n}{3}\|\varphi\| \quad \text{for every} \quad \varphi \in [\varphi_1, \ldots, \varphi_n].$$

Let $\varphi = \sum_{j=1}^{\infty} \varphi_j'(x)\varphi_j \in [\varphi_j]$, with $\|\varphi\| \leq 1$, and let

$$\psi = S_n\varphi = \sum_{j=1}^{n} \varphi_j'(\varphi)\varphi_j.$$

Observe that $\|\psi\| \leq 2$. Since $u \in [\varphi_1', \ldots, \varphi_n']$, we see that $u(\varphi) = u(S_n\varphi) = u(\psi)$. Thus

$$\begin{aligned} |(u - Tx)(\varphi)| &= |u(\varphi) - \varphi(x)| = |u(\varphi) - \sum_{j=1}^{\infty} \varphi_j'(\varphi)\varphi_j(x)| \\ &\leq |u(\varphi) - \sum_{j=1}^{n} \varphi_j'(\varphi)\varphi_j(x)| + |\sum_{j=n+1}^{\infty} \varphi_j'(\varphi)\varphi_j(x)| \\ &= |u(\psi) - \psi(x)| + \sum_{j=n+1}^{\infty} |\varphi_j'(\varphi)|\,|\varphi_j(x)| \\ &\leq \frac{\varepsilon_n}{3}\|\psi\| + \sum_{j=n+1}^{\infty} 4\frac{\varepsilon_j}{3} \leq \frac{2}{3}\varepsilon_n + \frac{4}{3}\varepsilon \leq 2\varepsilon. \end{aligned}$$

Thus we have shown that $T(E) = [\varphi_j']$. Now it follows readily from (1.1) that

$$\frac{a}{b}\sum_{j=1}^{\infty}|\lambda_j| \leq \|\sum_{j=1}^{\infty}\lambda_j\varphi_j'\| \leq \frac{b}{a}\sum_{j=1}^{\infty}|\lambda_j|$$

for every $(\lambda_j) \in \ell^1$, and therefore $[\varphi_j']$ is isomorphic to ℓ^1. Thus we have found a surjective operator $V \in \mathcal{L}(E;\ell^1)$. By the open mapping theorem there is a bounded sequence (x_n) in E such that $Vx_n = e_n$ for every n. If we define $U \in \mathcal{L}(\ell^1; E)$ by $Ue_n = x_n$ for every n, then $V \circ U$ is the identity, and the proof is complete.

Proof of Theorem 1.9. Let E be a separable, infinite dimensional Banach space. Then there is a sequence (M_n) of finite dimensional subspaces of E such that $M_n \subsetneq M_{n+1}$ for every n and $M = \bigcup_{n=1}^{\infty} M_n$ is dense in E. By the Hahn-Banach theorem there is a sequence (ψ_n) in E' such that $\|\psi_n\| = 1$ and $\psi_n = 0$ on M_n for every n. Whence

$$\sum_{n=1}^{\infty} |\psi_n(x)| < \infty \quad \text{for every} \quad x \in M.$$

Let (ε_n) be a sequence such that $0 < \varepsilon_n < 1$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Since M is dense in E, the proof of Theorem 1.8 yields a strictly increasing sequence (p_n) in $\mathbb{N}$ and an increasing sequence of finite sets $A_n \subset B_E \cap M$ which verify conditions (i) and (ii) there. If we set $\varphi_n = \psi_{p_n}$ for every n, then it follows as before that (φ_n) is a basic sequence in E'. If (φ'_n) is the corresponding sequence of coordinate functionals, and $T : E \to [\varphi_n]'$ is defined by $Tx(\varphi) = \varphi(x)$ for every $x \in E$ and $\varphi \in [\varphi_n]$, then it follows as before that T maps E onto $[\varphi'_n]$.

Theorem 1.9 shows that Problem 1.7 is equivalent to the following problem.

1.10. Problem. *Does every infinite dimensional Banach space have a separable, infinite dimensional quotient?*

Problem 1.10 was mentioned by Rosenthal [30] in 1969, but the problem is probably much older. Actually a variant of Problem 1.10 was mentioned by Banach in [2, p. 244].

2 Quasi-Complemented Subspaces of Banach Spaces

Let M be a closed subspace of a Banach space E. Recall that M is said to be *complemented* in E if there is a closed subspace N of E such that $M \cap N = \{0\}$ and $M + N = E$. By using the closed graph theorem one can readily prove that M *is complemented in* E *if and only if there is a continuous projection* P *from* E *onto* M.

Following Murray [22] we will say that M is *quasi-complemented* in E if there is a closed subspace N of E such that $M \cap N = \{0\}$ and $M + N$ is dense in E. One can readily prove that M *is quasi-complemented in*

E if and only if there is a closed, densely defined projection P with range M.

Murray [22] posed the problem of whether every closed subspace of a Banach space is quasi-complemented, and he himself gave a partial, positive solution in [23], where he proved that *every closed subspace of a separable and reflexive Banach space is quasi-complemented.* Shortly afterwards Mackey [21] improved that result as follows.

2.1. Theorem. *Every closed subspace of a separable Banach space is quasi-complemented.*

Before proving Theorem 2.1 we need two auxiliary lemmas.

2.2. Lemma. *Let E be a vector space, and let F be a subspace of E^* which separates the points of E. Suppose that E and F are at most countable dimensional. Then there are a Hamel basis (x_n) of E and a Hamel basis (φ_n) of F such that $\varphi_m(x_n) = \delta_{mn}$ for all m, n.*

Proof. (a) First assume F finite dimensional. It is well know that if $\varphi_1, \ldots, \varphi_n$, ψ are linear functionals on E such that $\cap_{i=1}^n Ker\varphi_i \subset Ker\psi$, then ψ is a linear combination of $\varphi_1, \ldots, \varphi_n$. Now let $(\varphi_1, \ldots, \varphi_n)$ be any Hamel basis of F. By the preceding remark we can find $x_1, \ldots, x_n \in E$ such that $\varphi_i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Whence it follows that $x - \sum_{j=1}^n \varphi_j(x)x_j \in {}^\perp F$ for every $x \in E$. But since F separates the points of E, we see that ${}^\perp F = \{0\}$. Whence $(x_1, \ldots, x_n)$ and $(\varphi_1, \ldots, \varphi_n)$ are Hamel bases of E and F with the required property.

(b) Next assume F infinite dimensional. Let (y_n) and (ψ_n) be Hamel bases of E and F, respectively. Let $m_1 = 1$ and $x_1 = y_{m_1}$. Let n_1 be the first integer such that $\psi_{n_1}(x_1) \neq 0$ and let $\varphi_1 = \psi_{n_1}/\psi_{n_1}(x_1)$. Next let n_2 be the first integer such that $\psi_{n_2} \notin span\{\varphi_1\}$, and let

$$\varphi_2 = \psi_{n_2} - \psi_{n_2}(x_1)\varphi_1.$$

Let m_2 be the first integer such that $\varphi_2(y_{m_2}) \neq 0$, and let

$$x_2 = \frac{y_{m_2} - \varphi_1(y_{m_2})x_1}{\varphi_2(y_{m_2})}.$$

Next let m_3 be the first integer such that $y_{m_3} \notin span\{x_1, x_2\}$, and let

$$x_3 = y_{m_3} - \varphi_1(y_{m_3})x_1 - \varphi_2(y_{m_3})x_2.$$

Let n_3 be the first integer such that $\psi_{n_3}(x_3) \neq 0$, and let

$$\varphi_3 = \frac{\psi_{n_3} - \psi_{n_3}(x_1)\varphi_1 - \psi_{n_3}(x_2)\varphi_2}{\psi_{n_3}(x_3)}.$$

We continue in this way, by constructing x_k before φ_k in step k if k is odd, and the other way around if k is even. It is clear that $\varphi_j(x_k) = \delta_{jk}$ for all j, k. It is also clear that $\{m_1, \ldots, m_{2p}\} \supset \{1, \ldots, p\}$ and $\{n_1, \ldots, n_{2p}\} \supset \{1, \ldots, p\}$, and therefore

$$span\{x_1, \ldots, x_{2p}\} \supset span\{y_1, \ldots, y_p\}$$

and

$$span\{\varphi_1, \ldots, \varphi_{2p}\} \supset span\{\psi_1, \ldots, \psi_p\}$$

for every p.

