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Biorthogonal systems in certain Banach spaces.

M. VALDIVIA

Dedicated to Professor Baltasar Rodríguez Salinas

Abstract

Let X be an Asplund space. We show in this paper that X admits a total biorthogonal system, $(x_i, u_i)_{i\in I}$ such that the closed linear hull of $\{x_i : i \in I\}$ is a weakly compactly generated Banach space. We also prove that if Y is a weakly compactly convex-determined normed subspace of a Banach space X with dens $Y \geq$ dens X^*_σ then there is a total biorthogonal system $(x_i, u_i)_{i\in I}$ for X such that the linear hull $\{x_i : i \in I\}$ is a dense subspace of Y.

Throughout this paper all vector spaces are suppose to be real ones. The set of positive integers is denote by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$.

When no confusion occurs, $\|\cdot\|$ will represent the norm in a normed space X; unless stated, $B(X)$ is the closed unit ball of X. We write X^* for the conjugate space of X; X^{**} for the conjugate of X^*, and we identify X, in the usual way, with a subspace of X^{**}. If A is a subset of X^*, A_σ stands for such a subset equiped with the induced weak-star topology of X^*; $A_\perp$ is the subspace of X orthogonal to A, and A° is the polar set of A in X. By $< \cdot, \cdot >$ we represent the usual duality between X and X^*, i.e., if $x \in X$ and $u \in X^*$, $< x, u >= u(x)$. For a subset M of X, $M^\perp$ will be the subspace of X^* orthogonal to M, and M° the polar of M in X^*; also, lin M will denote the linear span of M. The

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space X endowed with the weak topology will be represented by X_σ.

For a dense subspace Y of X, $\sigma(X^*, Y)$ will be the topology on X^* defined by pointwise convergence respect to Y; $\mu(X^*, Y)$ will denote the topology on X^* given by uniform convergence respect to each weakly compact absolutely convex subset of Y. We shall shorten X^*_μ to mean X^* with the topology $\mu(X^*, X)$. Given two closed subspaces X_1 and X_2 of X, we say X_1 is an orthogonal complement of X_2 in $X_1 + X_2$ provided $X_1 \cap X_2 = \{0\}$ and the projection of $X_1 + X_2$ onto X_2 along X_1 has norm one.

For a continuous projection T in the normed space X, T^* denotes its conjugate projection in X^*.

$|A|$ will be the cardinal number of the set A. Also, $|\alpha|$ will stand for the cardinal number of the ordinal α; ω is the first infinite ordinal, while $\aleph_0$ is the first infinite cardinal number.

The density character of a topological space M is the smallest cardinal number λ for which there is a dense subset A of M with $|A| = \lambda$. We then write dens $M = \lambda$.

A projective resolution of the identity operator in a Banach space X is a family

$$\{T_\alpha : \omega \leq \alpha \leq \mu\}$$

of continuous projections in X, where μ is the first ordinal number such that $|\mu| = \operatorname{dens} X$, and T_μ is the identity operator on X,

$$\| T_\alpha \| = 1, \ \operatorname{dens} T_\alpha(X) \leq |\alpha|,$$

$$T_\alpha \circ T_\beta = T_\beta \circ T_\alpha = T_\beta, \text{ if } \omega \leq \beta \leq \alpha \leq \mu,$$

and for each limit ordinal α, $\omega < \alpha \leq \mu$, the closure of

$$\cup \{T_\beta(X) : \omega \leq \beta < \alpha\}$$

in X coincides with $T_\alpha(X)$. It is known that given $x \in X$ and $\varepsilon > 0$ there is only a finite number of ordinals α satisfying

$$\| (T_{\alpha+1} - T_\alpha)(x) \| > \varepsilon,$$

A Banach space X is said to be Asplund provided that the conjugate Y^* of each separable subspace Y of X is also separable. For each Asplund space X, M. Fabian and G. Godefroy show in [2] that the identity operator of X^* has a projective resolution. Some of the ideas in [2] are used in this paper.

If $(x_i)_{i\in I}$ is a collection of vectors in the normed space X, $[x_i : i \in I]$ is the closure of $\text{lin}\{x_i : i \in I\}$. A biorthogonal system for X,

$$(x_i, u_i)_{i\in I}, x_i \in X, u_i \in X^*, i \in I,$$

is total whenever $\text{lin}\{u_i : i \in I\}$ is dense in X^*_σ. When $[x_i : i \in I]$ is X such a system said to be complete. Whenever $(x_i, u_i)_{i\in I}$ is both complete and total then it is called a Markushevich basis. If besides $[u_i : i \in I] = X^*$ such a basis is said to be shrinking.

A polish space M is a separable topological space admitting a compatible metric d such that (M, d) is complete.

