

A short intervals result for linear equations in two prime variables.

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Abstract

Given A and B integers relatively prime, we prove that almost all integers n in an interval of the form $[N, N+H]$, where $N^{1/3+\varepsilon} \leq H \leq N$, can be written as a sum $Ap_1 + Bp_2 = n$, with p_1 and p_2 primes and ε an arbitrary positive constant. This generalizes the results of [PP] established in the classical case $A = B = 1$ (Goldbach's problem).

1 Introduction

Let us consider the linear equation $Ap_1 + Bp_2 = n$, where p_1 and p_2 are prime numbers and A, B, n are positive integers satisfying the following conditions:

- 1) A, B are relatively prime, $(A, B) = 1$,
- 2) $n \in \mathcal{A} = \{n : (AB, n) = 1, ABn \equiv 0 \pmod{2}\}$.

In the classical case $A = B = 1$ (Goldbach's problem) Perelli and Pintz [PP] proved that the above equation has solutions for all even integers $2n$ in an interval of the form $[N, N+H]$ with $O(HL^{-E})$ exceptions

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and

$$R(2n) = 2n\mathfrak{S}(2n) + O(NL^{-C}),$$

where

$$R(2n) = \sum_{h+k=2n} \Lambda(h)\Lambda(k), \quad \mathfrak{S}(2n) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|n \\ p>2}} \left(\frac{p-1}{p-2}\right),$$

Λ is von Mangoldt's function, $L = \log N$, $N^{1/3+\varepsilon} \leq H \leq N$ and $E, C > 0$, $0 < \varepsilon < 2/3$ are arbitrary constants.

In this paper we want to establish the same results in the general case for A and B . Let us denote

$$R(n) = R(n, A, B) = \sum_{Ah+Bk=n} \Lambda(h)\Lambda(k),$$

$$M(n) = M(n, A, B) = \sum_{Ah+Bk=n} 1,$$

$$\mathfrak{S}(n) = \mathfrak{S}_{A,B}(n) = \prod_{p|ABn} \left(1 + \frac{1}{p-1}\right) \prod_{p \nmid ABn} \left(1 - \frac{1}{(p-1)^2}\right).$$

Theorem 1. *Let $0 < \varepsilon < 2/3$, $C > 0$ be arbitrary constants and $N^{1/3+\varepsilon} \leq H \leq N$. Then*

$$\sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} |R(n) - M(n)\mathfrak{S}(n)|^2 \ll_{\varepsilon, A, B, C} H N^2 L^{-C}.$$

Clearly Theorem 1 implies

Corollary. *Let $E, C > 0$ be arbitrary constants and $N^{1/3+\varepsilon} \leq H \leq N$. Then, for all $n \in [N, N+H] \cap \mathcal{A}$ with at most $O(HL^{-E})$ exceptions, the equation $Ap_1 + Bp_2 = n$ is solvable and we have*

$$R(n) = M(n)\mathfrak{S}(n) + O_{\varepsilon, A, B, C}(NL^{-C}).$$

Theorem 2. *Let $0 < \varepsilon < 5/6$, $E > 0$ be arbitrary constants. Then, for all $n \in [N, N+H] \cap \mathcal{A}$ with at most $O_{\varepsilon, A, B, C}(HL^{-E})$ exceptions, the equation $Ap_1 + Bp_2 = n$ is solvable, provided $N^{7/36+\varepsilon} \leq H \leq N$.*

Remarks.

I. The condition $(A, B) = 1$ is natural. In fact, if $(A, B) = d > 1$, then the equation $Ap_1 + Bp_2 = n$ has no solutions when $d \nmid n$. If $d \mid n$, then the equation is equivalent to $A'p_1 + B'p_2 = n'$, where $A' = A/d$, $B' = B/d$, $n' = n/d$ with $(A', B') = 1$. If $n \notin \mathcal{A}$, then our equation has at most one solution, and our method is not able to detect it.

II. It is easy to see [A; ch.2, ex.9] that for fixed A and B , $(A, B) = 1$, we have

$$|[N, N + H] \cap \mathcal{A}| \sim \frac{\varphi(AB)}{AB}H,$$

where φ is Euler's function. Hence the exceptional set in the above results is of order smaller than $|[N, N + H] \cap \mathcal{A}|$.