Some readers will recognize in the proof of Lemma 2.2 the proof of a result of Markushevich (see [20, pp. 43-44]) on the existence of biorthogonal systems.

2.3. Lemma. *Let E be a vector space, and let F be a subspace of E^* which separates the points of E. Suppose that E and F are at most countable dimensional. Then for each $\sigma(E, F)$-closed subspace M of E, there is a $\sigma(E, F)$-closed subspace N of E such that $M + N = E$ and $M^\perp + N^\perp = F$. Here $M^\perp$ denotes the orthogonal of M with respect to the dual pair $\langle E, F \rangle$.*

Proof. Let $R : E^* \to M^*$ be the restriction mapping, and let $G = R(F)$. By applying Lemma 2.2 to M and G we can find a Hamel basis (x_n) of M and a Hamel basis (g_n) of G such that $g_m(x_n) = \delta_{mn}$ for all m, n. Since G can be identified with $F/M^\perp$, the sequence $(g_n) \subset G$ yields a sequence $(f_n) \subset F$ such that

$$M^\perp + span(f_n) = F \quad , \quad f_m(x_n) = \delta_{mn}.$$

Next let $S : F^* \to (M^\perp)^*$ be the restriction mapping, and let $H = S(E)$. By applying Lemma 2.2 to $M^\perp$ and H we can find a Hamel basis

(φ_n) of $M^\perp$ and a Hamel basis (h_n) of H such that $h_m(\varphi_n) = \delta_{mn}$ for all m, n. Since M is $\sigma(E, F)$-closed, $M = M^{\perp\perp}$, and H can be identified with $E/M^{\perp\perp} = E/M$. Hence the sequence $(h_n) \subset H$ yields a sequence $(z_n) \subset E$ such that

$$M + span(z_n) = E \quad , \quad \varphi_m(z_n) = \delta_{mn}.$$

Next define

$$y_n = z_n - \sum_{j=1}^{n} f_j(z_n)x_j \quad , \quad \psi_n = f_n - \sum_{j=1}^{n} f_n(z_{j-1})\varphi_{j-1} \ ,$$

where $z_0 = 0$ and $\varphi_0 = 0$. Keeping in mind that $\varphi_m(x_n) = 0$ for all m, n, it follows easily that $\psi_m(y_n) = 0$ for all m, n.

Let $N = span(y_n)$. Then $(\psi_n) \subset N^\perp$ and

$$\begin{aligned} F &= M^\perp + span(f_n) = span(\varphi_n) + span(f_n) \\ &= span(\varphi_n) + span(\psi_n) = M^\perp + N^\perp. \end{aligned}$$

Likewise we get that

$$\begin{aligned} E &= M + span(z_n) = span(x_n) + span(z_n) \\ &= span(x_n) + span(y_n) = M + N. \end{aligned}$$

Since $F = M^\perp + N^\perp$, it follows that $M^{\perp\perp} \cap N^{\perp\perp} = \{0\}$. Since $E = M + N = M^{\perp\perp} + N^{\perp\perp}$, it follows that $N = N^{\perp\perp}$. Thus N is $\sigma(E, F)$-closed and the proof is complete.

Proof of Theorem 2.1. Let M be a closed subspace of a separable Banach space E. Since E is separable, the Hahn-Banach theorem yields a sequence (φ_n) in E' which separates the points of E. Likewise there is a sequence (μ_n) in $(E/M)'$ which separates the points of E/M. Since $(E/M)'$ can be identified with $M^\perp$, the sequence $(\mu_n) \subset (E/M)'$ yields a sequence $(\psi_n) \subset M^\perp$ such that

$$M = \bigcap_{n=1}^{\infty} Ker\psi_n.$$

Let

$$F_1 = span(\varphi_n) + span(\psi_n).$$

Since E is separable, there are sequences $(x_n) \subset M$ and $(y_n) \subset E$ such that $span(x_n)$ is dense in M, whereas $span(y_n)$ is dense in E. Let

$$E_1 = span(x_n) + span(y_n)$$

and let

$$M_1 = M \cap E_1 = \bigcap_{n=1}^{\infty} (E_1 \cap \ Ker\psi_n),$$

so that M_1 is $\sigma(E_1, F_1)$-closed. By Lemma 2.3 there is a $\sigma(E_1, F_1)$-closed subspace N_1 of E_1 such that

$$M_1 + N_1 = E_1 \quad , \quad M_1^{\perp} + N_1^{\perp} = F_1.$$

Let $N = \overline{N_1}^E$. Clearly $M + N$ is dense in E. We claim that $M \cap N = \{0\}$. Indeed let $x \in M \cap N$. Since $N = \overline{N_1}^E$ we see that $\varphi(x) = 0$ for every $\varphi \in N_1^{\perp}$. Since we can readily verify that $M = \overline{M_1}^E$, it follows that $\varphi(x) = 0$ for every $\varphi \in M_1^{\perp}$. Since $F_1 = M_1^{\perp} + N_1^{\perp}$, it follows that $\varphi(x) = 0$ for every $\varphi \in F_1$, and therefore $x = 0$, as asserted.

Later on Lindenstrauss [18] gave another partial positive solution to Murray's problem by proving that *every closed subspace of a reflexive Banach space is quasi-complemented.* But shortly afterwards Lindenstrauss [19] ended up solving Murray's problem in the negative by proving that *if I is any uncountable set, then $c_0(I)$ is not quasi-complemented in $\ell^{\infty}(I)$.* But the following variant of Murray's problem, posed by Rosenthal [30], seems to remain open.

2.4. Problem. *Does every infinite dimensional Banach space have a separable, infinite dimensional, quasi-complemented subspace?*

The following result of Rosenthal [30] shows that Problem 2.4 is equivalent to Problem 1.10.

2.5. Theorem. *Let E be a Banach space. Then E has a separable, infinite dimensional, quasi-complemented subspace if and only if E has a separable, infinite dimensional quotient.*

Proof. First assume that E has a separable, infinite dimensional, quasi-complemented subspace M. Let N be a closed subspace of E such that $M \cap N = \{0\}$ and $M + N$ is dense in E. Let $Q : E \to E/N$ be the quotient mapping. Then one can readily see that $Q(M)$ is dense in E/N and $Q|M$ is injective. Hence E/N is separable and infinite dimensional.

Next assume that there is a closed subspace N of E such that E/N is separable and infinite dimensional. Let $Q : E \to E/N$ be the quotient mapping, and let (b_n) be a countable, dense subset of E/N. Choose $(a_n) \subset E$ such that $Qa_n = b_n$ for every n, and let $M = [a_n]$.

We claim that $M + N$ is dense in E. Indeed given $x \in E$ we have that $Qx = \lim b_{n_k}$ for a suitable sequence (n_k). By the open mapping theorem there is a sequence $(x_k) \subset E$ such that $\lim x_k = x$ and $Qx_k = b_{n_k}$ for every k. Thus $Qx_k = b_{n_k} = Qa_{n_k}$, $x_k - a_{n_k} \in N$ and $x_k \in M + N$ for every k.

If $M \cap N = \{0\}$, the proof is complete, so assume $M \cap N \neq \{0\}$. By Theorem 2.1 the subspace $N_1 = M \cap N$ is quasi-complemented in M. Thus there is a closed subspace M_1 of M such that $M_1 \cap N_1 = \{0\}$ and $M_1 + N_1$ is dense in M. Hence

$$E = \overline{M + N} = \overline{M_1 + N_1 + N} = \overline{M_1 + N}$$

and

$$M_1 \cap N = M_1 \cap M \cap N = M_1 \cap N_1 = \{0\}.$$

If M_1 were finite dimensional, then E/N would be isomorphic to M_1 and hence finite dimensional. Thus M_1 is infinite dimensional and the proof is complete.