For a given set I, $\Sigma^{(I)}$ denotes the subspace of the topological space $\boldsymbol{R}^I$ formed by all elements $(x_i : i \in I)$ such that the set

$$\{i \in I : x_i \neq 0\}$$

is countable. Every compact topological space homeomorphic to a subspace of $\Sigma^{(I)}$, for some I, is called a Corson compact.

Let P and Y be two topological spaces. A mapping φ from P on the power set of Y is said to be upper semicontinuous if, for a given p in P and a neighbourhood V of the set $\varphi(p)$ there is a neighbourhood U of p such that $\varphi(q)$ is contained in V for every q in U.

A topological space Y is defined as K-analytic, [3], if there is a polish space P and a mapping φ from P in the compact subsets of Y satisfying the following conditions

(a) φ is upper-semicontinuous.

(b) $\cup\{\varphi(p) : p \in P\} = Y$.

In the above definition we want P to be only separable and metrizable, instead of polish, then Y is said to be countably determined, [12].

We say that the normed space X is weakly convex-K-analytic if there is a mapping φ from a polish space P on the weakly compact absolutely convex subsets of X such that it is upper semicontinuous respect to the weak topology of X and $\cup\{\varphi(p) : p \in P\} = X$. If in this definition we want P to be separable and metrizable, instead of polish, we obtain the definition of a wekly countable convex-determined normed space X, [11].

1 Biorthogonal system in Asplund spaces

Before starting the construction of certain biorthogonal systems in Asplund spaces we give three preliminary propositions.

Proposition 1 *Let X be a Banach space. Let $A_\circ$ and $B_\circ$ be infinite subsets of X and X^*, respectively, such that $\mid A_\circ \mid = \mid B_\circ \mid$. Let (ψ_n) be a sequence of continuous mappings from X into X^* which converges pointwise to the mapping ψ. Then there are two closed subspaces E and F of X and X^*, respectively, satisfying the following conditions:*

(a) $\operatorname{dens} E \leq \mid A_\circ \mid$, $\operatorname{dens} F \leq \mid B_\circ \mid$, $A_\circ \subseteq E$, $B_\circ \subseteq F$.

(b) $E^\perp$ *is an orthogonal complement of F in $F + E^\perp$.*

(c) *For each x* in *E, $\psi_n(x)$ and $\psi(x)$, $n = 1, 2 \cdots$, are in F.*

Proof. For each u in X^* and each positive integer n, we choose in X an element $x(u,n)$ such that

$$\| x(u,n) \| = 1, < x(u,n), u > \geq \| u \| - \frac{1}{n}.$$

We proceed inductively and suposse that, for a non-negative integer m, we have already found the sets

$$A_m \subseteq X,\ B_m \subseteq X^*,\ \mid A_\circ \mid = \mid A_m \mid = \mid B_m \mid .$$

Let C_n and D_m be the lineat spans over the field of rationals of A_m and B_m, respectively. We write

$$A_{m+1} : C_m \cup \{x(u,n) : u \in D_m,\ n = 1, 2, \cdots\}$$

$$B_{m+1} := D_m \cup \{\psi_n(x) : x \in C_m,\ n = 1, 2, \cdots\}$$

Let E and F denote the closures of $\bigcup_{m=0}^{\infty} A_m$ and $\bigcup_{m=0}^{\infty} B_m$ in X and X^*, respectively. Clearly, E and F are Banach spaces such that

$$\operatorname{dens} E \leq \mid A_\circ \mid,\ \operatorname{dens} F \leq \mid B_\circ \mid,\ A_\circ \subset E,\ B_\circ \subset F.$$

We take now $v \in E^{\perp}$, $w \in F$ and $\varepsilon > 0$. We may find a positive integer m, $\frac{1}{m} < \varepsilon$, and element u in B_m so that $\| w - u \| < \varepsilon$. Then

$$\| w \| \leq \| w - u \| + \| u \| < \varepsilon + < x(u,m), u > + \frac{1}{m}$$

$$\leq 2\varepsilon + < x(u,m), u + v > \leq 2\varepsilon + |< x(u,m), u - w >|$$

$$+ |< x(u,m), v + w >| \leq 2\varepsilon + \| u - w \| + \| v + w \| \leq 3\varepsilon + \| v + w \|$$

and consequently

$$\| w \| \leq \| v + w \|,$$

hence we conclude that $E^{\perp}$ is an orthogonal complement of F in $F + E^{\perp}$.

Take now x in E. We many find a sequence (x_m) convergent to x so that x_m is an A_m, $m = 1, 2 \cdots$ If we fix a positive integer n, we have

$$\psi_n(x_m) \in F, \; m = 1, 2 \cdots,$$

and thus

$$\psi_n(x) = \lim \psi_n(x_m) \in F,$$

hence

$$\psi(x) = \lim \psi_n(x) \in F.$$

q.e.d.