III. The number of non-negative integer solutions of the linear equation $Ax + By = n$, with $(A, B) = 1$, is given by

$$M(n) = \left[\frac{n}{AB} \right] \quad \text{or} \quad \left[\frac{n}{AB} \right] + 1.$$

This result can be proved by appealing to the following theorem (see [NZM; Theorem 5.1]) and recalling that $[\alpha] - [\beta] = [\alpha - \beta]$ or $[\alpha - \beta] + 1$ (where $[\alpha]$ denotes the integer part of α):

Theorem 3. *Let (x_o, y_o) be an integer solution of the linear equation $Ax + By = n$, with $(A, B) = 1$. Then the solutions of $Ax + By = n$ are given by:*

$$x = x_o + Bt, \quad y = y_o - At,$$

where t is an integer.

IV. Following the proofs of Theorem 1 and 2 below and computing the implicit constant in $\ll_{\varepsilon, A, B, C}$, it is easy to remark that the fore-mentioned results holds uniformly for $A, B \ll L^{C_1}$, for a suitable constant $C_1 = C_1(C) > 0$.

2 Notation

C - an arbitrary positive constant,

ε - an arbitrarily small positive constant,

$\delta = 1/3 + \varepsilon$ or $7/36 + \varepsilon$,

N, H - positive integers such that $N^\delta \leq H \leq N$, $N > N_0 = N_0(A, B, C, \varepsilon)$,

$D = \frac{3}{\varepsilon}(2C + 25)$, $Q = \frac{\sqrt{H}}{2}$.

$$I_{q,a} = \left\{ \frac{a}{q} + \eta, \eta \in \xi_{q,a} \right\}, \quad \text{where } \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ} \right),$$

$$I'_{q,a} = \left\{ \frac{a}{q} + \eta, \eta \in \xi'_q \right\}, \quad \text{where } \xi'_q = \left(-\frac{L^{4D}}{qY}, \frac{L^{4D}}{qY} \right),$$

$$\mathfrak{M} = \bigcup_{q \leq L^D} \bigcup_{\substack{a=1 \\ (a,q)=1}}^q I'_{q,a} \quad \text{major arcs,} \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathfrak{M} \quad \text{minor arcs,}$$

$$I'_{B,q,a} = \{ \alpha \in [0, 1] \mid \exists \beta \in I'_{q,a} : B\alpha \equiv \beta \pmod{1} \} = B^{-1} \bigcup_{r=0}^{B-1} (r + I'_{q,a}),$$

$$\mathfrak{M}_B = \bigcup_{q \leq L^D} \bigcup_{\substack{a=1 \\ (a,q)=1}}^q I'_{B,q,a}, \quad \mathfrak{m}_B = [0, 1] \setminus \mathfrak{M}_B.$$

$$R^*(n) = R^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk=n \\ N-Y < Ah \leq N \\ Bk \leq Y}} \Lambda(h)\Lambda(k),$$

$$M^*(n) = M^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk=n \\ N-Y < Ah \leq N \\ Bk \leq Y}} 1.$$

$$e(\alpha) = e^{2\pi i \alpha}, \quad e_q(a) = e(a/q),$$

$$S_A(\alpha) = \sum_{N-Y < Ah \leq N} \Lambda(h)e(Ah\alpha), \quad S_B(\alpha) = \sum_{Bk \leq Y} \Lambda(k)e(Bk\alpha),$$

$$S(\beta) = \sum_{Bk \leq Y} \Lambda(k)e(k\beta), \quad T(\eta) = \sum_{Bk \leq Y} e(k\eta),$$

$$R(\eta, q, a) = S\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\varphi(q)} T(\eta),$$

$$W(\chi, \eta) = \sum_{Bk \leq Y} \Lambda(k) \chi(k) e(k\eta) - \delta_\chi T(\eta),$$

with $\delta_\chi = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$

$$\sum_{a=1}^q \frac{1}{(a, q)^s} = \sum_{\substack{a=1 \\ (a, q)=1}}^q \frac{1}{(a, q)^s}, \quad \|\beta\| = \text{distance of } \beta \text{ from the nearest integer}, \quad d(q) = \sum_{d|q} 1.$$

$$c_q(m) = \sum_{a=1}^q e\left(\frac{ma}{q}\right), \text{ Ramanujan's sum}$$

$$\tau(\chi) = \sum_{a=1}^q \chi(a) e\left(\frac{a}{q}\right), \text{ Gauss' sum}$$

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n).$$

The constants in the \ll and O -symbols might depend on ε, A, B, C , even in an ineffective way.

