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Bessel potentials in Orlicz spaces.

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Abstract

It is shown that Bessel potentials have a representation in term of measure when the underlying space is Orlicz. A comparison between capacities and Lebesgue measure is given and geometric properties of Bessel capacities in this space are studied. Moreover it is shown that if the capacity of a set is null, then the variation of all signed measures of this set is null when these measures are in the dual of an Orlicz-Sobolev space.

Introduction

In [4,6,7] a theory of capacity and potential in Orlicz spaces was extensively studied and applications to Bessel kernels were announced. Here we begin these applications.

We give a representation of Bessel potentials in terms of measure. Estimations at two sides for Bessel capacities of a ball are obtained in term of radii. This uses a representation of Orlicz-Sobolev spaces in term of Orlicz spaces and Bessel kernels; namely: $W^m L_A(R^n) = g_m * L_A(R^n)$ when A and A^* satisfy the Δ_2 condition.

We study also the relation between Bessel capacities and Hausdorff measure and give a condition in term of Hausdorff measure for a Bessel capacity of a set to be null.

For attaining this goal, we are combining the theory of capacities in Orlicz spaces and the methods of the nonlinear potential theory, developed by N. G. Meyers [17] and L. I. Hedberg [13]. (See also D. R.

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Adams and L. I. Hedberg [1] and W. P. Ziemer [20] for good and comprehensive survey of this theory).

On the other hand we show that if the capacity of a set is null, then the variation of all signed measures of this set is null when these measures are in the dual of an Orlicz-Sobolev space. This generalizes the corresponding result in Sobolev spaces by M. Grun-Rehomme [12].

1 Preliminaries

Let $A : R \to R^+$ be an $N-$function, $i.e. A$ is continuous, convex, with $A(t) > 0$ for $t > 0$, $\lim_{t\to 0} \frac{A(t)}{t} = 0$, $\lim_{t\to +\infty} \frac{A(t)}{t} = +\infty$ and A is even.

Equivalently, A admits the representation: $A(t) = \int_0^{|t|} a(x)dx$, where $a : R^+ \to R^+$ is non-decreasing, right continuous, with $a(0) = 0, a(t) > 0$ for $t > 0$ and $\lim_{t\to +\infty} a(t) = +\infty$. The $N-$function A^* conjugate to A is defined by $A^*(t) = \int_0^{|t|} a^*(x)dx$, where a^* is given by $a^*(s) = \sup\{t : a(t) \leq s\}$.

Let A be an $N-$function and let Ω be an open set in R^n. We note $\mathcal{L}_A(\Omega)$ the set, called an Orlicz class, of measurable functions f, on Ω, such that $\rho(f, A, \Omega) = \int_\Omega A(f(x))dx < \infty$.

Let A and A^* be two conjugate $N-$functions and let f be a measurable function defined almost everywhere in Ω. The Orlicz norm of f, $\| f \|_{A,\Omega}$ or $\| f \|_A$ if there is no confusion, is defined by

$$\| f \|_A = \sup\left\{\int_\Omega | f(x)g(x) | \, dx : g \in \mathcal{L}_{A^*}(\Omega) \text{ and } \rho(g, A^*, \Omega) \leq 1\right\}$$

The set $L_A(\Omega)$ of measurable functions f, such that $\| f \|_A < \infty$ is called an Orlicz space.

When $\Omega = R^n$, we set L_A in place of $L_A(R^n)$.

The Luxemburg norm $||| f |||_{A,\Omega}$, or $||| f |||_A$ if there is no confusion, is defined in $L_A(\Omega)$ by:

$$||| f |||_A = \inf\left\{s > 0 : \int_\Omega A\left[\frac{f(x)}{s}\right] dx \leq 1\right\}.$$

Let A be an $N-$function. We say that A verifies the Δ_2 condition if there exists a constant $C > 0$ such that $A(2t) \leq CA(t)$ for all $t \geq 0$.

We recall the following results. Let A be an $N-$function and a its derivative. Then

1) A verifies the Δ_2 condition if and only if one of the following holds

   i) $\forall r > 1, \exists k = k(r) : (\forall t \geq 0, A(rt) \leq kA(t))$;
   ii) $\exists \alpha > 1 : (\forall t \geq 0, ta(t) \leq \alpha A(t))$;
   iii) $\exists \beta > 1 : (\forall t \geq 0, ta^*(t) \geq \beta A^*(t))$;
   iv) $\exists d > 0 : (\forall t \geq 0, (A^*(t)/t)' \geq da^*(t)/t)$.

   Moreover α in ii) and β in iii) can be chosen such that

$$\alpha^{-1} + \beta^{-1} = 1.$$

   We note $\alpha(A)$ the smallest α such that ii) holds.

2) If A verifies the Δ_2 condition, then

   i) $\forall t \geq 1,\ A(t) \leq A(1)t^\alpha$ and $\forall t \leq 1, A(t) \geq A(1)t^\alpha$
   ii) $\forall t \geq 1,\ A^*(t) \geq A^*(1)t^\beta$ and $\forall t \leq 1,\ A^*(t) \leq A^*(1)t^\beta$.

   See for instance [10, 15, 18].

Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition. We note $\alpha(A) = \alpha$ and $\alpha(A^*) = \alpha^*$. Then we have from 2) below

$$\forall t \geq 0,\ \alpha^* A^*(t) \geq ta^*(t) \geq \beta A^*(t).$$

Hence $\beta \leq \alpha^*$.

If $\beta = \alpha^*$, then $\forall t \geq 0,\ \alpha^* A^*(t) = ta^*(t)$.

This implies that there exists a constant C, such that: $\forall t > 0,\ A^*(t) = Ct^{\alpha^*}$.

This means that we are in the case of Lebesgue classes L^p, which is treated in the literature.

Hence we suppose in the sequel that $\beta < \alpha^*$.