Proposition 2 *Let X be a Banach space that does not contain a copy of ℓ_1. Let M be an absolutely convex bounded and closed subset of X^*. Let ψ be a mapping of the first Baire class from X to X^* such that*

$$\psi(x) \in M, \; < x, \psi(x) > = \sup \{< x, u >: u \in M\}, \; x \in X.$$

Then M is weak star compact and it coindices wit the closed absolutely convex hull in X^* of $(1) \{\psi(x) : x \in X\}$.

Proof. Let us assume that the stated property does not hold. Let P denote the weak-star closure of M in X^*. We take an element $u_\circ$ in P not contained in the closed absolutely convex hull of (1) in X^*. Let (ψ_n) be a sequence of continuous mappings from X to X^* pointwise convergent to ψ. We find two countably infinite subsets $A_\circ$ and $B_\circ$ of X and X^*, respectively, so that $u_\circ$ be in $B_\circ$. By applying the former proposition we get hold of two closed subspaces E and F of X and

X^*, respectively, with the properties there stated. We identify E^* with $X^*/E^\perp$ in the usual manner. Let f denote the canonical mapping from X^* onto $X^*/E^\perp$. Let A be the closure of $f(M)$ in $X^*/E^\perp$. If φ is the restriction of $f \circ \psi$ to E, then, for every x of E,

$$\varphi(x) = f(\psi(x)) \in A,$$

$$< x, \varphi(x) >=< x, \psi(x) >= \sup\{< x, u >: u \in M\} =$$
$$= \sup\{< x, u >: u \in A\}.$$

Since E is separable with no copy of ℓ_1, we may apply [4, Th. III.4] to conclude that $f(u_o)$ is in the closed absolutely convex hull in $X^*/E^\perp$ of

$$\{\varphi(x) : x \in E\}.$$

Since F is isometric to the subspace $f(F)$ of $X^*/E^\perp$ and u_o belongs to F we have that u_o belongs to the closed absolutely convex hull in F of

$$\{\psi(x) : x \in E\},$$

which is a contradiction.

q.e.d.

Proposition 3 *Let X be a Banach space not containing copy of ℓ_1. Let ψ be a mapping of the first Baire class from X to X^* such that*

$$\| \psi(x) \|= 1, < x, \psi(x) >=\| x \|, x \in X.$$

Let E and F be two closed subspaces of X and X^*, respectively, such that $E^\perp$ is an orthogonal complement of F in $F + E^\perp$. If $\psi(x)$ is in F for each x in E then $E^\perp$ is an orthogonal complement of F in X^*.

Proof. As before, we identify E^* with $X^*/E^\perp$. Again, let f denote the canonical mapping from X^* onto $X^*/E^\perp$. Let φ be the restriction of $f \circ \psi$ to E. Then $\varphi : E \longrightarrow X^*/E^\perp$ is of the first Baire class,

$$\| \varphi(x) \|= 1, < x, \varphi(x) >=\| x \|, x \in E,$$

hence, by Proposition 2, we have that $B(X^*/E^\perp)$ coincides with the closed absolutely convex hull in $X^*/E^\perp$ of

$$\{\varphi(x) : x \in E\}.$$

Thus, if M denotes the closed absolutely convex hull in F of

$$\{\psi(x) : x \in E\}$$

it follows that $F(M) = B(X^*/E^\perp)$. But $M \subset B(F)$ and $f(B(F))$ is contained in $B(X^*/E^\perp)$. Therefore, $f(B(F)) = B(X^*/E^\perp)$, and so $E^\perp$ is an orthogonal complement of F in X^*.

q.e.d.

In the next two propositions we consider an infinite dimensional Asplund space X such that $\operatorname{dens} X = \operatorname{dens} X^*_\sigma$. Let Y be a closed subspace of X with $\operatorname{dens} Y^*_\sigma = \operatorname{dens} X$. Let μ be the first ordinal of $\operatorname{dens} X$. We set

$$\{\omega_\eta : 0 \leq \eta < \mu\}$$

to be a dense subset of X^*. Let (ψ_n) be a sequence of continuous mappings from X to X^* converging pointwise to the mapping ψ such that

$$\| \psi(x) \| = 1, < x, \psi(x) > = \| x \|, \ x \in X$$

The existence of these mappings in guaranteed by [6]. We represent by $\mathcal{H}$ the family of all triples

$$((x_i, u_i)_{i \in I}, E, \alpha)$$

which accomplish the following conditions:

1. $(x_i, u_i)_{i \in I}$ is a biorthogonal system for X, $| I | \geq \aleph_0$.

2. E is a closed subspace of X such that

$$x_i \in E \cap Y, \ i \in I, \ \operatorname{dens} E = | I | .$$

3. $\psi_n(x) \in [u_i : i \in I]$, $n = 1, 2, \cdots, x \in E$, and $[u_i : i \in I]$ has $E^\perp$ as orthogonal complement in X^*.