Let us remark that Rosenthal [30] proved that *c_0 is quasi-complemented in ℓ^∞*. It follows from Theorem 2.5 that *ℓ^∞ has a separable, infinite dimensional quotient.* Since $\ell^\infty = C(\beta I\!N)$, this follows also from a result of Lacey [17], who proved that *if X is any infinite compact, Hausdorff space, then $C(X)$ has a separable, infinite dimensional quotient.* In Section 4 we will see that *ℓ^∞ has a quotient isomorphic to ℓ^2.*

3 Nonbarrelled Subspaces of Banach Spaces

Every closed, convex, balanced and absorbing subset of a locally convex space is called a *barrel*. A locally convex space is said to be *barrelled* if every barrel is a neighborhood of zero. It follows from the open mapping theorem that closed subspaces of Banach spaces are always barrelled, and the following proposition furnishes a simple procedure for constructing nonbarrelled subspaces of Banach spaces.

3.1. Proposition. *Let (M_n) be a sequence of closed subspaces of a Banach space E such that $M_n \subsetneqq M_{n+1}$ for every n. Then the subspace $M = \bigcup_{n=1}^{\infty} M_n$, with the induced topology, is not barrelled.*

Proof. Assume that M, with the induced topology τ_0, is a barrelled space. Let τ_1 denote the inductive limit topology on M, that is $(M, \tau_1) = \operatorname{ind} M_n$.

We first show that $(M,\tau_0)' = (M,\tau_1)'$. Certainly $\tau_0 \leq \tau_1$ and therefore $(M,\tau_0)' \subset (M,\tau_1)'$. To show the reverse inclusion let $\varphi \in (M,\tau_1)'$ and let $\varphi_n = \varphi | M_n$ for every n. Thus $\varphi_n \in M_n'$ and by the Hahn-Banach theorem there is $\tilde{\varphi}_n \in (M,\tau_0)'$ such that $\tilde{\varphi}_n | M_n = \varphi_n$ for every n. Since $M = \bigcup_{n=1}^{\infty} M_n$ we see that $\varphi(x) = \lim \tilde{\varphi}_n(x)$ for every $x \in M$. Since (M,τ_0) is barrelled, we conclude that $\varphi \in (M,\tau_0)'$, by the Banach-Steinhaus theorem.

We next show that $\tau_0 = \tau_1$. Indeed let V be a closed, convex, balanced neighborhood of zero in (M,τ_1). Since $(M,\tau_1)' = (M,\tau_0)'$, V is also closed, and therefore a barrel, in (M,τ_0). Since (M,τ_0) is barrelled, V is a neighborhood of zero in (M,τ_0).

To complete the proof choose $x_n \in M_n \backslash M_{n-1}$ for every n. Since (M,τ_0) is a normed space, the sequence $(x_k/\|x_k\|)$ is bounded in $(M,\tau_0) = (M,\tau_1)$. But since $(M,\tau_1) = \operatorname{ind} M_n$ is a strict inductive limit, the sequence $(x_k/\|x_k\|)$ is contained and bounded in some M_n, a contradiction.

The next result is due to Saxon and Wilanšky [32]. See also [38, p. 255].

3.2. Theorem. *A Banach space E has a separable, infinite dimen-*

sional quotient if and only if E has a dense, nonbarrelled subspace.

This theorem shows that Problem 1.10 is equivalent to the following problem.

3.3. Problem. *Does every infinite dimensional Banach space have a dense, nonbarrelled subspace?*

Proof of Theorem 3.2. We first assume that E has a separable, infinite dimensional quotient space E/S. Let $Q : E \to E/S$ be the quotient mapping. There is a sequence (N_n) of finite dimensional subspaces of E/S such that $N_n \subsetneq N_{n+1}$ for every n and $\bigcup_{n=1}^{\infty} N_n$ is dense in E/S. Set $M_n = Q^{-1}(N_n)$ for every n. Clearly $M_n \subsetneq M_{n+1}$ for every n, and thus by Proposition 3.1 the subspace $\bigcup_{n=1}^{\infty} M_n$ is not barrelled. We claim that $\bigcup_{n=1}^{\infty} M_n$ is dense in E. Indeed given $x \in E$ we have that $Qx = \lim y_k$, where $y_k \in N_{n_k}$ for every k. By the open mapping theorem there is a sequence $(x_k) \subset E$ such that $x = \lim x_k$ and $Qx_k = y_k$ for every k. Thus $x_k \in M_{n_k}$ for every k, as we wanted.

Conversely assume that E has a dense, nonbarrelled subspace M_0. Let B_0 be a barrel in M_0 which is not a 0-neighborhood in M_0. Without loss of generality we way assume that B_0 is closed in E. Indeed if B_0 were not closed in E, then we would consider $\overline{B_0}$ instead of B_0. For $\overline{B_0}$ is a barrel in span $\overline{B_0}$, and $\overline{B_0}$ is not a 0-neighborhood in span $\overline{B_0}$, since $B_0 = \overline{B_0} \cap M_0$.

Since $M_0 =$ *span* B_0 is not barrelled, $M_0 \neq E$, and hence there is $x_1 \in S_E$ such that $x_1 \notin M_0$. Since in particular $x_1 \notin 2B_0$, the Hahn-Banach separation theorem yields $\varphi_1 \in E'$ such that $\varphi_1(x_1) = 1$ and $|\varphi_1| \leq 1/2$ on B_0.

Let

$$B_1 = B_0 + \{\alpha_1 x_1 : |\alpha_1| \leq 1\}$$

and let $M_1 = \text{span } B_1$. Then B_1 is closed in E, B_1 is a barrel in M_1 and B_1 is not a 0-neighborhood in M_1, since $B_0 = B_1 \cap M_0$. Whence M_1 is not barrelled, and in particular $M_1 \neq E$.

We claim that Ker $\varphi_1 \not\subset M_1$. Indeed Ker $\varphi_1 \subset M_1$ and $M_1 \neq E$ would imply Ker $\varphi_1 = M_1$ and M_1 would be barrelled. Then choose $x_2 \in S_E$ such that $x_2 \in$ Ker φ_1 and $x_2 \notin M_1$. In particular $x_2 \notin$

$4B_1$ and the Hahn-Banach separation theorem yields $\varphi_2 \in E'$ such that $\varphi_2(x_2) = 1$ and $|\varphi_2| \le 1/4$ on B_1.

Let

$$B_2 = B_1 + \{\alpha_2 x_2 : |\alpha_2| \le 1\} = B_0 + \{\alpha_1 x_1 + \alpha_2 x_2 : |\alpha_1| \le 1, |\alpha_2| \le 1\}$$

and let $M_2 = \text{span } B_2$. As before M_2 is not barrelled and therefore $M_2 \neq E$.

We claim that $\text{Ker } \varphi_1 \cap \text{ Ker } \varphi_2 \not\subset M_2$. Indeed if we assume $\text{Ker } \varphi_1 \cap \text{ Ker } \varphi_2 \subset M_2$ then

$$\begin{aligned} \text{Ker } \varphi_1 &= \mathbb{K} x_2 \oplus \text{ Ker } (\varphi_2 | \text{Ker } \varphi_1) \\ &= \mathbb{K} x_2 \oplus (\text{ Ker } \varphi_1 \cap \text{ Ker } \varphi_2) \subset \mathbb{K} x_2 + M_2 = M_2 \end{aligned}$$

but $\text{Ker } \varphi_1 \subset M_2$ and $M_2 \neq E$ would imply $\text{Ker } \varphi_1 = M_2$ and M_2 would be barrelled. Then choose $x_3 \in S_E$ such that $x_3 \in \text{ Ker } \varphi_1 \cap \text{ Ker } \varphi_2$ and $x_3 \notin M_2$. In particular $x_3 \notin 8B_2$ and the Hahn-Banach separation theorem yields φ_3 e E' such that $\varphi_3(x_3) = 1$ and $|\varphi_3| \le 1/8$ on B_2.