Let A be an $N-$function. We put $\widehat{A}$ an $N-$function equal to A in a neighbourhood of infinity and such that (see[3, lemma 4.4]): $\int_0^1 [\widehat{A}^{-1}(t)/t^{1+1/n}]dt < \infty$.

If $\int_1^{+\infty}[A^{-1}(t)/t^{1+1/n}]dt = \infty$, we define a new N-function $\widehat{A}_1$ by the formula

$$\widehat{A}_1^{-1}(X) = \int_0^x [\widehat{A}^{-1}(t)/t^{1+1/n}]dt$$

and we let A_1 to be an N-function equal to A in neighbourhood of 0 and to $\widehat{A}_1$ in a neighbourhood of infinity (see[3, lemma 4.5] for the constructin of such N-function). If $\int_1^{+\infty}[A_1^{-1}(t)/t^{1+1/n}]dt = \infty$, we start again the same construction and we put $A_2 = (A_1)_1, \cdots$

Let $j = J(A, n)$ be the smallest integer such that $\int_1^{+\infty}[A_j^{-1}(t)/t^{1+1/n}]dt < \infty$. If $\int_0^{+\infty}[A^{-1}(t)/t^{1+1/n}]dt < \infty$, we put $J(A, n) = 0$.

Observe that $J(A, n) \leq n$ because there exists a constant C, such that $A^{-1}(t) \leq Ct$, $\forall t \geq 1$.

Let m be a positive integer. The Orlicz-Sobolev space $W^m L_A(\Omega)$ is the space of real functions f, such that f and its distributional derivatives up to the order m, are in $L_A(\Omega)$.

$W^m L_A(\Omega)$ is a Banach space equipped with the norm:

$$|||f|||_{mA} = \sum_{|i| \leq m} |||D^i f|||_A, f \in W^m L_A(\Omega)$$

Let $W^{-m} L_{A^*}(\Omega)$ denote the space of distributions on Ω, which can be written as sums of derivatives up to order m of functions in $L_{A^*}(\Omega)$. It is a Banach space under the usual quotient norm.

Recall that if A and A^* satisfy the Δ_2 condition, the dual of $W^m L_A(R^n)$ coincides with $W^{-m} L_{A^*}(R^n)$.

For more details, one can cosult the classical references [2, 14, 15, 16, 18].

We define a capacity as a positive set function C given on a σ-additive class of sets τ, which contains compact sets and has the properties:

(i) $C(\emptyset) = 0$.

(ii) If X and Y are in τ and $X \subset Y$, then $C(X) \leq C(Y)$.

(iii) If $X_i, i = 1, 2, \cdots$ are in τ, then $C\left(\bigcup_{i\geq 1} X_i\right) \leq \sum_{i\geq 1} C(X_i)$.

Let k be a positive and measurable function in R^n and let A be an N−function. For $X \subset R^n$, we define

$$C_{kA}(X) = \inf\{A(|||\, f \,|||_A) : f \in L_A{}^+ \text{ and } k * f \geq 1 \text{ on } X\}$$

$$C'_{kA}(X) = \inf\{|||\, f \,|||_A : f \in L_A{}^+ \text{ and } k * f \geq 1 \text{ on } X\}$$

where $k * f$ is the usual convolution. The sign $+$ deals with positive elements in the considered space. Then C'_{kA} is a capacity and C_{kA} is called A−capacity.

If a statement holds except on a set X where $C_{kA}(X) = 0$, then we say that the statement holds C_{kA}−quasi everywhere (abbreviated C_{kA}−q.e or (k, A)−q.e is there is no confusion).

We call a function f in $L_A{}^+$ such that $k * f \geq 1$ on X, a test function for $C'_{kA}(X)$. Moreover, a test function, say f, for $C'_{kA}(X)$ such that $C'_{kA}(X) = |||\, f \,|||_A$ is called a C'_{kA}−capacitary distribution of X and $k * f$ a C'_{kA}−capacitary potential of X.

For the properties of C'_{kA} and C_{kA}, see [6], and for the existence and uniqueness of a C'_{kA}−capacitary distribution of a set, see [7].

M will be the vector space of all Radon measures. The cone of positive elements of M will be denoted by M^+.

$M_1(R^n)$ will be the Banach space of measures equipped with the norm $\| \mu \|$=total variation of $\mu < \infty$.

Recall that if X is a measurable set in R^n, then $\| \mu \| (X) = \sup \sum_{i\geq 1} | \mu | (X_i)$, the sup being taken over all decompositions $(X_i)_i$ of X.

Recall also that if $\mu \in M_1(R^n)$, then $\mu^+ = \frac{1}{2}(\| \mu \| + \mu)$ and $\mu^- = \frac{1}{2}(\| \mu \| - \mu)$.

Bessel kernel is of principal interest in this paper. As classical references, see [8, 9, 19].

For $m > 0$, the Bessel kernel, g_m, is most easily defined through its Fourier transform $F(g_m)$ as :

$$[F(g_m)](x) = (2\pi)^{-n/2}(1 + | x |^2)^{-m/2},$$

where

$$[F(f)](x) = (2\pi)^{-n/2} \int f(y) e^{-ixy} dy \text{ for } f \in L_1.$$

g_m is positive, in L_1 and verifies the equality

$$g_{r+s} = g_r * g_s.$$

We put

$$B_{mA} = C_{g_m A} \text{ and } B'_{mA} = C'_{g_m A}.$$

2 Representation of potentials and comparison with Lebesgue measure

Theorem 1. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition. Let m be a positive integer and X a set in R^n such that $0 < B'_{mA}(X) < \infty$.*

Let f be the $B'_{mA}-$capacitary distribution of X. Then there exists a positive measure μ_X such that:

1) $g_m * f = B'_{mA}(X) \cdot g_m * [a^{-1} \circ (g_m * \mu_X)]$, *where a is the derivative of A.*

2) $\operatorname{supp} \mu_X \subset \bar{X}$.