4. α denotes an infinite ordinal not exceeding μ and such that $| \alpha | = | I |$,

$$\omega_\eta \in [u_i : i \in I], \ 0 \leq \eta < \alpha.$$

Proposition 4 *$\mathcal{H}$ is non empty.*

Proof. We choose a countably infinite set $A_\circ$ formed by linearly independent vectors of Y and a countably infinite subset $B_\circ$ of X^* containing $\{\omega_\eta : 0 \leq \eta < \omega\}$. Proposition 1 yields two subspaces E and F of X and X^*, respectively, with the properties there stated. Proposition 3 then tells us that F has $E^\perp$ as orthogonal complement in X^*. We apply now the method described in [8, Prop. 1. f.3] to find a biorthogonal system $(x_n, u_n)_{n\in N}$ for X such that

$$\operatorname{lin}\{x_n : n \in \mathbb{N}\} = \operatorname{lin} A_\circ,\ [u_n : n \in \mathbb{N}] = F.$$

Then

$$\left((x_n, u_n)_{n\in\mathbb{N}},\ E, \omega\right) \in \mathcal{H}.$$

q.e.d.

Given two elements of $\mathcal{H}$, we write

$$((x_i, u_i)_{i\in I},\ E, \alpha) < ((y_i, v_j), G, \gamma)$$

whenever $\{(x_i, u_i) : i \in I\}$ and E are strictly contained in $\{(y_i, v_j) : j \in J\}$ and G, respectively, and $\alpha < \gamma$.

Proposition 5 *$(\mathcal{H}, \leq)$ is an inductive ordered set.*

Proof. Clearly, $(\mathcal{H}, \leq)$ is an ordered set. Let $\mathcal{L}$ be a non-empty subset of $\mathcal{H}$ such that $(\mathcal{L}, \leq)$ is linearly ordered. We denote by $\{(c_k, u_k) : k \in K\}$, G and γ the union of the sets $\{(x_i, u_i) : i \in I\}$, the closure of the union of the sets E and the supremum of the ordinals α, respectively, when $((x_i, u_i)_{i\in I},\ E, \alpha)$ ranges covers $\mathcal{L}$. We then have that $(x_k, u_k)_{k\in K}$ is a biorthogonal system for X, G is a closed subspace of X such that

$$x_k \in G \cap Y,\ k \in K, \operatorname{dens} G = | K |,$$

$$\psi_n(x) \in [u_k : k \in K],\ , x \in G,\ n = 1, 2 \cdots,$$

and, in light of Proposition 3, $G^\perp$ is an orthogonal complement of $[u_k : k \in K]$ in X^*. Finally

$$\omega_\eta \in [u_k : k \in K], 0 \leq \eta < \gamma,$$

and $| \gamma | = | K |$.

q.e.d.

Theorem 1 *Let X be an Asplund space such that* $\operatorname{dens} X = \operatorname{dens} X^*_\sigma$. *Let Y be a closed subspace of X with* $\operatorname{dens} Y^*_\sigma = \operatorname{dens} X$. *Then there is a biorthogonal sytem $(x_i, u_i)_{i\in I}$ for X such that*

$$x_i \in Y,\ j \in I,\ [u_i : i \in I] = X^*.$$

Proof. We base our discussion on the density character of X. For separable Banach spaces the property is certainly true, as it follows from the proof of [8, Prop. 1. f. 3]. Let us assume now taht X is not separable and that, for each Asplund space Z such that $\operatorname{dens} Z = \operatorname{dens} Z^*_\sigma < \operatorname{dens} X$ and each closed subspace W of Z, $\operatorname{dens} W^*_\sigma = \operatorname{dens} Z$, there is a biorthogonal system $(z_j, v_j)_{j\in J}$ such that

$$z_j \in W,\ j \in J,\ [v_j : j \in J] = Z^*.$$

Let μ be the first ordinal of $\operatorname{dens} X$. We take a dense subset $\{\omega_\eta : 0 \leq \eta < \mu\}$ of X^*. In light of [6], we may find a mapping ψ of the first Baire class from X to X^* such that

$$\| \psi(x) \| = 1,\ < x, \psi(x) > = \| x \|,\ x \in X.$$

Let (ψ_n) be a sequence of continuous mappings from X to X^* which converges pointwise to ψ. We consider now the inductive ordered set $(\mathcal{H}, \leq)$ formerly defined. Let $((x_i, u_i)_{i\in I}, E, \alpha)$ be a maximal element of $(\mathcal{H}, \leq)$. Assume $\alpha < \mu$. We open up a transfinite induction process by setting $E_\alpha := E$ and $F_\alpha := [u_i : i \in I]$. Suppose also that, for a given ordinal ρ, $\alpha < \rho \leq \mu$, we have already defined E_ν and F_ν, $\alpha \leq \nu < \rho$, so that $E_\nu^\perp$ is an orthogonal complement of F_ν in X^*, $\operatorname{dens} E_\nu = \operatorname{dens} F_\nu = |\, \nu \,|$, $\omega_\eta \in F_\nu$, $0 \leq \eta < \nu$,