Proceeding inductively we can find sequences $(x_n) \subset E$ and $(\varphi_n) \subset E'$, and a sequence of closed, convex, balanced sets $B_n \subset E$ such that $||x_n|| = 1, \varphi_n(x_n) = 1, \varphi_m(x_n) = 0$ whenever $m < n$, and $|\varphi_n| \le 2^{-n}$ on B_{n-1}. Furthermore for $n \ge 1$

$$B_n = B_{n-1} + \{\alpha_n x_n : |\alpha_n| \le 1\} = B_0 + \{\sum_{i=1}^{n} \alpha_i x_i : |\alpha_i| \le 1\}.$$

If $M_n = \text{span } B_n$, then $x_n \notin M_{n-1}$ for every n. Moreover

$$E = \mathbb{K} x_1 \oplus \ldots \oplus \mathbb{K} x_n \oplus \bigcap_{i=1}^{n} \text{ Ker } \varphi_i$$

For every $n \in \mathbb{N}$ set

$$N_n = \bigcap_{i=n}^{\infty} \text{ Ker } \varphi_i \quad , \quad N = \bigcup_{n=1}^{\infty} N_n$$

We claim that given $b \in B_0$ and $n \in \mathbb{N}$, there is $y_n \in N_n$ such that $||y_n - b|| \le 2^{-n+1}$. Indeed if we define $\alpha_n = -\varphi_n(b)$, then $|\alpha_n| \le 2^{-n}$.

If we next define $\alpha_{n+1} = -\varphi_{n+1}(b + \alpha_n x_n)$, then $|\alpha_{n+1}| \leq 2^{-n-1}$. Thus we may inductively define $(\alpha_j)_{j=n}^{\infty}$ by

$$\alpha_j = -\varphi_j(b + \sum_{i=n}^{j-1} \alpha_i x_i)$$

and $|\alpha_j| \leq 2^{-j}$ for every $j \geq n$. If we set $y_n = b + \sum_{i=n}^{\infty} \alpha_i x_i$ then $\|y_n - b\| \leq 2^{-n+1}$. Furthermore $y_n \in N_n$ since for $j \geq n$ we have that

$$\varphi_j(y_n) = \varphi_j(b + \sum_{i=n}^{j-1} \alpha_i x_i + \alpha_j x_j + \sum_{i=j+1}^{\infty} \alpha_i x_i) = -\alpha_j + \alpha_j + 0 = 0$$

Thus our claim has been proved, and whence it follows that N is dense in $M_0 =$ span B_0. Thus N is dense in E.

We next show that dim $(N_n/N_{n-1}) = 1$ for every n. To show this observe that $B_0 \not\subset$ Ker φ_{n-1} since $M_0 =$ span B_0 is dense in E. Let $b \in B_0$ such that $b \notin$ Ker φ_{n-1}. Then the previous claim yields $y_n \in N_n$ such that $\|y_n - b\| \leq 2^{-n+1}$. Since $y_n = b + \sum_{i=n}^{\infty} \alpha_i x_i$ we see that $\varphi_{n-1}(y_n) = \varphi_{n-1}(b) \neq 0$. Thus $y_n \in N_n$ but $y_n \notin N_{n-1}$, and it follows that $N_n = N_{n-1} \oplus I\!K y_n$, as we wanted.

To complete the proof of the theorem we show that the quotient E/N_1 is separable. To see this write $N_n = N_{n-1} \oplus I\!K y_n$ for every $n \geq 2$. Let $\psi \in (E/N_1)'$ and suppose that $\psi \circ Q_1(y_n) = 0$ for every $n \geq 2$, where $Q_1 : E \to E/N_1$ is the quotient mapping. Since

$$N_n = N_1 \oplus I\!K y_2 \oplus \ldots \oplus I\!K y_n$$

every $z_n \in N_n$ can be uniquely written as

$$z_n = z_1 + \sum_{i=2}^{n} \lambda_i x_i$$

with $z_1 \in N_1$ and $\lambda_i \in I\!K$. Whence $\psi \circ Q_1(z_n) = 0$. Thus $\psi \circ Q_1(z) = 0$ for every $z \in N$, and therefore for every $z \in E$. By the Hahn-Banach theorem span $(Q_1 y_n)$ is dense in E/N_1.

3.4. Corollary. *Let $T \in \mathcal{L}(E;F)$, and suppose that $T(E)$ is dense in F, but $T(E) \neq F$. Then F has a separable, infinite dimensional quotient.*

Proof. If $T(E)$ were barrelled, then $T : E \to T(E)$ would be an open mapping. Thus $T(E)$ would be complete, and therefore $T(E) = F$. We have thus shown that $T(E)$ is a dense, nonbarrelled subspace of F. By Theorem 3.2 F has a separable, infinite dimensional quotient.

A Banach-space E is said to be *weakly compactly generated* if there is a convex, balanced, weakly compact set $K \subset E$ such that the Banach space E_K is dense in E. Reflexive Banach spaces and separable Banach spaces are weakly compactly generated.

3.5. Corollary. *Every infinite dimensional, weakly compactly generated Banach space has a separable, infinite dimensional quotient space.*

Proof. Let E be an infinite dimensional, weakly compactly generated Banach space, and let K be a convex, balanced, weakly compact subset of E such that the Banach space E_K is dense in E. By a result of Davis et al. [5] (see also [6, pp. 227-228]) every weakly compact operator between Banach spaces factors through some reflexive Banach space. Thus the inclusion mapping $E_K \hookrightarrow E$ factors through some reflexive Banach space F. If the mapping $F \to E$ is surjective, then E is also reflexive, and hence has a separable, infinite dimensional quotient by Corollary 1.6. If the mapping $F \to E$ is not surjective, then it has a dense image, and E has a separable, infinite dimensional quotient by Corollary 3.4.

Corollaries 3.4 and 3.5 can be found in the book of Wilansky [38, p.256].

4 Other Banach Spaces with Separable Quotients

In this section we prove the following theorems.

4.1. Theorem. *If E has a subspace isomorphic to ℓ^1, then E has a quotient isomorphic to ℓ^2.*

4.2. Theorem. *Let E be a real Banach space. If E' has a subspace isomorphic to ℓ^1, then E has a quotient isomorphic to c_0 or ℓ^2.*

First proof of Theorem 4.1. Since E has a subspace isomorphic to ℓ^1, it follows from a theorem of Pelczynski [29] that E' has a subspace isomorphic to $L^1[0,1]$. Actually Pelczynski proved that theorem under the additional assumption that E be separable, but an exercise in Diestel's book [6, pp. 211-212] allows us to drop the separability assumption. Now $L^1[0,1]$ has a subspace isomorphic to ℓ^2. This follows from the Khintchine's inequalities: If (r_n) is the sequence of Rademacher functions, then for each p, $1 \le p < \infty$, there are constants $b_p \ge a_p > 0$ such that

$$a_p(\sum_{j=1}^{n}|\lambda_j|^2)^{1/2} \le (\int_0^1 |\sum_{j=1}^{n}\lambda_j r_j(t)|^p dt)^{1/p} \le b_p(\sum_{j=1}^{n}|\lambda_j|^2)^{1/2}$$

for all $\lambda_1 \dots \lambda_n \in \mathbb{K}$ and $n \in \mathbb{N}$. The Rademacher functions r_n are defined by $r_1 = h_1, r_2 = h_2, r_3 = h_3 + h_4, r_4 = h_5 + h_6 + h_7 + h_8$, etc. where (h_n) is the Haar system. See [6, pp. 105-107] or [20, p.66]. Thus E' has a subspace isomorphic to ℓ^2, and the proof of Corollary 1.6 shows that E has a quotient isomorphic to ℓ^2.