If in addition we suppose that X is compact, then

3) $g_m * f \leq 1$ *on* $\operatorname{supp} \mu_X$.

Proof. We follow Hedberg's method in [13].

1) From [7], for all $g \in L_A$ such that $g_m * g \geq 0$ on X, we have:

$$\int [a \circ (f / ||| f |||_A)] \cdot g dx \geq 0.$$

On the other hand, from [10] there exists $T \in W^{-m} L_{A^*}(R^n)$ such that

$$a \circ (f / ||| f |||_A) = g_m * T.$$

Hence

$$\forall g \in {L_A}^+, \int (g_m * T) \cdot g dx \geq 0.$$

Thus T is a positive measure which we note μ_X.

We have the representation $f = |||f|||_A[a^{-1} \circ (g_m * \mu_X)]$, and 1) follows by convolution with g_m.

2) Let $S(R^n)$ be the Schwartz space of C^∞ functions of rapidly decrease. Since A verifies the Δ_2 condition, $S(R^n)$ is dense in L_A. Let $g \in S(R^n)$ be such that $\operatorname{supp} g_m * g \subset {}^c\bar{X}$. Then

$$\forall t \in R,\ g_m * (f + tg) \geq 1 \text{ on } X.$$

By a similar calculus than the one in [7, Théorème 1], we obtain

$$< T, g_m * g >=< \mu_X, g_m * g >= 0.$$

Hence

$$\operatorname{supp} \mu_X \subset \bar{X}$$

3) We remark that the set $O = \{x : (g_m * f)(x) > 1\}$ is an open. This implies that for $g \in S(R^n)$ such that $g_m * g \subset O$, we have

$$g_m*(f+tg) \geq 1 \text{ on } X \text{ for all } t \text{ such that } |t| \text{ is sufficiently small}.$$

Again, by the same argument than the one in 2) we find that $< \mu_X, g_m * g >= 0$.

And thus $\operatorname{supp} \mu_X \subset {}^cO = \{x : (g_m * f)(x) \leq 1\}$.

This completes the proof.

Remark 1. Let A be an $N-$function. If X is a compact set, then

$$B'_{mA}(X) = \sup\{\mu(X)/\, |||g_m * \mu|||_{A^*} : \mu > 0,\ \operatorname{supp} \mu \subset X\}. \qquad (*)$$

It is a consequence of [6, Théorème 11].

An application of 3) in theorem 1 gives also (*), but with the condition that A and A^* satisfy the Δ_2 condition.

Theorem 2. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 codition. Let m be a positive integer. Then*

1) *If $m \leq J(A,n)$ there exists a constant $C = C(A,n,m) > 0$ such that*

$$B'_{m\mathcal{A}}(X) \geq C[A_m{}^{-1}(1/m^*(X))]^{-1}$$

for all set X such that $m^*(X) \neq 0$. (Here m is the Lebesgue measure on R^n and m^* is the outer measure associated to m).

2) *If $m > J(A,n)$, there exists a constant $C = C(A,n,m) > 0$ such that*

$$B'_{m\mathcal{A}}(X) \geq C$$

for all set X such that $X \neq \oslash$.

Proof.

1) It is enough to prove 1) when X is a non-empty, bounded and open set. This implies that $B'_{m\mathcal{A}}(X) < \infty$.

Let $m \leq J(A,n)$. Then from [3] (see also [11] for the case of a bounded and open set) the space $W^m L_A(R^n)$ is embedded in L_{Am}. Thus there exists a constant C which depends only on A, n and m, such that: $\forall h \in W^m L_A(R^n)$, $|||\, h \,|||_{Am} \leq C^{-1} |||\, h \,|||_{m\mathcal{A}}$.

We put: $g_m * L_A(R^n) = \{g_m * u : u \in L_A\}$.

From [10] we have

$$W^m L_A(R^n) = g_m * L_A(R^n) \text{ and } |||\, g_m * u \,|||_{m\mathcal{A}} = |||\, u \,|||_A \ \forall u \in L_A.$$

Hence

$$\forall f \in L_A,\ |||\, g_m * f \,|||_{Am} \leq C^{-1} |||\, f \,|||_A .$$

Let f be a test function for $B'_{m\mathcal{A}}(X)$. Then Hölder inequality in Orlicz spaces gives

$$m(X) \leq \int_X (g_m * f) dx \leq \| \chi_X \|_{(Am)^*} |||\, g_m * f \,|||_{Am} .$$

Here $(A_m)^*$ is the conjugate to A_m and χ_X is the characteristic function of X. From the equality $\| \chi_X \|_{(Am)^*} = m(X) A_m{}^{-1}(1/m(X))$, we deduce that

$$C \leq A_m{}^{-1}(1/m(X)) \cdot |||\, f \,|||_A .$$

Whence

$$B'_{m\mathcal{A}}(X) \geq C[A_m{}^{-1}(1/m(X))]^{-1}.$$

2) if $m > J(A,n)$, then from [3] the space $W^m L_A(R^n)$ is imbedded in $C(R^n) \cap L^\infty(R^n)$. Then there exists a constant C which depends only on A, n and m, such that

$$\forall f \in L_A,\ \forall x \in R^n,\ (g_m * f)(x) \leq C^{-1} |||\, f \,|||_A .$$

Thus if f is a test function for $B'_{mA}(X)$ with $X \neq \emptyset$, then $C \leq |||\, f \,|||_A$.

This implies $B'_{mA}(X) \geq C$.

The theorem follows.

The following lemma is proved in [5]. For completeness we give the proof.