$$\psi_n(x) \in F_\nu,\ x \in E_\nu,\ n = 1, 2 \cdots,$$

If ρ is an isolated ordinal, since $\operatorname{dens} Y^*_\sigma > |\, \rho - 1 \,|$, we have that $F_{\rho-1}$ cannot separate points in Y and hence there is a non-zero element z_ρ in $Y \cap (F_{\rho-1})_\perp$. Take M and P to be two dense subsets of $E_{\rho-1}$ and $F_{\rho-1}$, respectively, with $|\, M \,| = |\, P \,| = \operatorname{dens} F_{\rho-1}$. Applying Propositions 1 and 3 to $A_\circ := M \cup \{z_\rho\}$ and $B_\circ := P \cup \{\omega_{\rho-1}\}$ we obtain two closed subspaces E_ρ and F_ρ of X and X^*, respectively, such that

(a) $A_\circ \subset E_\rho$, $B_\circ \subset F_\rho$, $\operatorname{dens} E_\rho = \operatorname{dens} F_\rho = |\, A_\circ \,|$.

(b) $\psi_n(x) \in F_\rho$, $x \in E_\rho$, $n = 1, 2, \cdots$

(c) $E_\rho^\perp$ is an orthogonal complement of F_ρ in X^*.

If ρ is a limit ordinal, we define E_ρ and F_ρ as the closures of $\cup\{E_\nu : \alpha \leq \nu < \rho\}$ and $\cup\{F_\nu : \alpha \leq \nu < \rho\}$ in X and X^*, respectively. Then

$$\operatorname{dens} E_\rho = \operatorname{dens} F_\rho = | \rho |$$

and conditions (b) and (c) are thus satisfied.

We write $T_\rho, \alpha \leq \rho < \mu$, for the projection of X^* onto F_ρ along $E_\rho^\perp$, and T_μ for the identity operator in X^*. The collection

$$\{T_\rho,\ \alpha < \rho \leq \mu\}$$

fulfills then all the requirements of a projective resolution of the identity operator in X^*, except that α may be different from ω. Consequently, given $\delta > 0$ and u in X^*, the set of ordinals ρ for which

$$\| \left(T_{\rho-1} - T_\rho\right)(u) \| > \delta$$

is finite.

By the procedure we have just followed, given $\alpha \leq \xi < \mu$, the subspace $Y \cap E_{\xi+1} \cap (F_\xi)_\perp$ is different from $\{0\}$. Let u be any element of X^* and let us assume that the set of ordinals ξ for which

$$Y \cap E_{\xi+1} \cap (F_\xi)_\perp \subset \{u\}_\perp$$

is not countable. We may then find

$$\alpha \leq \xi_1 < \xi_1 + 1 < \xi_2 < \xi_2 + 1 < \cdots < \xi_m < \xi_m + 1 < \cdots < \mu,$$

$\varepsilon > 0$ and $x_m \in Y \bigcap E_{\xi_m+1} \bigcap (F_{\xi_m})_\perp$, $\| x_m \| = 1$, such that

$$< x_m, u >> \varepsilon,\ m = 1, 2, \cdots$$

Clearly,

$$< x_m, T_{\xi_m+1} u > = < x_m, u >, < x_m, T_{\xi_m} u > = 0,\ m = 1, 2 \cdots$$

and thus

$$\| < T_{\xi_m+1} - T_{\xi_m})(u) \| > \varepsilon,\ m = 1, 2, \cdots,$$

a contradiction. Therefore, the set of such ordinals ξ must be countable.

If now G stands for the closure in X of the linear

$$\bigcup\left\{Y\bigcap E_{\xi+1}\bigcap(F_\xi)_\perp : \alpha\leq\xi<2\alpha\right\},$$

then G is a closed subspace of the Asplund space $E_{2\alpha}$. Let f be the canonical mapping from $E_{2\alpha}$ onto $E_{2\alpha}/E_\alpha$. In the case $|\alpha|=\aleph_0$, we have $\operatorname{dens} G_\sigma^* = \operatorname{dens} E_{2\alpha}$. On the other hand, if $|\alpha|>\aleph_0$, for a subset A of X^*, we have in light of the argument above used that the set of ordinals $\xi, \alpha\leq\xi<2\alpha$, for which there is some u in A not vanishing in $Y\cap E_{\xi+1}\cap(F_\xi)_\perp$ has a cardinal number not greater than $|A|\aleph_0$. Consequently, if A separates points in G, then $|A|=|\alpha|$, and $\operatorname{dens} G_\sigma^* = \operatorname{dens} E_{2\alpha}$. Now, since $F_{2\alpha}$ has $E_{2\alpha}^\perp$ as orthogonal complement in X^*, we may identify $F_{2\alpha}$ with the conjugate of $E_{2\alpha}$ and $F_{2\alpha}\cap F_\alpha^\perp$ with the conjugate of $E_{2\alpha}/E_\alpha$. Since G is an orthogonal complement of F_α in $E_\alpha+G$, it follows that $f(G)$ is a subspace of the Asplund space $E_{2\alpha}/E_\alpha$ isometric to G, hence