Second proof of Theorem 4.1. The following, more "modern" proof of Theorem 4.1 was suggested by the referee. If $1 \le p < \infty$ then an operator $T \in \mathcal{L}(E;F)$ is said to be *absolutely p-summing* if there is a constant $c_p > 0$ such that

$$(\sum_{j=1}^{n}||Tx_j||^p)^{1/p} \le c_p \sup\{(\sum_{j=1}^{n}|x'(x_j)|^p)^{1/p} : x' \in B_{E'}\}$$

for all $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$. By a result of Banach and Mazur (see [6, pp. 73-74] or [20, p. 108]) there is a surjective operator $T \in \mathcal{L}(\ell^1;\ell^2)$. By a result of Grothendieck (see [20, pp. 69-70]) the operator T is absolutely 1-summing, and therefore absolutely 2-summing. By a result of Pietsch (see[20, pp. 64-65]) there is a probability measure μ and there are operators $U \in \mathcal{L}(\ell^1; L^\infty(\mu))$ and $V \in \mathcal{L}(L^2(\mu);\ell^2)$ such that the following diagram commutes.

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & \ell^2 \\ U\downarrow & & \uparrow V \\ L^\infty(\mu) & \underset{I}{\hookrightarrow} & L^2(\mu) \end{array}$$

Now let $S \in \mathcal{L}(\ell^1; E)$ be an isomorphic embedding. By a result of Nachbin [24] the space $L^\infty(\mu)$ has the Hahn-Banach extension property. Thus there is an operator $\tilde{U} \in \mathcal{L}(E; L^\infty(\mu))$ such that $\tilde{U} \circ S = U$. Since the operator $T \in \mathcal{L}(\ell^1; \ell^2)$ is surjective, the operator $V \circ I \circ \tilde{U} \in \mathcal{L}(E; \ell^2)$ is surjective as well.

Since ℓ^∞ has a subspace isomorphic to ℓ^1 (this follows easily from Proposition 4.5 below), Theorem 4.1 shows that *ℓ^∞ has a quotient isomorphic to ℓ^2.*

Let (e_n) and (f_n) be Schauder bases of E and F, respectively. (e_n) and (f_n) are said to be *equivalent* if there is a topological isomorphism $T : E \to F$ such that $Te_n = f_n$ for every n. One can readily see that *a sequence (x_n) in E is a basic sequence equivalent to the canonical Schauder basis of ℓ^1 if and only if there are constants $b \geq a > 0$ such that*

$$a \sum_{j=1}^{n} |\lambda_j| \leq \|\sum_{j=1}^{n} \lambda_j x_j\| \leq b \sum_{j=1}^{n} |\lambda_j|$$

for all $\lambda_1 \dots \lambda_n \in \mathbb{K}$ and $n \in \mathbb{N}$. Likewise one can readily see that *a sequence (x_n) in E is a basic sequence equivalent to the canonical Schauder basis of c_0 if and only if there are constants $b \geq a > 0$ such that*

$$a \sup |\lambda_j| \leq \|\sum_{j=1}^{n} \lambda_j x_j\| \leq b \sup |\lambda_j|$$

for all $\lambda_1 \dots \lambda_n \in \mathbb{K}$ and $n \in \mathbb{N}$.

Theorem 4.2 follows from the following two more precise theorems.

4.3. Theorem. *Suppose that E' contains a normalized basic sequence (ψ_n) such that*

(i) (ψ_n) is equivalent to the canonical Schauder basis of ℓ^1.

(ii) $\lim \psi_n(x) = 0$ for every $x \in E$.

Then E has a quotient isomorphic to c_0.

4.4. Theorem. *Let E be a real Banach space such that*

(i) E' has a subspace isomorphic to ℓ^1.

(ii) Whenever (ψ_n) is a basic sequence in E' which is equivalent to the canonical Schauder basis of ℓ^1, then $(\psi_n(x))$ does not converge to

zero for some $x \in E$.
Then:

(a) E has a subspace isomorphic to ℓ^1.
(b) E has a quotient isomorphic to ℓ^2.

Theorem 4.3 is due to Johnson and Rosenthal [14], whereas Theorem 4.4 is due to Hagler and Johnson [11]. See also [6, pp. 219-222].

Proof of Theorem 4.3. We can find sequences (p_n) and (A_n) as in the proof of Theorem 1.8, since there we did not use the full fact that $\sum_{n=1}^{\infty} |\psi_n(x)| < \infty$ for every $x \in E$, but only the weaker fact that $\lim \psi_n(x) = 0$ for every $x \in E$. If we set $\varphi_n = \psi_{p_n}$ for every n,, then it follows as before that (φ_n) is a basic sequence in E'. Since (ψ_n) is equivalent to the canonical Schauder basis of ℓ^1, there are constants $b \geq a > 0$ such that

$$a \sum_{j=1}^{n} |\lambda_j| \leq || \sum_{j=1}^{n} \lambda_j \psi_j || \leq b \sum_{j=1}^{n} |\lambda_j|$$

for all $\lambda_1 \ldots \lambda_n \in \mathbb{K}$ and $n \in \mathbb{N}$, and clearly the subsequence $(\varphi_n) = (\psi_{\varphi_n})$ satisfies the same inequalities. If (φ'_n) is the corresponding sequence of coordinate functionals, then one can readily see that

$$\frac{1}{b} \sup |\lambda_j| \leq || \sum_{j=1}^{n} \lambda_j \varphi'_j || \leq \frac{1}{a} \sup |\lambda_j|$$

for all $\lambda_1 \ldots \lambda_n \in \mathbb{K}$ and $n \in \mathbb{N}$. Thus the basic sequence (φ'_n) is equivalent to the canonical Schauder basis of c_0. As before let $T : E \to [\varphi_n]'$ be defined by $Tx(\varphi) = \varphi(x)$ for every $x \in E$ and $\varphi \in [\varphi_n]$.

We claim that $T(E) \subset [\varphi'_n]$. Indeed if $x \in E$ and $\varphi = \sum_{n=1}^{\infty} \varphi'_n(\varphi)\varphi_n \in [\varphi_n]$, then

$$Tx(\varphi) = \varphi(x) = \sum_{n=1}^{\infty} \varphi'_n(\varphi)\varphi_n(x)$$

Thus $Tx = \sum_{n=1}^{\infty} \varphi_n(x)\varphi'_n \in [\varphi'_n]$, since $(\varphi_n(x)) \in c_0$.

Finally the proof of Theorem 1.8 show that $[\varphi'_n] \subset T(E)$, and the proof is complete.

The proof of Theorem 4.4 is more involved and will be given after some preliminaries. I do not know if Theorem 4.4 is true in the case of complex Banach spaces.

A sequence (X_n) of nonvoid subsets of a set X is said to be a *tree* if for every $n \in \mathbb{N}, X_{2n}$ and X_{2n+1} are disjoint subset of X_n. The notion of tree and the next result are due to Pelczynski [29]. See also [6, pp. 204-205].

4.5. Proposition. *Let $(X_k)_{k=1}^{\infty}$ be a tree of subsets of a set X. Let $(f_n)_{n=0}^{\infty}$ be a bounded sequence in real $\ell^{\infty}(X)$. Suppose there is $\delta > 0$ such that*

$$(-1)^k f_n(x) \geq \delta \quad \text{whenever} \quad x \in X_k, 2^n \leq k < 2^{n+1}, n \geq 0. \tag{4.1}$$

Then (f_n) is a basic sequence equivalente to the canonical Schauder basis of ℓ^1.