3 Lemmas

Lemma 1. *Let $0 < \varepsilon < 5/12$, $D, G > 0$ be arbitrary constants, and N be a sufficiently large integer. Suppose that $N^{7/12+\varepsilon} \leq Y \leq N$ and $q \leq L^D$. We have*

$$\sum_{\substack{N-Y < h \leq N \\ h \equiv a \pmod{q}}} \Lambda(h) = \frac{Y}{\varphi(q)} + O_{\varepsilon, D, G}(Y L^{-G}),$$

for $(a, q) = 1$.

Proof. We have (see [PPS])

$$\psi(N, \chi) - \psi(N - Y, \chi) = \delta_\chi Y + O_{\varepsilon, D, G}(Y L^{-G}), \quad \text{for every } \chi \pmod{q},$$

with $G > 0$, $N^{7/12+\varepsilon} \leq Y \leq N$, $q \leq L^D$.

Then the Lemma follows from the property

$$\sum_{\substack{N-Y < h \leq N \\ h \equiv a \pmod{q}}} \Lambda(h) = \frac{1}{\varphi(q)} \sum_X \bar{\chi}(a) (\psi(N, \chi) - \psi(N - Y, \chi)).$$

Lemma 2. *If $(A, B) = 1$ and $n \in A$, we have*

$$\prod_{p|An} \left(1 + \frac{1}{p-1}\right) \prod_{p \nmid ABn} \left(1 - \frac{1}{(p-1)^2}\right) = \sum_{q=1}^{+\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q),$$

where $f_{A,B}(q) = \mu(q(q, B))\mu(q, A)\varphi(q, A)$.

Proof. As in [V; p.34], using the well-known property $c_q(n) = \frac{\mu(q/(q,n))\varphi(q)}{\varphi(q/(q,n))}$, we obtain

$$\begin{aligned} \sum_{X < q \leq Y} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q) &= \sum_{\substack{t|n \\ u|A}} \frac{\mu(tu)}{\varphi(tu)} \sum_{\substack{X \\ tu < q \leq Y \\ (q, ABn)=1}} \frac{\mu(q)}{\varphi(q)^2} \ll \\ &\ll \sum_{d|An} \frac{\mu(d)^2}{\varphi(d)} \min\left(\frac{d}{X}, 1\right). \end{aligned}$$

Hence $\sum_{q=1}^{+\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q)$ converges. Furthermore, it is easy to see that

$$\frac{\mu(p^s)}{\varphi(p^s)^2} c_{p^s}(n) f_{A,B}(p^s) = \begin{cases} 1, & \text{if } s = 0, \\ \frac{1}{\varphi(p)}, & \text{if } s = 1 \text{ and } p|An, \\ -\frac{1}{\varphi(p)^2}, & \text{if } s = 1 \text{ and } p \nmid ABn, \\ 0, & \text{if } s = 1 \text{ and } p|B \text{ or } s > 1. \end{cases}$$

Then, the Lemma is proved.

Lemma 3. *Let X and Y be natural numbers such that $X < Y$. If $(A, B) = 1$ then for every real $\xi \neq 0$ we have that*

$$\sum_{\substack{X < t \leq Y \\ t \in A}} e(t\xi) \ll_{A,B} \min\left(\frac{Y-X}{\hat{\delta}AB}, \frac{1}{\|\hat{\delta}AB\xi\|}\right),$$

where $\hat{\delta} = \frac{3-(-1)^{AB}}{2}$.