$$\operatorname{dens} f(G)_\sigma^* = \operatorname{dens}(E_{2\alpha}/E_\alpha) < \operatorname{dens} X$$

and we may thus find a biorthogonal system $(t_i,u_i)_{i\in I_1}$ for $E_{2\alpha}/E_\alpha$, I_1 disjoint from I, such that

$$t_i\in f(G), i\in I_1,\ [u_i : i\in I_1] = F_{2\alpha}\cap E_\alpha^\perp.$$

By selecting x_i in G with $f(x_i)=t_i$, $i\in I_1$, and setting $J := I\cup I_1$, it turns out that

$$((x_j,u_j)_{j\in J}, E_{2\alpha}, 2\alpha)\in\mathcal{H}$$

and

$$((x_i,u_i)_{i\in I}, E, \alpha) < ((x_j,u_j)_{j\in J}, E_{2\alpha}, 2\alpha),$$

thus attaining a contradiction. Thereby $\alpha=\mu$ and $(x_i,u_i)_{i\in I}$ is a biorthogonal system for X such that

$$x\in Y,\ i\in I,\ [u : i\in I] = X^*.$$

q.e.d.

Theorem 2 *If X is an Asplund space, then there is a total biorthogonal system for X, $(x_i,u_i)_{i\in I}$, such that, if v_i is the restriction of u_i to $[x_i : i\in I]$, $i\in I$, then $(x_i,v_i)_{i\in I}$ is a shrinking Markushevich basis for $[x_i : i\in I]$.*

Proof. If X_σ^* is separable, the result then follows from [8, Prop. 1. f. 3]. Assuming that X_σ^* is not separable, let $\{\omega_\eta : \circ \leq \eta < \mu\}$ be a dense subset of X_σ^*, with μ the first ordinal number of dens X_σ^*. Let (ψ_n) be a sequence of continuous mappings from X to X^* converging pointwise to ψ and such that

$$\| \psi(x) \| = 1, < x, \psi(x) > = \| x \|, x \in X.$$

We consider a countably infinite set $A_\circ$ of linearly independent vectors of X, as well as a countably infinite subset $B_\circ$ of X^* such that $\omega_\eta \in B_\circ, \circ \leq \eta < \omega$. We apply Proposition 1 to obtain subspaces E and F with the properties there stated. We open up a new transfinite induction process by setting $E_\omega := E$ and $F_\omega := F$. Suppose that for an ordinal $\rho, \omega < \rho \leq \mu$, a collection of subspaces E_ν and $F_\nu, \omega \leq \nu < \rho$ have been defined in such a way that $E_\nu^\perp$ is an orthogonal complement of $F_\nu$in X^*,

$$\operatorname{dens} E_\nu = \operatorname{dens} F_\nu = | \nu |, \omega_\eta \in F_\nu, \circ \leq \eta > \nu,$$

$$\psi_n(x) \in F_\nu, x \in E_\nu, n = 1, 2, \cdots$$

Proceeding analogously as in the proof of our last theorem, taking $Y = X$, we are able to construct E_ρ and E_ρ. Since $E_\mu^\perp$ is an orthogonal complement of F_μ in X^*, we may identify F_μ with the conjugate of E_μ. For every ordinal $\alpha, \omega \leq \alpha \leq \mu$, let S_α be the mapping from F_μ onto F_α along to $E_\alpha^\perp \cap F_\mu$. Then

$$\{S_\alpha : \omega \leq \alpha \leq \mu\}$$

is a resolution of the identity operator in F_μ. The argument followed in the proof of the mentioned theorem leads us to

$$\operatorname{dens} E_\mu = \operatorname{dens} (E_\mu)_\sigma^* = | \mu | .$$

By appling Theorem 1 we obtain a biorthogonal system $(x_i, u_i)_{i \in I}$ for E_μ such that $[u_i : i \in I] = X^*/E_\mu^\perp$. Hence, if v_i denotes the restriction of u_i to $[x_i : i \in I]$ we have that $(x_i, v_i)_{i \in I}$ is a shrinking Markushevich for $[x_i : i \in I]$.

q.e.d.

Corollary *If X is an Asplund space, then there is a total biorthogonal system for $X, (x_i, u)_{i \in I}$, such that $[x_i : i \in I]$ is a weakly compactly generated Banach space*

Proof. A straightorward consequence of the former theorem and [9, p. 700].

q.e.d.

2 Weakly countably convex-determined normed spaces

We give here a few results, some of which will be used in the next section.