Proof. Let $\lambda_0 \ldots, \lambda_n \in \mathbb{R}$. Then certainly

$$\|\sum_{j=0}^{n} \lambda_j f_j\| \leq \sup \|f_j\| \sum_{j=0}^{n} |\lambda_j| \, .$$

To complete the proof we will show that

$$\|\sum_{j=0}^{n} \lambda_j f_j\| \geq \delta \sum_{j=0}^{n} |\lambda_j| \, . \tag{4.2}$$

Since $\|\sum_{j=0}^{n} \lambda_j f_j\| = \sum_{j=0}^{n}(-\lambda_j) f_j\|$, we may assume that $\lambda_0 < 0$. By (4.1)

$$\lambda_0 f_0 \geq |\lambda_0|\delta \qquad \text{on} \qquad X_1.$$

By (4.1) again, $\lambda_1 f_1 \geq |\lambda_1|\delta$ on one of the sets X_2 or X_3. Hence on that set we have that

$$\lambda_0 f_0 + \lambda_1 f_1 \geq (|\lambda_0| + |\lambda_1|)\delta.$$

Similarly we see that on one of the sets X_4, X_5, X_6 or X_7 we have that

$$\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \geq (|\lambda_0| + |\lambda_1| + |\lambda_2|)\delta.$$

Proceeding inductively we get (4.2) for every n.

Let (φ_n) be a bounded sequence in E'. A sequence (ψ_n) in E' is said to be a *block* of (φ_n) if for every $n \in I\!N$

$$\psi_n = \sum_{j\in A_n} \alpha_j \varphi_j$$

where (A_n) is a sequence of finite subsets of $I\!N$ such that $A_n < A_{n+1}$ and $\sum_{j\in A_n} |\lambda_j| = 1$ for every n. For $A, B \subset I\!N$, $A < B$ means that $p < q$ for all $p \in A$ and $q \in B$. If we define

$$\delta(\varphi_n) = \sup_{\|x\|=1} \limsup_{n\to\infty} |\varphi_n(x)|$$

then

$$\begin{aligned} \delta(\psi_n) &= \sup_{\|x\|=1} \limsup_{n\to\infty} \Big| \sum_{j\in A_n} \alpha_j \varphi_j(x)\Big| \\ &\le \sup_{\|x\|=1} \limsup_{n\to\infty} \sup_{j\in A_n} |\varphi_j(x)| \\ &\le \sup_{\|x\|=1} \limsup_{n\to\infty} |\varphi_j(x)| = \delta(\varphi_n) \end{aligned}$$

Thus we have shown that $\delta(\psi_n) \le \delta(\varphi_n)$ *whenever* (ψ_n) *is a block of* (φ_n).

4.6. Lemma. *Every bounded sequence* (φ_n) *in* E' *has a block* (ψ_n) *with the property that* $\delta(\theta_n) = \delta(\psi_n)$ *for every block* (θ_n) *of* (ψ_n).

Proof. Define

$$\varepsilon(\varphi_n) = \inf\{\delta(\psi_n) : (\psi_n) \text{ is a block of } (\varphi_n)\}.$$

for each bounded sequence (φ_n) in E'. Since the block relation is reflexive and transitive, it is plain that $\varepsilon(\varphi_n) \le \delta(\varphi_n)$ for every (φ_n) and $\varepsilon(\varphi_n) \le \varepsilon(\psi_n)$ whenever (ψ_n) is a block of (φ_n).

To prove the lemma, if suffices to find a block (ψ_n) of (φ_n) such that $\varepsilon(\psi_n) = \delta(\psi_n)$. Now let $(\psi_n^{(1)})$ be a block of (φ_n) such that

$$\delta(\psi_n^{(1)}) \le \varepsilon(\varphi_n) + 1.$$

Next let $(\psi_n^{(2)})$ be a block of $(\psi_n^{(1)})$ such that

$$\delta(\psi_n^{(2)}) \le \varepsilon(\psi_n^{(1)}) + \frac{1}{2}.$$

In general let $(\psi_n^{(k)})$ be a block of $(\psi_n^{(k-1)})$ such that

$$\delta(\psi_n^{(k)}) \leq \varepsilon(\psi_n^{(k-1)}) + \frac{1}{k}.$$

let (ψ_n) be the diagonal sequence $(\psi_n^{(n)})$. Since the block relation is transitive, we see that $(\psi_n)_{n=k}^{\infty}$ is a block of $(\psi_n^{(k)})_{n=k}^{\infty}$ for every k, and $(\psi_n)_{n=1}^{\infty}$ is a block of $(\varphi_n)_{n=1}^{\infty}$. Thus

$$\delta(\psi_n) \leq \delta(\psi_n^{(k)}) \qquad \text{and} \qquad \varepsilon(\psi_n^{(k)}) \leq \varepsilon(\psi_n)$$

for every k and by our selection of the blocks $(\psi_n^{(k)})$, we have that

$$\limsup_{k\to\infty} \ \delta(\psi_n^{(k)}) \leq \limsup_{k\to\infty} \ \varepsilon(\psi_n^{(k)}).$$

Thus

$$\delta(\psi_n) \leq \limsup_{k\to\infty} \ \delta(\psi_n^{(k)}) \leq \limsup_{k\to\infty} \ \varepsilon(\psi_n^{(k)}) \leq \varepsilon(\psi_n) \leq \delta(\psi_n)$$

and $\delta(\psi_n) = \varepsilon(\psi_n)$.

Proof of Theorem 4.4. By Theorem 4.1 it suffices to prove (a). By (i) there is a basic sequence (φ_n) in E' which is equivalent to the canonical Schauder basis of ℓ^1. Thus there are $b \geq a > 0$ such that

$$a\sum_{j=1}^{n} |\lambda_j| \leq || \sum_{j=1}^{n} \lambda_j \varphi_j || \leq b \sum_{j=1}^{n} |\lambda_j| \tag{4.3}$$

for all $\lambda_1, \ldots, \lambda_n \in I\!R$ and $n \in I\!N$. Without loss of generality we may assume that $b = 1$. By Lemma 4.6 there is a block (ψ_n) of (φ_n) with the property that $\delta(\theta_n) = \delta(\psi_n)$ for every block (θ_n) of (ψ_n). Observe that every block of (φ_n), and in particular (ψ_n), verifies (4.3). Moreover it follows from (ii) that $\delta(\theta_n) > 0$ for each sequence (θ_n) in E' which verifies (4.3).

Now let $\delta = \delta(\psi_n) > 0$; and let $0 < \varepsilon < \delta$. By definition of $\delta(\psi_n)$, there are $x_0 \in S_E$ and an infinite set $N_1 \subset I\!N$ such that

$$|\psi_n(x_0)| \leq \delta - \varepsilon \qquad \text{for all} \quad n \in N_1.$$

Without loss of generality we may assume that

$$\psi_n(x_0) \le -\delta + \varepsilon \qquad \text{for all} \;\; n \in N_1.$$

Let $0 < \varepsilon' < \varepsilon/3$. Write $N_1 = A \cup B$, where $A = (m_j)$ and $B = (n_j)$, with $m_j < n_j < m_{j+1}$ for every $j \in \mathbb{N}$. Since the sequence $(\frac{1}{2}(\psi_{m_j} - \psi_{n_j}))$ is a block of (ψ_n), it follows that $\delta(\frac{1}{2}(\psi_{m_j} - \psi_{n_j})) = \delta(\psi_n) = \delta$. Hence there are $x_1 \in S_E$ and an infinite set $J \subset \mathbb{N}$ such that

$$|\tfrac{1}{2}(\psi_{m_j} - \psi_{n_j})(x_1)| \ge \delta - \varepsilon' \;\; \text{for} \quad j \in J.$$

Without loss of generality we may assume that

$$\tfrac{1}{2}(\psi_{m_j} - \psi_{n_j})(x_1) \ge \delta - \varepsilon' \;\; \text{for} \;\; j \in J.$$