Proof. This is well-known for $A = B = 1$ (see [Vi]). Let $h_1, h_2, \dots, h_{\varphi(\hat{\delta}AB)}$ be a reduced residue system mod $\hat{\delta}AB$. Define $h'_i = \min\{X < t \leq Y : t \equiv h_i \pmod{\hat{\delta}AB}\}$, for every $i = 1, 2, \dots, \varphi(\hat{\delta}AB)$. Then we write

$$\begin{aligned} \sum_{\substack{X < t \leq Y \\ t \in \mathcal{A}}} e(t\xi) &= \sum_{i=1}^{\varphi(\hat{\delta}AB)} e(h'_i \xi) \sum_{0 \leq s \leq \frac{Y-X}{\hat{\delta}AB}} e(s\hat{\delta}AB\xi) \ll \\ &\ll \varphi(\hat{\delta}AB) \min\left(\frac{Y-X}{\hat{\delta}AB}, \frac{1}{\|\hat{\delta}AB\xi\|}\right). \end{aligned}$$

Thus, the Lemma is proved.

Lemma 4. *Let us denote*

$$I''_{q,a} = \begin{cases} I_{q,a} \setminus I'_{q,a} & \text{if } q \leq L^D, \\ I_{q,a} & \text{if } q > L^D. \end{cases}$$

We have that

$$\max_{\substack{q \leq Q \\ (a,q)=1}} \int_{I''_{q,a}} |S(\beta)|^2 d\beta \ll YL^{-2C-3}.$$

Proof. See [PP; §5].

4 Proof of Theorems 1 and 2

We note that $M^*(n) = \frac{N+Y-2n+1}{AB} + O(1)$. Moreover, in the case $Y = N$, for $N < n \leq N + H$ we have that $R^*(n) = R(n) + O_{A,B}(HL)$, $M^*(n) = M(n) + O_{A,B}(HL)$. Hence, from $\mathfrak{S}(n) \ll L$ we get that

$$\begin{aligned} \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} |R(n) - M(n)\mathfrak{S}(n)|^2 &\ll \\ \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} |R^*(n) - M^*(n)\mathfrak{S}(n)|^2 + H^3 L^2 &= \end{aligned}$$

$$= \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| R^*(n) - \frac{Y}{AB} \mathfrak{S}(n) \right|^2 + O(H^3 L^2).$$

Moreover, writing

$$\begin{aligned} \sum_{\mathfrak{m}_B} &= \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| \int_{\mathfrak{m}_B} S_A(\alpha) S_B(\alpha) e(-n\alpha) d\alpha \right|^2, \\ \sum_{\mathfrak{M}_B} &= \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| \int_{\mathfrak{M}_B} S_A(\alpha) S_B(\alpha) e(-n\alpha) d\alpha - \frac{Y}{AB} \mathfrak{S}(n) \right|^2, \end{aligned}$$

we have

$$\sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| R^*(n) - \frac{Y}{AB} \mathfrak{S}(n) \right|^2 \ll \sum_{\mathfrak{m}_B} + \sum_{\mathfrak{M}_B}.$$

In order to prove Theorems 1 and 2 it suffices to choose $H = Y^{1/3+\varepsilon}$, where $Y = N$ in Theorem 1 and $Y = N^{7/12+\varepsilon}$ in Theorem 2 and show that

$$\sum_{\mathfrak{M}_B} \ll HY^2 L^{-C}, \quad (1)$$

$$\sum_{\mathfrak{m}_B} \ll HY^2 L^{-C}. \quad (2)$$

Proof of (1). In this section we show that (1) holds for $N^{7/12+\varepsilon} \leq Y \leq N$. For any $\alpha \in I'_{B,q,a}$, we have that

$$S_B(\alpha) = S(\beta) = \sum_{Bk \leq Y} \Lambda(k) e(k\beta),$$

where $\beta \in I'_{q,a}$ with $B\alpha \equiv \beta \pmod{1}$. Then, for $\beta = \frac{a}{q} + \eta$, $\eta \in \xi'_q$, we write

$$S_B(\alpha) = S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta, q, a).$$