Proposition 6 *For a Banach space X the following assertions are equivalents:*

(a) *X is weakly countably convex-determined.*

(b) *X is weakly countably determined.*

(c) *There exists a metrizable separable topological space P and a mapping φ from P to the compact subsets of X_σ which is upper semi-continuous, respect to the weak topology of X, and such that $\cup\{\varphi(p) : p \in P\}$ is dense in X.*

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are obvious. We show that $(c) \Rightarrow (a)$. Since every polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$ equipped with the discrete topology, it means no restriction to assume that P is a subspace of $\mathbb{N}^{\mathbb{N}}$. Let B denote the closed unit ball of X. For an element $(n_1, n_2, \cdots, n_q, \cdots)$ in P, let $\psi(n_1, n_2, \cdots, n_q \cdots)$ represent the closed absolutely convex hull of $\varphi(n_1, n_2, \cdots, n_q \cdots)$. By Krein's theorem, [7, p.325], $\psi(n_1, n_2, \cdots, n_q \cdots)$ is weakly compact. Besides, it is not difficult to verify that ψ is upper semicontinuous respect to the weak topology of X. Now, given a positive integer n and a double sequence (n_{rq}) such that, for each positive integer r, $(n_{r1}, n_{r2}, \cdots, n_{rq}, \cdots)$ is in P, we define

$$\Lambda(n, (n_{rq})) := (nB) \cap (\bigcap_{r=1}^{\infty} \psi(n_{r1}, n_{r2}, \cdots, n_{rq}, \cdots) + \frac{1}{r}B)).$$

Clearly, this set is absolutely convex, closed and bounded in X. We consider a sequence

$$(n, (b_{rq}^{(m)})), m = 1, 2, \cdots, \tag{2}$$

such that

$$b_{rq}^{(m)} = n_{rq}, r, q = 1, 2, \cdots, m, \; m = 1, 2, \cdots,$$

and, for fixed positive integers m and r, $(b_{rq}^{(m)})$ be in P. For each positive integer m, we choose an element x_m in $\Lambda(n, (b_{rq}^{(m)}))$. Let $x_\circ$ be a point in the weak-star adherence of (x_m) in X^{**}. For any fixed positive integer r, we may write

$$x_m = y_m + z_m, \; y_m \in \psi(b_{r1}^{(m)}, \; b_{r2}^{(m)} \cdots, \; b_{rq}^{(m)}, \cdots), z_m \in \frac{1}{r}B.$$

Then there is a point $z_\circ$ in the weak-star adherence of (z_m) in X^{**} such that $y_\circ := x_\circ - z_\circ$ is in the weak-star closure of (y_m) in X^{**} and, since

$$((b_{r1}^{(m)}, b_{r2}^{(m)}, \cdots, \; b_{rq}^{(m)}, \cdots), \; m = 1, 2, \cdots,$$

is a sequence in P converging to $(n_{r1}, n_{r2}, \cdots, n_{rq}, \cdots)$, it follows that $y_\circ$ belongs to $\psi(n_{r1}, n_{r2}, \cdots, n_{rq} \cdots)$. Besides this, distance in X^{**} from $x_\circ$ to X is less or equal than $\| x_\circ - y_\circ \| = \| z_\circ \| \leq \frac{1}{r}$ and, being the inequality true for any positive integer r, we have that $x_\circ$ is in X. it is now clear that $x_\circ$ belongs to $\Lambda(n, (n_{rq}))$. If the sequence (2) is taken such that

$$b_{rq}^{(m)} = n_{rq}, \; m, r, q = 1, 2, \cdots$$

then x_m is in $\Lambda(n, (n_{rq})), m = 1, 2, \cdots$, and thus has its weak adherent point $x_\circ$ in this set. Hence $\Lambda(n, (n_{rq}))$ is weakly compact.

Now, let M be the set of all pairs $(n, (n_{rq}))$, where n is a positive integer such that, for each r, $(n_{r1}, n_{r2}, \cdots, r_{rq} \cdots,)$ is in P. We consider in M the topology of termwise convergence. We then have that M is metrizable separable topological space, and Λ is a mapping from M to the weakly compact absolutely convex subsets of X. We see next that Λ is upper semicontinuous respect to the weak topology of X. Take $(n, (n_{rq}))$ in M. Let V be a neighbourhood of $\Lambda(n, (n_{rq}))$ in X_σ. On assuming that no image under Λ of any neighbourhood of $(n, (n_{rq}))$ in M is contained in V, we may find a sequence in M convergent to $(n, (n_{rq}))$ and such that the image under Λ of each of its terms is not contained in V. Since this sequence has a subsequence of the type (2), we may select

$$x_m \in \Lambda(n, (b_{rq}^{(m)}) \backslash V, \; m = 1, 2, \cdots$$

Thus, if $x_\circ$ is a weak-star adherent point of (x_m) in X^{**}, by the argument formerly employed, we have $x_\circ$ in $\Lambda(n, (n_{rq}))$, which is a contradiction.