Since (ψ_{m_j}) and (ψ_{n_j}) also are blocks of (ψ_n), it follows that $\delta(\psi_{m_j}) = \delta(\psi_{n_j}) = \delta(\psi_n) = \delta$. Hence $\limsup_{j\to\infty} |\psi_{m_j}(x_1)| \le \delta$ and $\limsup_{j\to\infty} |\psi_{n_j}(x_1)| \le \delta$. Hence there is $j_0 \in \mathbb{N}$ such that

$$|\psi_{m_j}(x_1)| \le \delta + \varepsilon' \;\; \text{for} \;\; j \ge j_0,$$
$$|\psi_{n_j}(x_1)| \le \delta + \varepsilon' \;\; \text{for} \;\; j \ge j_0.$$

We claim that

$$\psi_{m_j}(x_1) \ge \delta - 3\varepsilon' \;\; \text{for} \;\; j \in J, \;\; j \ge j_0,$$
$$\psi_{n_j}(x_1) \le -\delta + 3\varepsilon' \;\; \text{for} \;\; j \in J, \;\; j \ge j_0.$$

Indeed

$$\begin{aligned} \psi_{m_j}(x_1) &= (\psi_{m_j} - \psi_{n_j})(x_1) + \psi_{n_j}(x_1) \\ &\ge (\psi_{m_j} - \psi_{n_j})(x_1) - |\psi_{n_j}(x_1)| \\ &\ge 2(\delta - \varepsilon') - (\delta + \varepsilon') = \delta - 3\varepsilon' \end{aligned}$$

and the second inequality is proved similarly. We have thus found two disjoint, infinite subsets N_2 and N_3 of N_1 such that

$$\psi_n(x_1) \ge \delta - \varepsilon \qquad \text{for} \qquad n \in N_2,$$
$$\psi_n(x_1) \le -\delta + \varepsilon \qquad \text{for} \qquad n \in N_3.$$

Next let $0 < \varepsilon' < \varepsilon/7$. Then $N_2 \supset P \cup Q$ and $N_3 \supset R \cup S$, where $P = (p_k)$, $Q = (q_k)$, $R = (r_k)$ and $S = (s_k)$, with $p_k < q_k < r_k < s_k < p_{k+1}$ for every $k \in \mathbb{N}$. Then the sequence $(\frac{1}{4}(\psi_{p_k} - \psi_{q_k} + \psi_{r_k} - \psi_{s_k}))$ is a block of (ψ_n) and proceeding as before we can find $x_2 \in S_E$ and an infinite set $K \subset \mathbb{N}$ such that

$$\tfrac{1}{4}(\psi_{p_k} - \psi_{q_k} + \psi_{r_k} - \psi_{s_k}) \geq \delta - \varepsilon' \qquad \text{for} \quad k \in K.$$

Then as before we can find $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} \psi_{p_k}(x_2) &\geq \delta - 7\varepsilon' && \text{for} \quad k \in K, k \geq k_0, \\ \psi_{q_k}(x_2) &\leq -\delta + 7\varepsilon' && \text{for} \quad k \in K, k \geq k_0, \\ \psi_{r_k}(x_2) &\geq \delta - 7\varepsilon' && \text{for} \quad k \in K, k \geq k_0, \\ \psi_{s_k}(x_2) &\leq -\delta + 7\varepsilon' && \text{for} \quad k \in K, k \geq k_0. \end{aligned}$$

We have thus found disjoint infinite subsets N_4 and N_5 of N_2 and N_6 and N_7 of N_3 such that

$$\begin{aligned} \psi_n(x_2) &\geq \delta - \varepsilon && \text{for} \quad n \in N_4 \cup N_6, \\ \psi_n(x_2) &\leq -\delta + \varepsilon && \text{for} \quad n \in N_5 \cup N_7. \end{aligned}$$

Proceeding inductively we can find a tree $(N_p)_{p=1}^{\infty}$ of subsets of $\mathbb{N}$ such that

$$(-1)^p \psi_n(x_m) \geq \delta - \varepsilon \ \text{ whenever } \ n \in N_p, 2^m \leq p < 2^{m+1}, m \geq 0.$$

Set $\Psi_p = \{\psi_n : n \in N_p\}$ for every $p \in \mathbb{N}$. Then $(\Psi_p)_{p=1}^{\infty}$ is a tree of subsets of $B_{E'}$ such that

$$(-1)^p \psi(x_m) \geq \delta - \varepsilon \ \text{ whenever } \ \psi \in \Psi_p, 2^m \leq p < 2^{m+1}, m \geq 0.$$

By Proposition 4.5 (x_m) is a basic sequence equivalent to the canonical Schauder basis of ℓ^1.

4.7. Corollary. *Let E be a real Banach space which has a subspace isomorphic to c_0. Then:*

(a) E' has a complemented subspace isomorphic to ℓ^1.

(b) E has a quotient isomorphic to c_0 or ℓ^2.

4.8. Corollary. *Let E be an infinite dimensional Banach space. Then there is a sequence (φ_n) in E' such that $\|\varphi_n\| = 1$ for every $n \in \mathbb{N}$ and*

$\lim \varphi_n(x) = 0$ *for every* $x \in E$.

Proof. First consider the case of a real Banach space E. There are two possibilities:

(a) First assume that E' has a subspace isomorphic to ℓ^1. Then by Theorem 4.2 E has an infinite dimensional quotient E/M with a Schauder basis (f_n). Let $(f'_n) \subset (E/M)'$ be the sequence of coordinate functionals, and let $Q : E \to E/M$ be the quotient mapping. Then $Qx = \sum_{n=1}^{\infty} f'_n \circ Q(x) f_n$ for every $x \in E$, and we may assume that $\|f_n\| = 1$ for every n. Since $\|f_n\| = 1$ we can find $e_n \in E$ such that $Qe_n = f_n$ and $\|e_n\| \leq 2$. Since $f'_n \circ Q(e_n) = f'_n(f_n) = 1, \|f'_n \circ Q\| \geq \frac{1}{2}$ for every n. Thus it suffices to take $\varphi_n = f'_n \circ Q / \|f'_n \circ Q\|$ for every n.

(b) Next assume that E' has no subspace isomorphic to ℓ^1. Then by a result of Rosenthal [31] (see [6, pp. 201-211] or [20, pp. 99-101]), every bounded sequence in E' has a weakly Cauchy subsequence. Now by Riesz' lemma there is a sequence (x'_n) in E' such that $\|x'_n\| = 1$ for every n, and $\|x'_m - x'_n\| \geq \frac{1}{2}$ whenever $m \neq n$. By Rosenthal's theorem we may assume that (x'_n) is weakly Cauchy. Whence it follows that $\lim_{n\to\infty} x''(x'_{n+1} - x'_n) = 0$ for each $x'' \in E''$. Let $y'_n = x'_{n+1} - x'_n$ for every n. Then $\|y'_n\| \geq \frac{1}{2}$ for every n, and $\lim_{n\to\infty} x''(y'_n) = 0$ for every $x'' \in E''$. Thus it suffices to normalize each y'_n. This completes the proof in the case of a real Banach space.

If E is a complex Banach space, then there is a sequence (φ_n) of real linear functionals on E such that $\|\varphi_n\| = 1$ for every n and $\lim \varphi_n(x) = 0$ for every $x \in E$. If we define $(\psi_n) \subset E'$ by $\psi_n(x) = \varphi_n(x) - i\varphi_n(ix)$, then $\lim \psi_n(x) = 0$ for every $x \in E$, and $\|\psi_n\| \geq 1$ for every n. Thus it suffices to normalize each ψ_n to complete the proof.

4.9. Corollary. *Let E be a complex, infinite dimensional Banach space. Then for each $r > 0$ there is a holomorphic function $f : E \to \mathbb{C}$ which is unbounded on the ball $B(0; r)$.*

Proof. Let $0 < \rho < r$. By Corollary 4.8 there is a sequence (φ_n) in E' such that $\|\varphi_n\| = 1/\rho$ for every n and $\lim_{n\to\infty} \varphi_n(x) = 0$ for every $x \in E$. By a result of Dineen [7], the function $f : E \to \mathbb{C}$ defined by $f(x) = \sum_{n=1}^{\infty} (\varphi_n(x))^n$ is holomorphic on E and is unbounded on the ball $B(0; \rho + \varepsilon)$ for every $\varepsilon > 0$.