Now, we have that

$$R(\eta, q, a) = \sum_{Bk \leq Y} \Lambda(k) e\left(k\left(\frac{a}{q} + \eta\right)\right) - \frac{\mu(q)}{\varphi(q)} T(\eta) =$$

$$\begin{aligned} & \sum_{b=1}^q e\left(\frac{ab}{q}\right) \sum_{\substack{Bk \leq Y \\ k \equiv b \pmod{q}}} \Lambda(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)}T(\eta) + O(\log^2 Y) = \\ & \sum_{b=1}^q e\left(\frac{ab}{q}\right) \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(b) \sum_{Bk \leq Y} \Lambda(k)\chi(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)}T(\eta) + O(\log^2 Y) = \\ & \frac{1}{\varphi(q)} \sum_{\chi} \chi(a)\tau(\bar{\chi})W(\chi, \eta) + O(\log^2 Y). \end{aligned}$$

By partial summation, using the Siegel-Walfisz theorem, we have that

$$W(\chi, \eta) \ll YL^{-6D},$$

uniformly for $q \leq L^D$, $\frac{a}{q} + \eta \in I'_{q,a}$. Thus, from the well-known estimate $\tau(\chi) \ll q^{1/2}$, we obtain

$$R(\eta, q, a) \ll YL^{-\frac{11D}{2}}, \quad (3)$$

uniformly for $\eta \in \xi'_{q,a}$, $q \leq L^D$, $(a, q) = 1$. Hence, we write

$$\begin{aligned} \int_{\mathfrak{M}_B} S_A(\alpha)S_B(\alpha)e(-n\alpha)d\alpha &= \frac{1}{B} \sum_{q \leq L^D} \sum_{a=1}^q \sum_{r=0}^{B-1} e\left(-\frac{n}{B}\left(\frac{a}{q} + r\right)\right) \times \\ & \int_{\xi'_q} S_A\left(\frac{1}{B}\left(\frac{a}{q} + r + \eta\right)\right) \left(\frac{\mu(q)}{\varphi(q)}T(\eta) + R(\eta, q, a)\right) e\left(-\frac{n}{B}\eta\right) d\eta \ll \\ & \sum_1 + \sum_2 + \sum_3, \end{aligned}$$

where

$$\begin{aligned} \sum_1 &= \frac{1}{B} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^q \sum_{r=0}^{B-1} e\left(-\frac{n}{B}\left(\frac{a}{q} + r\right)\right) \times \\ & \int_{-\frac{1}{2}}^{\frac{1}{2}} S_A\left(\frac{1}{B}\left(\frac{a}{q} + r + \eta\right)\right) T(\eta) e\left(-\frac{n}{B}\eta\right) d\eta, \\ \sum_2 &= \frac{1}{B} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^q \sum_{r=0}^{B-1} e\left(-\frac{n}{B}\left(\frac{a}{q} + r\right)\right) \times \end{aligned}$$

$$\begin{aligned}
& \int_{\frac{LAD}{qY}}^{1/2} S_A\left(\frac{1}{B}\left(\frac{a}{q} + r + \eta\right)\right) T(\eta) e\left(-\frac{n}{B}\eta\right) d\eta, \\
\sum_3 &= \frac{1}{B} \sum_{q \leq L^D} \sum_{a=1}^q \sum_{r=0}^{B-1} e\left(-\frac{n}{B}\left(\frac{a}{q} + r\right)\right) \times \\
& \int_{\xi'_q} S_A\left(\frac{1}{B}\left(\frac{a}{q} + r + \eta\right)\right) R(\eta, q, a) e\left(-\frac{n}{B}\eta\right) d\eta.
\end{aligned}$$

Recalling that

$$\frac{1}{B} \sum_{r=0}^{B-1} e_B(r(Ah - n)) = \begin{cases} 1 & \text{if } B \mid Ah - n, \\ 0 & \text{if } B \nmid Ah - n, \end{cases}$$

we have

$$\begin{aligned}
\sum_1 &= \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{N-Y < Ah \leq N} c_{qB}(Ah - n) \Lambda(h) \times \\
& \frac{1}{B} \sum_{r=0}^{B-1} e_B(r(Ah - n)) \sum_{Bk \leq N} \int_{-\frac{1}{2}}^{\frac{1}{2}} e_B(\eta(Ah + Bk - n)) d\eta \\
&= \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{N-Y < Ah \leq N \\ Ah \equiv n \pmod{B}}} c_q(B^{-1}(Ah - n)) \Lambda(h). \tag{4}
\end{aligned}$$