Lemma 1. *Let A be an N-function and $0 < \beta \leq 1$. Let φ be defined on R^n by $\varphi(x) = \beta x$. Then*

$$|||\, f \,|||_A \leq |||\, f o\varphi \,|||_A \leq \beta^{-n} |||\, f \,|||_A \ \forall f \in L_A.$$

Proof. We have $\int A[(fo\varphi)(t)/\, |||\, fo\varphi \,|||_A] dt \leq 1$. Then

$$\int A[f(z)/\, |||\, fo\varphi \,|||_A] dz \leq \beta^n \leq 1.$$

This implies that $|||\, f \,|||_A \leq |||\, fo\varphi \,|||_A$.

On the other hand, let λ such that $\int A(f(x)/\lambda) dx \leq 1$.

Then $\int \beta^n A(f(\beta z)/\lambda) dx \leq 1$.

Hence $\int A(\beta^n f(\beta z)/\lambda) dx \leq 1$.

Consequently $|||\, fo\varphi \,|||_A \leq \beta^{-n} |||\, f \,|||_A$ and the lemma follows.

Lemma 2. *1) Let A be an N-function and let m be such that $0 < m < n$. Let $S_\rho = B(x,\rho)$ be the open ball centred at x and with radius ρ. Then there exists a constant C independent of ρ such that*

$$B'_{mA}(S_\rho) \leq C\rho^{-m} \textit{ for } 0 < \rho \leq 1.$$

2) Let A be an $N-$function satisfying the Δ_2 condition and let m be such that $0 < m < n$. Let $C(A)$ be the smallest constant C' such that: $A(2t) \leq C'A(t)\,\forall t$.

Then there exists a constant C independent of ρ such that: $B'_{mA}(S_\rho) \leq C2^{-q}\rho^{-m}$ for $0 < \rho \leq 1$, where q is the greatest positive integer such that $q \leq \operatorname{Log}\rho^{-n}/\operatorname{Log}C(A)$.

Proof. We follow the argument given in [17].

1) Let f be a test function for $B'_{mA}(S_4)$. Then $\int g_m(x-y)f(y)dy \geq 1$ on S_4.

By a change of variable we obtain

$$\int \rho^{-n} g_m[(x-z)/\rho]f(z/\rho)dz \geq 1 \text{ on } S_{4\rho}.$$

From the following asymptotique behaviours of g_m (see for instance [8, 9, 19])

$$g_m(x) = 2^{-m}\pi^{-n/2}\Gamma[(n-m)/2)]\Gamma(m/2)^{-1} \mid x \mid^{m-n} + + o(\mid x \mid^{m-n}),$$
$$0 < m < n, \text{ as } x \to o$$
$$g_m(x) \sim 2^{-(m+n-1)/2}\pi^{-(n-1)/2}\Gamma(m/2)^{-1} \mid x \mid^{(m-n-1)/2} e^{-|x|}$$
$$\text{as } x \to \infty$$

we deduce the existence of a constant C_1, such that

$$C_1{}^{-1}r^{m-n}e^{-2r} \leq g_m(r) \leq C_1 r^{m-n}e^{-r/2},\ g_m(r) = g_m(r, 0, \cdots, 0).$$

Therefore

$$\begin{aligned} g_m(r/\rho) &\leq C_1\rho^{n-m}r^{m-n}e^{-r/2\rho} \leq C_1\rho^{n-m}r^{m-n}e^{-2r} \leq \\ &\leq C_1^2\rho^{n-m}g_m(r), \text{ for } 0 < \rho \leq 1/4. \end{aligned}$$

This implies

$$C_1\rho^{-m}\int \rho^{-n}g_m(x-z)f(z/\rho)dz \geq 1 \text{ on } S_{4\rho}, \text{ for } 0 < \rho \leq 1/4.$$

Thus, for $0 < \rho \leq 1/4$, we have

$$B'_{mA}(S_{4\rho}) \leq C_1\rho^{-m}\,|||\, fou\, |||_A$$

where $u(z) = z/\rho = \varphi^{-1}(z)$.

We put $fou = g$. Then lemma 1 gives $|||\, g\, |||_A \leq |||\, go\varphi\, |||_A \leq \rho^{-n}\, |||\, g\, |||_A$.

Thus

$$|||\, fou\, |||_A \leq |||\, f\, |||_A \leq \beta^{-n}\, |||\, fou\, |||_A .$$

Hence

$$B'_{mA}(S_{4\rho}) \leq C_1 \rho^{-m}\, |||\, f\, |||_A .$$

This implies that

$$B'_{mA}(S_{4\rho}) \leq C_1 \rho^{-m} B'_{mA}(S_4), \text{ for } 0 < \rho \leq 1/4.$$

The desired inequality follows if we replace ρ by $\rho/4$.

2) In this case we evaluate $|||\, fou\, |||_A$ in term of $|||\, f\, |||_A$.

We have $\int A[f(z)/\, |||\, f\, |||_A] dz \leq 1$.

Put $z = u(t)$. Then $\rho^{-n} \int A[(fou)(t)/\, |||\, f\, |||_A] dt \leq 1$.

Whence

$$\begin{aligned} \int A[2^q (fou)(t)/\, |||\, f\, |||_A] dt &\leq C(A)^q \int A[(fou)(t)/\, |||\, f\, |||_A] dt \\ &\leq \rho^{-n} \int A[(fou)(t)/\, |||\, f\, |||_A] dt \leq 1. \end{aligned}$$

This means that $|||\, fou\, |||_A \leq 2^{-q}\, |||\, f\, |||_A$.

Hence $B'_{mA}(S_{4\rho}) \leq C_1 \rho^{-m} 2^{-q} B'_{mA}(S_4)$, for $0 < \rho \leq 1/4$.

The desired inequality follows if we replace ρ by $\rho/4$. This completes the proof.