A set $B \supset E$ is said to be *bounding* if every holomorphic function $f : E \to \mathbb{C}$ is bounded on B. A set $L \subset E$ is said to be *limited* if $\lim_{n\to\infty} \sup_{x\in L} |\varphi_n(x)| = 0$ for every sequence (φ_n) in E' such that $\lim_{n\to\infty} \varphi_n(x) = 0$ for every $x \in E$. Let $\mathcal{D}(E)$ denote the family of all holomorphic functions $f : E \to \mathbb{C}$ of the form $f(x) = \sum_{n=1}^{\infty}(\varphi_n(x))^n$, where $(\varphi_n) \subset E'$ and $\lim_{n\to\infty} \varphi_n(x) = 0$ for every $x \in E$. Then one can prove that *a set $L \subset E$ is limited if and only if every $f \in \mathcal{D}(E)$ is bounded on L*. Whence every bounding set is limited, and Josefson [16] and Schlumprecht [35] have given examples of limited sets which are not bounding. Moreover the proof of Corollary 4.9 readily yields the following corollary.

4.10. Corollary. *Let E be a complex, infinite dimensional Banach space. Then bounding sets and limited sets in E are nowhere dense.*

Corollary 4.8 is due independently to Josefson [15] and Nissenzweig [26] (see also [11] or [6, pp. 219-223]), and answered a question raised by Thorp and Whitley [37]. Corollary 4.8 was also the missing link in Dineen's method of proof [7] of Corollaries 4.9 and 4.10, which answered questions raised by Nachbin [25] and Alexander [1].

It follows from Corollary 1.6 and Theorems 1.8 and 4.2 that *a real Banach spaces E has an infinite dimensional quotient with a Schauder basis if its dual E' has an infinite dimensional subspace which is either reflexive or isomorphic to c_0 or ℓ^1*. Thus Problem 1.7 is closely connected with the problem of whether every infinite dimensional Banach space has an infinite dimensional subspace which is either reflexive or isomorphic to c_0 or ℓ^1. This problem, mentioned in [20, p.104], remained open for a long time, and was recently solved in the negative by Gowers [10]. Actually Gowers [10] constructed an infinite dimensional Banach space E such that neither E nor E' contain any infinite dimensional subspace which is reflexive or isomorphic to c_0 or ℓ^1.

References

[1] H. ALEXANDER, Analytic functions on Banach spaces, doctoral

thesis, University of California, Berkeley, 1968.

[2] S. BANACH, Theorie des Opérations Linéaires, Warsaw, 1932, reprinted by Chelsea, New York.

[3] C. BESSAGA and A. PELCZYNSKI, On bases and unconditional convergence of series in Banach spaces, Studia Math. 17(1958), 151-164.

[4] M. DAY, On the basis problem in normed spaces, Proc. Amer. Math. Soc. 13(1962), 655-658.

[5] W. DAVIS, T. FIGIEL, W. JOHNSON and A. PELCZYNSKI, Factoring weakly compact operators. J. Funct. Anal. 17(1974), 311–327.

[6] J. DIESTEL, Sequences and Series in Banach Spaces, Springer, New York, 1984.

[7] S. DINEEN, Unbounded holomorphic functions on a Banach space, J. London Math. Soc.(2), 4 (1972), 461-465.

[8] P. ENFLO, A counterexample to the approximation property in Banach spaces, Acta Math. 130(1973), 309-317.

[9] B. GELBAUM, Banach spaces and bases, An. Acad. Brasil. Ciênc. 30(1958), 29-36.

[10] W. GOWERS, A Banach space not containing c_0, ℓ^1 or a reflexive subspace, Trans. Amer. Math. Soc. 344(1994), 407-420.

[11] J. HAGLER and W. JOHNSON, On Banach spaces whose dual balls are not weak* sequentially compact, Israel J. Math. 28(1977), 325-330.

[12] R. JAMES, Bases and reflexivity in Banach spaces, Ann. of Math. 52(1950), 518-527.

[13] R. JAMES, Bases in Banach spaces, Amer. Math. Monthly 89(1982), 625-640.

[14] W. JOHNSON and H. ROSENTHAL, On w*-basic sequences and their applications to the study of Banach spaces, Studia Math. 43(1972), 77-92.

[15] B. JOSEFSON, Weak sequential convergence in the dual of a Banach space does not imply norm convergence, Ark. Mat. 13(1975), 79-89.

[16] B. JOSEFSON, A Banach space containing non-trivial limited sets but no non-trivial bounding sets, Israel J. Math. 71(1990), 321-327.

[17] E. LACEY, Separable quotients of Banach spaces, An. Acad. Brasil Ciênc. 44(1972), 185-189.

[18] J. LINDENSTRAUSS, On a theorem of Murray and Mackey, An. Acad. Brasil. Ciênc. 39(1967), 1-6.

[19] J. LINDENSTRAUSS, On subspaces of Banach spaces without quasi-complements, Israel J. Math. 6(1968), 36-38.

[20] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces I, Springer, Berlin, 1977.

[21] G. MACKEY, Note on a theorem of Murray, Bull. Amer. Math. Soc. 52(1946), 322-325.

[22] F. MURRAY, The analysis of linear transformations, Bull. Amer. Math. Soc. 48(1942), 76-93.

[23] F. MURRAY, Quasi-complements and closed projections in reflexive Banach spaces, Trans. Amer. Math. Soc. 58(1945), 77-95.

[24] L. NACHBIN, A theorem of the Hahn-Banach type for linear transformations, Trans. Amer. Math. Soc. 68(1950), 28-46.

[25] L. NACHBIN, Concerning spaces of holomorphic functions, lecture notes, Rutgers University, New Brunswick, New Jersey, 1970.

[26] A. NISSENZWEIG, w^* sequential convergence, Israel J. Math. 22(1975), 266-272.

[27] A. PELCZYNSKI, A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces", Studia Math. 21(1962), 371-374.

[28] A. PELCZYNSKI, Some problems on bases in Banach and Fréchet spaces, Israel J. Math. 2(1964), 132-138.

[29] A. PELCZYNSKI, On Banach spaces containing $L_1(\mu)$, Studia Math. 30(1968), 231-246.

[30] H. ROSENTHAL, On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$, J. Funct. Anal. 4(1969), 176-214.

[31] H. ROSENTHAL, A characterization of Banach spaces containing ℓ^1, Proc. Nat. Acad. Sci. USA 71(1974), 2411-2413.

[32] S. SAXON and A. WILANSKY, The equivalence of some Banach space problems, Colloq. Math. 37(1977), 217-226.

[33] J. SCHAUDER, Zur Theorie Stetiger Abbildungen in Funktionalräumen, Math. Z. 26(1927), 47-65.

[34] J. SCHAUDER, Eine Eigenschaft der Haarschen Orthogonal systems, Math. Z. 28(1928), 317-320.

[35] T. SCHLUMPRECHT, A limited set which is not bounding, Proc. Roy. Irish Acad. 90(1990), 125-129.

[36] A. TAYLOR, Introduction to Functional Analysis, John Wiley, New York, 1958.

[37] E. THORP and R. WHITLEY, Operator representation theorems, Illinois J. Math. 9 (1965), 595-601.

[38] A. WILANSKY, Modern Methods in Topologial Vector Spaces, McGraw-Hill, New York, 1978.

Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postal 6065
13081-970 Campinas, SP, Brazil
e-mail address: mujica@ime.unicamp.br

Recibido: 24 de Junio de 1996
Revisado: 11 de Noviembre de 1996