Using the property $c_q(m) = \sum_{d|(q,m)} d\mu\left(\frac{q}{d}\right)$, we write

$$\sum_{\substack{N-Y < Ah \leq N \\ Ah \equiv n \pmod{B}}} c_q(B^{-1}(Ah - n)) \Lambda(h) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{N-Y < Ah \leq N \\ Ah \equiv n \pmod{Bd}}} \Lambda(h).$$

We observe that, since $(A, B) = 1$ and $(AB, n) = 1$, the linear congruence $Ah \equiv n \pmod{Bd}$ has a solution if and only if $(A, d) = 1$ and, if it exists, this solution is unique. Then, from Lemma 1 we obtain

$$\begin{aligned}
& \sum_{\substack{N-Y < Ah \leq N \\ Ah \equiv n \pmod{B}}} c_q(B^{-1}(Ah - n)) \Lambda(h) = \\
& \frac{Y}{A} \sum_{\substack{d|q \\ (d, An)=1}} \frac{d}{\varphi(dB)} \mu\left(\frac{q}{d}\right) + O_{\varepsilon, D, \alpha}(qd(q)Y A^{-1}L^{-G}) =
\end{aligned}$$

$$\frac{Y}{A\varphi(B)} \sum_{\substack{d|q \\ (d, An)=1}} \frac{\varphi(d, B)}{(d, B)} \frac{d}{\varphi(d)} \mu\left(\frac{q}{d}\right) + O_{\varepsilon, D, G}(qd(q)YA^{-1}L^{-G}). \quad (5)$$

For every square-free q , we have

$$\sum_{\substack{d|q \\ (d, An)=1}} \frac{\varphi(d, B)}{(d, B)} \frac{d}{\varphi(d)} \mu\left(\frac{q}{d}\right) = \frac{c_q(n)}{\varphi(q)} f_{\Lambda, B}(q), \quad (6)$$

where $f_{\Lambda, B}(q) = \mu(q(q, B))\mu(q, A)\varphi(q, A)$ (see Lemma 2).

Since $T(\eta) \ll \min(YB^{-1}, \|\eta\|^{-1})$, using the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} & \left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} S_{\Lambda}(B^{-1}(a/q + r + \eta))T(\eta)e(-nB^{-1}\eta)d\eta \right| \leq \\ & \left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} \left| S_{\Lambda}(B^{-1}(a/q + r + \eta)) \right|^2 d\eta \right|^{1/2} \times \\ & \left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} \|\eta\|^{-2} d\eta \right|^{1/2} \ll q^{1/2}YL^{1-2D}. \end{aligned}$$

Analogously, from (3) we have that

$$\int_{\mathcal{I}'_q} S_{\Lambda}(B^{-1}(a/q + r + \eta))R(\eta, q, a)e(-nB^{-1}\eta)d\eta \ll q^{-1/2}YL^{1-\frac{7D}{2}}$$

Therefore, the contribution of \sum_2 and \sum_3 to $\sum_{\mathfrak{M}_B}$ is

$$\ll_{\Lambda, B} HY^2 \max(L^{2-D}, L^{2-4D}) \ll HY^2L^{-C}. \quad (7)$$

Recalling that $\frac{B}{\varphi(B)} = \prod_{p|B} \left(1 + \frac{1}{p-1}\right)$, from (4)-(7) it follows that

$$\begin{aligned} \sum_{\mathfrak{M}_B} & \ll \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| \sum_1 - \frac{Y}{AB} \mathfrak{S}(n) \right|^2 + O(HY^2L^{-C}) \ll \\ & \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| \frac{Y}{AB} \left(\frac{B}{\varphi(B)} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)^2} f_{\Lambda, B}(q) c_q(n) - \mathfrak{S}(n) \right) \right|^2 + \end{aligned}$$

$$O\left(\sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| Y A^{-1} L^{-G} \sum_{q \leq L^D} \overline{\frac{qd(q)}{\varphi(q)}} \right|^2\right) + O(HY^2 L^{-C}).$$

The contribution of the first O -term is

$$\ll HY^2 L^{3D-2G} \ll HY^2 L^{-C},$$

provided $2G > 3D + C$. Then, from the proof of Lemma 2, it is easy to see that

$$\sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \left| \frac{B}{\varphi(B)} \sum_{q \leq L^D} \frac{\mu(q)}{\varphi(q)^2} f_{\mathcal{A}, B}(q) c_q(n) - \mathfrak{S}(n) \right|^2 \ll HL^{2-D} \ll HL^{-C}.$$

Thus, we conclude that (1) is proved.