3 Relation between capacity and Hausdorff measure

Lemma 3. *Let $u(r)$, $0 \leq r < \infty$, be strictly positive, decreasing and continuous from the right. Let $\mu \in M^+$ such that $\int u(|\, x - y\, |) d\mu(y) \in L_A$.*

Then there exists a function $\widehat{u}(r)$, $0 \leq r < \infty$, where

1) $\widehat{u}(r)$ *is strictly positive, decreasing and continuous from the right,*

2) $\int \widehat{u}(| x - y | d\mu(y) \in L_A$,

3) $\widehat{u}(r) \geq u(r)$ *and* $\lim_{r\to 0} \widehat{u}(r)u(r)^{-1} = +\infty$.

Proof. We follow Meyer's idea in [17] in the case of Lebesgue classes.

Define $f_0(x) = \int_{\{x=y\}} u(| x - y |)d\mu(y)$ and

$$f_i(x) = \int_{\{2^{-i}\leq|x-y|<2^{-i+1}\}} u(| x - y |)d\mu(y), i = 1, 2, \cdots.$$

Note that $f_0(x) = 0$ for almost all x, since $\mu(\{x\}) > 0$ for at most a countable number of points x.

We will prove the existence of an increasing sequence $(a_i)_i$ of finite real numbers such that $\forall i, a_i \geq 1$ and $\sum_{i\geq 1} a_i f_i \in L_A$.

We remark that $\sum_{i\geq l} f_i \to 0$ strongly in L_A as $l \to \infty$.

Therefore there exists a sequence of positive integers, $(l_j)_j, j = 1, \cdots$ such that

$$\sum_{j\geq 1} ||| \sum_{i\geq l_j} f_i |||_A < \infty.$$

Moreover there exists an increasing sequence of real numbers $(b_j)_j$ such that $\forall j, b_j \geq 1$ and

$$\sum_{j\geq 1} b_j ||| \sum_{i\geq l_j} f_i |||_A < \infty.$$

We define $a_i = b_j$ for $l_j \leq i < l_{j+1}$. Then

$$||| \sum_{i\geq 1} a_i f_i |||_A \leq \sum_{j\geq 1} ||| \sum_{l_j\leq i\leq (l_j+1)-1} a_i f_i |||_A \leq \sum_{j\geq 1} b_j ||| \sum_{i\geq l_j} f_i |||_A < \infty.$$

This shows that $\forall i, a_i \geq 1$ and

$$\sum_{i\geq 1} a_i f_i \in L_A.$$

We define $\widehat{u}$ by:

$$\widehat{u}(0) = +\infty,$$
$$\widehat{u}(r) = u(r) \text{ for } 2^{-i} \leq r < 2^{-i+1}; i = 1, 2, \cdots,$$
$$\widehat{u}(r) = u(r) \text{ for } 1 \leq r.$$

Then $\widehat{u}$ verifies properties 1), 2) and 3).
This completes the proof.

Theorem 3. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let m be a positive real such that $\alpha m < n$, where $\alpha = \alpha(A)$. Let $\widehat{u}$ be a positive, decreasing function defined on R, continuous from the right and such that*

$$g_m(r) \leq \widehat{u}(r) \text{ and } \lim_{r\to 0} \widehat{u}(r) g_m(r)^{-1} = +\infty.$$

If $B' = B'_{\widehat{u}A}$, then

$$\lim_{\rho\to 0} B'(S_\rho) B'_{mA}(S_\rho)^{-1} = 0.$$

Proof. Since B' is invariant under translation, the centre of S_ρ is of no importante and we can take it to be zero.

Let f be a test function for $B'_{mA}(S_\rho)$ such that

$$|||f|||_A \leq 2B'_{m,A}(S_\rho).$$

Let θ be a finite constant greater than one. Then we have

$$\int_{\{|y|<\rho\theta\}} g_m(x-y) f(y) dy + \int_{\{|y|\geq\rho\theta\}} g_m(x-y) f(y) dy \geq 1,\ x \in S_\rho.$$

We pose: $I = \int_{\{|y|\geq\rho\theta\}} g_m(x-y) f(y) dy.$
Then

$$I \leq 2 \inf\{\lambda : \int_{\{|y|\geq\rho\theta\}} A^*[g_m(x-y)/\lambda] dy \leq 1\}. |||f|||_A.$$

But if λ is such that

$$\int_{\{|y|\geq\rho(\theta-1)\}} A^*[g_m(x-y)/\lambda] dy \leq 1,$$

then

$$\int_{\{|y|\geq\rho\theta\}} A^*[g_m(x-y)/\lambda]dy \leq 1.$$

Hence

$$I \leq 2\inf\{\lambda : \int_{\{|y|\geq\rho(\theta-1)\}} A^*[g_m(x-y)/\lambda]dy \leq 1\}. \||f\||_A \cdot$$

We begin by estimate the integral

$$J_m = \int_{\{|y|\geq\rho(\theta-1)\}} A^*[g_m(y)]dy.$$

By a change of variable we have

$$J_m = C_1 \int_{[\rho(\theta-1),\infty]} A^*[g_m(t)]t^{n-1}dt.$$

On the other hand, there exists a constant C_2 such that

$$g_m(x) \leq C_2 \mid x \mid^{m-n}.$$

Hence

$$J_m \leq C_1' \int_{[\rho(\theta-1),\infty[} A^*(t^{m-n})t^{n-1}dt \leq C_1'(J_m' + J_m'')$$

where $J_m' = \int_{[\rho(\theta-1),1[} A^*(t^{m-n})t^{n-1}dt$ and $J_m'' = \int_{[1,\infty[} A^*(t^{m-n})t^{n-1}dt.$
We have supposed that $\rho(\theta-1) < 1$.
We must evaluate J_m' and J_m''.
First, note that $t^{m-n} \geq 1$ for $\rho(\theta-1) < t < 1$. So

$$A^*(t^{m-n}) \leq A^*(1)(t^{m-n})^{\alpha^*}.$$

Since $(m-n)\alpha^* + n \leq (m-n)\beta + n < 0$, we deduce that

$$\begin{aligned} J_m' &\leq A^*(1) \int_{[\rho(\theta-1),\infty[} (t^{m-n})^{\alpha^*} t^{n-1}dt \\ &\leq A^*(1)[(m-n)\alpha^* + n][1 - \{\rho(\theta-1)\}^{(m-n)\alpha^*+n}]. \end{aligned}$$

On the other hand, since A verifies the Δ_2 condition, we get for $t \geq 1$

$$A^*(t^{m-n}) \leq A^*(1)(t^{m-n})^{\beta}.$$

Hence

$$J''_m \leq A^*(1) \int_{[1,\infty[} (t^{m-n})^{\beta} t^{n-1} dt \leq A^*(1)(\alpha - 1)(n - \alpha m)^{-1}.$$

Then

$$J_m \leq C'_1 A^*(1)(\alpha - 1)(n - \alpha m)^{-1} [\rho(\theta - 1)]^{(m-n)\alpha^* + n}.$$

This shows that there exists a constant $K_1 \geq 1$, such that

$$J_m \leq K_1 [\rho(\theta - 1)]^{(m-n)\alpha^* + n}.$$

Put $E(\theta, \rho) = \{| y | \geq \rho(\theta - 1)\}$ and remark that $[\rho(\theta - 1)]^{(m-n)\alpha^* + n} \geq 1$.
Then

$$|||\, g_m \,|||_{A^*, E(\theta,\rho)} \leq K_1 [\rho(\theta - 1)]^{(m-n)\alpha^* + n}.$$

Hence there exists a constant K' such that

$$I \leq K' [\rho(\theta - 1)]^{(m-n)\alpha^* + n} \cdot |||\, f \,|||_A \,.$$

From lemma 2 we get

$$I \leq K'' [\rho(\theta - 1)]^{(m-n)\alpha^* + n} \cdot 2^{-q} \rho^{-m} \text{ for } 0 < \rho \leq 1$$

where q is the greatest positive integer such that

$$q \leq \operatorname{Log} \rho^{-n} / \operatorname{Log} C(A).$$

Put $C'' = \operatorname{Log} 2 / \operatorname{Log} C(A)$. Then

$$q + 1 \geq \operatorname{Log} \rho^{-n} / \operatorname{Log} C(A) \text{ implies that } 2^{-q} \leq 2\rho^{nC'}.$$

On the other hand, we know that for all $t > 0$, we have $\alpha/t \geq a(t)/A(t)$.

This implies

$$\operatorname{Log} A(2t)/A(t) = \int_t^{2t} \frac{a(s)}{A(s)} ds \leq \alpha \operatorname{Log} 2.$$

Hence $A(2t) \leq 2^{\alpha} A(t)$. So $2^{\alpha} \geq C(A)$.
This gives $\rho^{nC'} \leq \rho^{n/\alpha}$.

Hence

$$\begin{aligned} I &\le K''[\rho(\theta-1)]^{(m-n)\alpha^*+n}\cdot\rho^{-m+n/\alpha} \\ &\le K''(\theta-1)^{(m-n)\alpha^*+n}\cdot[\rho^{-m+n/\alpha}\rho^{(m-n)\alpha^*+n}]. \end{aligned}$$

Choose θ such that

$$\theta-1=K'''\rho^{-1+[m-n/\alpha]/[(m-n)\alpha^*+n]}.$$

K''' will be precise in the sequel.

Then

$$\rho(\theta-1)=K'''\rho^{[m-n/\alpha]/[(m-n)\alpha^*+n]}\to 0 \text{ as } \rho\to 0.$$

On the other hand we have

$$I\le K''(K''')^{(m-n)\alpha^*+n}.$$

We choose K''' such that

$$K''(K''')^{(m-n)\alpha^*+n}\le 1/2.$$

Then $I\le 1/2$.

This implies that

$$\int_{\{|y|<\rho\theta\}} g_m(x-y)f(y)dy\ge 1/2,\ x\in S_\rho.$$

We define

$$[g(\rho)]^{-1}=\inf\{\widehat{u}(r)[g_m(r)]^{-1}:0<r<\rho(\theta+1)\}.$$

Then there exists a constant D, such that

$$B'(S_\rho)\le Dg(\rho)B'_{mA}(S_\rho).$$

This completes the proof.

Remark 2. We have proved the inequality $2^{-q}\le 2\rho^{n/\alpha}$. This implies, in lemma 2, that there exists a constant C independent of ρ such that $B'_{mA}(S_\rho)\le C\rho^{-m+n/\alpha}$ for $0<\rho\le 1$.

Hence, if $\alpha m < n$, then

$$\lim_{\rho \to 0} B'_{mA}(S_\rho) = 0$$

Definition. *Let $\varphi(r)$ be a positive, increasing function in some interval $[0, r'[$ and such that*

$$\lim_{r \to 0} \varphi(r) = 0.$$

If X is an arbitrary set, the Hausdorff $\varphi-$measure of X is given by

$$H_{\varphi(r)}(X) = \lim_{s \to 0} \left\{ \inf \sum_{i \geq 1} \varphi(r_i) \right\},$$

where the above infimum is taken over all countable coverings of X by spheres $S(x_i, r_i)$ such that $r_i \leq s$.

Note that $H_{\varphi(r)}$ is a capacity which has the property

$$H_{\varphi(r)}(X) = H_{\varphi(r)}(Y), \tag{**}$$

where Y is a $G-$ set containing X.