Proof of (2). By the Cauchy-Schwarz inequality, Parseval's identity, the Brun-Titchmarsh inequality and the Lemma 3 we have that

$$\begin{aligned} \sum_{\mathfrak{m}_B} &= \sum_{\substack{N < n \leq N+H \\ n \in \mathcal{A}}} \int_{\mathfrak{m}_B} S_A(\xi) S_B(\xi) e(-n\xi) d\xi \int_{\mathfrak{m}_B} \overline{S_A(\alpha) S_B(\alpha)} e(n\alpha) d\alpha \ll \\ &\int_{\mathfrak{m}_B} |S_A(\xi) S_B(\xi)| \left(\int_{\mathfrak{m}_B} |S_A(\alpha) S_B(\alpha)| \times \right. \\ &\quad \left. \min\left(\frac{H}{\hat{\delta}_{AB}}, \frac{1}{\|\hat{\delta}_{AB}(\alpha - \xi)\|}\right) d\alpha \right) d\xi \ll \\ &(YL)^{\frac{3}{2}} \sup_{\xi \in \mathfrak{m}_B} \left(\int_{\mathfrak{m}_B} |S_B(\alpha)|^2 \min\left(\frac{H}{\hat{\delta}_{AB}}, \frac{1}{\|\hat{\delta}_{AB}(\alpha - \xi)\|}\right)^2 d\alpha \right)^{\frac{1}{2}} \ll \\ &(YL)^{\frac{3}{2}} \sup_{\xi \in [0,1]} \left(\int_{\mathfrak{m}} |S(\beta)|^2 \min\left(\frac{H}{\hat{\delta}_{AB}}, \frac{1}{\|\hat{\delta}_A(\beta - \xi)\|}\right)^2 d\beta \right)^{\frac{1}{2}} \ll \\ &(YL)^{\frac{3}{2}} \sum_{s=1}^A \sup_{\xi \in [0,1]} \left(\int_{\mathfrak{m}_s} |S(\beta)|^2 \min\left(\frac{H}{\hat{\delta}_{AB}}, \frac{1}{\|\hat{\delta}_A(\beta - \xi)\|}\right)^2 d\beta \right)^{\frac{1}{2}}, \end{aligned}$$

where $\mathfrak{m}_s = \mathfrak{m} \cap J_s$, with $J_s = \left[\frac{s-1}{A}, \frac{s}{A}\right]$, $s = 1, \dots, A$.

Hence, (2) is proved whenever

$$\int_{m_s \cap (\xi - \frac{1}{H}, \xi + \frac{1}{H})} |S(\beta)|^2 d\beta \ll YL^{-2C-3}, s = 1, \dots, A \quad (8)$$

for $Y_o(C, \varepsilon) \leq Y$, $Y^{1/3+\varepsilon} \leq H$, uniformly for $\xi \in [0, 1]$.
Since, for $\frac{a}{q} \neq \frac{a'}{q'}$ and $q, q' \leq Q$, we have that

$$\left| \frac{a}{q} - \frac{a'}{q'} \right| \geq \frac{1}{Q^2} = \frac{4}{H},$$

then there are at most two punctured arcs $I''_{q,a}$, with $q \leq Q$ and $(a, q) = 1$ (see Lemma 4) which intersect $(\xi - 1/H, \xi + 1/H)$. Then, in order to establish (8) it suffices to show that

$$\max_{\substack{q \leq Q \\ (a, q) = 1}} \int_{I''_{q,a}} |S(\beta)|^2 d\beta \ll YL^{-2C-3}.$$

But this is given by Lemma 4. Hence, (2) is proved and the proof of Theorems 1 and 2 is complete.

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