Theorem 4. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition and let X be a subset of R^n. Let m be a positive real such that $\alpha m < n$, where $\alpha = \alpha(A)$ and let $\varphi(r) = B'_{mA}(S_r)$. Then*

$$B_{mA}(X) = 0 \text{ if } H_{\varphi(r)}(X) < \infty.$$

Proof. In view of (**) and [4, Théorème 3], it is sufficient to consider the case $X = K$, K a compact. Assume that $B'_{mA}(K) > 0$. Then from [4, Théorème 4], there exists $\mu \in M^+$ such that

$$\mu \neq 0, \operatorname{supp} \mu \subset K \text{ and } g_m * \mu \in L_{A^*}.$$

Lemma 3 gives a kernel $\widehat{u}$ with properties 1), 2) and 3). If we set $\widehat{B}' = C'_{\widehat{u}A}$, by [4, Théorème 4], we must have

$$\widehat{B}'(K) > 0.$$

Lemma 3 and theorem 3 imply that

$$\lim_{\rho\to 0}\widehat{B}'(S_\rho)B'_{m\mathcal{A}}(S_\rho)^{-1}=0.$$

Now let $(S_{\rho i}(x_i))_i$ be a countable covering of K by spheres. Then

$$\widehat{B}'(K)\leq\sum_{i\geq 1}\widehat{B}'(S_{\rho i}(x_i))=\sum_{i\geq 1}[\widehat{B}'(S_{\rho i}(x_i))B'_{m\mathcal{A}}(S_{\rho i}(x_i))^{-1}]B'_{m\mathcal{A}}(S_{\rho i}(x_i)).$$

Since the ratio $[\widehat{B}'(S_{\rho i}(x_i))B'_{m\mathcal{A}}(S_{\rho i}(x_i))^{-1}]$ can be made as small as we wish while $\sum_{i\geq 1}B'_{m\mathcal{A}}(S_{\rho i}(x_i))$ remains bounded, we must have $\widehat{B}'(K)=0$ which gives a contradiction.

The proof is finished.

Remark 3. As we have noted in the remark 3, we consider the case $\beta<\alpha^*$. For L^p spaces, theorem 4 is true for $\varphi(r)=B_{m,p}(S_r)$, because $B_{m,p}$ is a capacity. (Here $B_{m,p}(X)=\inf\{\| f\|_p^p\colon f\in L^{p+}$ and $g_m*f\geq 1$ on $X\}$). In our case, unfortunately we don't know whether $B_{m\mathcal{A}}$ is a capacity, so the theorem is not sharp. The open question is to characterize the N–function A for which $B_{m\mathcal{A}}$ is a capacity.

I am very grateful to Professor L. I. Hedberg for pointing out this fact.

4 Capacities and measures in Orlicz-Sobolev spaces

Lemma 4. *Let $F'_{k\mathcal{A}}(X)=\inf\{|||\phi|||_A:\phi\in D^+(R^n)$ and $k*\phi\geq 1$ on $X\}$ and let $F_{k\mathcal{A}}(X)=A(F'_{k\mathcal{A}}(X))$. Then*

1) $\forall X\subset R^n, C_{k,\mathcal{A}}(X)\leq F_{k\mathcal{A}}(X)$.

2) $F_{k\mathcal{A}}(X)=0\Leftrightarrow C_{k\mathcal{A}}(X)=0$.

Proof. 1) It is obvious that

$$C_{k\mathcal{A}}(X)\leq F_{k\mathcal{A}}(X),\ \forall X\subset R^n.$$

This gives

$$F_{k\mathcal{A}}(X)=0\Rightarrow C_{k\mathcal{A}}(X)=0.$$

2) Suppose that $C_{kA}(X) = 0$. Then theorem 3c) in [6] gives a function $f \in {L_A}^+$ such that

$$k * f = \infty \text{ on } X.$$

On the other hand, there exists a sequence $(\phi_i)_i \subset D^+(R^n)$ which converges in modular and almost everywhere to f. By Fatou's Lemma we obtain

$$\lim_{i\to\infty} k * \phi_i = \infty \text{ on } X.$$

Hence

$$\forall N, \exists \phi \, : \, k * \phi_N \geq N \text{ on } X.$$

This implies that $N^{-1}\phi_N$ is a test function for $F'_{kA}(X)$.
Since $|||\, \phi_N \,|||_A$ is bounded, there exists a constant C, such that

$$C/N \geq N^{-1} |||\, \phi_N \,|||_A \geq F'_{kA}(X).$$

Whence

$$F'_{kA}(X) = 0.$$

The proof is finished.

Lemma 5. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A,n)$. Let $T \in W^{-m}L_{A^*}(R^n) \cap M_1(R^n)$ and let K be a compact set such that*

$$B_{kA}(K) = 0 \text{ and } T^-(K) = 0.$$

Then

$$\|\, T \,\|\, (K) = 0.$$

Proof. Let $\varepsilon > 0$ and O be an open set such that

$$K \subset O \text{ and } \|\, T \,\|\, (O \backslash K) < \varepsilon.$$

There exists a function $\xi \in D^+(R^n)$ such that

$$0 \leq \xi \leq 1,\ \xi = 1 \text{ on } K \text{ and } \operatorname{supp} \xi \subset O.$$

Since $B_{kA}(K) = 0$, lemma 4 gives $F_{kA}(K) = 0$ with

$F'_{kA}(X) = \inf\{|||\phi|||_A : \phi \in D^+(R^n) \text{ and } g_m * \phi \geq 1 \text{ on } X\}$ and $F_{kA}(X) = A(F'_{kA}(X))$.

There exists a sequence $(\psi_i)_i = (g_m * \phi_i)_i$ in $W^m L_A(R^n)$ such that $(\phi_i)_i \subset D^+(R^n)$, and

$$\forall i, \psi_i \geq 1 \text{ on } K, \psi_i \to 0 \text{ in } W^m L_A(R^n) \text{ and } \phi_i \to 0 \text{ on } L_A.$$

Let the function $H \in C_+{}^\infty(R^+)$ be defined by

$$\begin{array}{lll} H(t) = t & \text{if} & 0 \leq t \leq 1/2 \\ H(t) = 1 & \text{if} & t \geq 1 \\ H(t) \leq 1 & & \forall t. \end{array}$$

From [10] we have

$$\forall i, \exists f_i \in L_A \text{ such that } \psi_i = I_m * f_i \text{ and } |||f_i|||_A \leq C|||\psi_i|||_{mA}$$

where I_m is the Riesz kernel defined by: $I_m(x) = |X|^{m-n}$ and C is a constant independent of i.

Hence $|||f_i|||_A \to 0$.

We put for each $i, \Phi_i = H_o(I_m * |f_i|)$.

The same calculus that those given in [10] show that there exists a constant C' independent of i, such that $|||\Phi_i|||_{mA} \leq C'|||f_i|||_A$.

This implies

$$|||\Phi_i|||_{mA} \to 0.$$

Moreover, on K we have

$$1 \leq I_m * f_i \leq I_m * |f_i|.$$

Hence

$$\Phi_i \leq 1 \text{ and } \Phi_i = 1 \text{ on } K.$$

Now, we put $\varphi_i = \xi\Phi_i$.

Then

$$|||\varphi_i|||_{mA} \leq C''|||\Phi_i|||_{mA}$$

where C'' is a constant independent of i.

This implies

$$\varphi_i \to 0 \text{ in } W^m L_A(R^n).$$

Since $T^{-}(K) = 0$, we have

$$\int \varphi_i dT = \int_K \varphi_i dT + \int_{O-K} \varphi_i dT = \| T \| (K) + \int_{O-K} \varphi_i dT.$$

We remark that, for all $i, \varphi_i \in W^m L_A(R^n) \cap L^\infty, \varphi_i$ is continuous with compact support. By the approximation of unity we obtain

$$< T, \varphi_i >= \int \varphi_i dT.$$

Hence

$$| \int \varphi_i dT | \leq ||| \varphi_i |||_{mA} ||| T |||_{mA^*} \to 0,$$

and

$$\int_{O-K} \varphi_i dT \leq \| T \| (O \backslash K) < \varepsilon.$$

Hence for sufficiently large i, we have

$$\| T \| (K) < 2\varepsilon.$$

This means

$$\| T \| (K) = 0.$$

The proof is finished.

Lemma 6. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A, n)$. Let $T \in W^{-m} L_{A^*}(R^n) \cap M_1(R^n)$ and let X be a $\| T \| -$measurable set such that*

$$B_{kA}(X) = 0 \text{ and } T^{-}(X) = 0.$$

Then

$$\| T \| (X) = 0.$$

Proof. For each compact $K \subset X$, we have $B_{kA}(K) = 0$ and $T^{-}(K) = 0$. From lemma 5 we deduce that

$$\| T \| (K) = 0.$$

The inner measure of X is defined by $| T | *(X) = \sup\{\| T \| (K) : K \subset X$ and K compact $\}$.

Hence

$$| T | *(X) = 0.$$

Since X is a $\| T \|$ $-$measurable set, we conclude that

$$\| T \| (X) = | T | *(X) = 0.$$

The proof is finished.

Lemma 7. *Let A be an $N-$function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A, n)$. Let $T \in W^{-m}L_{A^*}(R^n) \cap M_1(R^n)$ and K a compact set such that $B_{kA}(K) = 0$.*

Then $\| T \| (K) = 0$.

Proof. There exists two $\| T \|$ $-$measurable sets, E and F, such that T^- is concentrated on E and T^+ on F. Hence

$$T^-(K \backslash E) = 0.$$

Lemma 6 gives

$$\| T \| (K \backslash E) = 0.$$

On the other hand

$$T^+(K \backslash F) = 0.$$

Lemma 6 is valid if we take T^+ in place of T^-.
Hence

$$\| T \| (K \backslash F) = 0.$$

This implies that

$$\| T \| (K) = 0.$$

The proof if finished.

Remark 4. From [6, Théorème 2] we know that C_{kA} and C'_{kA} are outer. This implies the following: If X is a set, there exists a G_δ set G, such that $X \subset G$ and

$$C_{kA}(X) = C_{kA}(G) \text{ and } C'_{kA}(X) = C'_{kA}(G).$$

Theorem 5. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A,n)$. Let $T \in W^{-m}L_{A^*}(R^n) \cap M_1(R^n)$ and let X be a set such that $B_{kA}(X) = 0$.*
Then $\| T \| (X) = 0$.

Proof. For all compact subset K of X, we have $B_{kA}(K) = 0$.
Lemma 7 implies $\| T \| (K) = 0$.
The inner measure of X is defined by $| T | *(X) = \sup\{\| T \| (K) : K \subset X \text{ and } K \text{ compact }\}$.
The outer measure of X is defined by $| T |^* (X) = \inf\{\| T \| (O) : X \subset O \text{ and } O \text{ open }\}$.
Hence

$$| T | *(X) = 0.$$

On the other hand, the above remark gives a G_δ set G, such that $X \subset G$ and $B_{kA}(G) = 0$.
Hence, if K is a compact set such that $K \subset G$, we have $B_{kA}(K) = 0$.
Lemma 7 implies that

$$\| T \| (K) = 0, \ \forall K \text{ compact such that } K \subset G.$$

From the definition of inner measure, we have $| T | *(G) = 0$.
Since G is a G_δ set, it is $\| T \|$ $-$measurable. Hence

$$\| T \| (G) = 0.$$

The inclusion $X \subset G$ gives

$$| T |^* (X) \leq | T |^* (G) = \| T \| (G) = 0.$$

This implies

$$| T |^* (X) = | T | *(X) = 0$$

and X is $\| T \|$ $-$measurable.
Hence $\| T \| (X) = 0$ and the theorem is proved.

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