Finally, for a point x in X, we find a positive integer m such that $x \in mB$ and, for each positive integer r, we find $(m_{r1}, m_{r2}, \cdots, m_{rq}, \cdots,)$ in P such that

$$(x + \frac{1}{r}B) \cap \varphi(m_{r1}, m_{r2}, \cdots, m_{rq} \cdots,) \neq \phi.$$

Hence,

$$x \in (m\,\mathrm{B}) \cap (\psi(m_{r1}, m_{r2}, \cdots, m_{rq} \cdots,) + \frac{1}{r}B).$$

Consequently,

$$\cup\{\Lambda(n, (n_{rq})) : (n, (n_{rq})) \in M\} = X.$$

q.e.d.

Proposition 7 *For a Banach space X, the following assertions are equivalent:*

(a) *X is weakly convex K-analytic.*

(b) *X is weakly K-analytic.*

(c) *There is polish space P and a mapping φ from P on the compact subsets of X_σ which is upper semicontinuous with respect to the weak topology on X, and such that $\cup\{\varphi(p) : p \in P\}$ is dense in X.*

Proof. It is totally analogous to the proof of last proposition, only noticing that P is now identifiable with $\mathbb{N}^{\mathbb{N}}$ and M is the set of all pairs $(n, (n_{rq}))$, n being any positive integer and (n_{rq}) any double sequence of positive integers.

q.e.d.

Note 1 Let X be a weakly compactly generated Banach space. Let D be a weakly compact absolutely convex hull of X such that lin D is dense in X. Let φ denote the mapping from $\mathbb{N}^{\mathbb{N}}$ on the compact subsets of X_σ such that

$$\varphi\,(n_1, n_2, \cdots, n_q, \cdots) = n_1 D, (n_1, n_2, \cdots, n_q, \cdots) \in \mathbb{N}^{\mathbb{N}}$$

Then, condition (c) of the former proposition holds, this yielding Talagrand's result that X is weakly K-analytic, [10].

Proposition 8 *For a normed space X the following assertions are equivalent:*

(a) *X is weakly countbly convex determined.*

(b) *X admits a sequence (U_n) of absolutely convex closed and bounded zero-neighbourhoods such that, for each x in X, there is a decreasing subsequence (U_{n_j}) of (U_n) such that x is in $\bigcap_{j=1}^{\infty} U_{n_j}$ and $\bigcup_{j=1}^{\infty} U_{n_j}^{\circ}$ is a zero neighbourhood in X_{μ}^{*}.*

(c) *In X_{σ}^{*}, there is a sequence (B_n) of compact absolutely convex subsets such that for each zero-neighbourhood W it is possible to find an increasing subsequence (B_{n_j}) of (B_n) so that $\bigcup_{j=1}^{\infty} B_{n_j}$ be a zero-neighbourhood in X_{μ}^{*} contained in W.*

Proof. $(a) \Rightarrow (b)$.B denotes the closed unit ball of X. Let φ be an upper semicontinuous mapping from a subspace P of $\mathbb{N}^{\mathbb{N}}$ on the weakly compact absolutely convex subset of X, X provided with the weak topology, such that

$$X = \cup \{\varphi(p) : p \in P\}$$

Given the positive integers $j, m_1, \cdots, m_j$, we write $Am_1, m_2, \cdots, m_j$ for the subset of P whose elements $(b_1, b_2, \cdots, b_q \cdots)$ satisfy

$$b_q = m_q,\ q = 1, 2, \cdots, j.$$

Given the positive integers $m, j, m_1, m_2, \cdots, m_j$, we define

$$C_{m,m_1,m_2,\cdots,m_j} := X$$

whenever $A_{m_1,m_2,\cdots,m_j}$ is empty. If this latter set is not empty, we write $B_{m_1,m_2,\cdots,m_j}$ for the absolutely convex hull of $\varphi(A_{m_1,m_2,\cdots,m_j})$ and denote by $C_{m,m_1,m_2,\cdots,m_j}$ the closure in X of the set

$$(m\, B) \cap (B_{m_1,m_2,\cdots,m_j} + \frac{1}{j} B).$$

We order all the sets $C_{m,m_1,m_2,\cdots,m_j}$ in a sequence (U_n).

For a point x in X, we find $(m_1, m_2, \cdots, m_q, \cdots)$ in P such that

$$x \in \varphi(m_1, m_2, \cdots, m_q, \cdots).$$

We determine also a positive integer m such that x is in mB. Then, from the sequence

$$C_{m,m_1,m_2,\cdots,m_j}, j = 1, 2, \cdots,$$

it is possible to find a subsequence which at the same time is a subsequence (U_{n_j}) of (U_n). This subsequence is clearly decreasing and

$$x \in \bigcap_{j=1}^{\infty} U_{n_j} = \bigcap_{j=1}^{\infty} C_{m,m_1,m_2,\cdots,m_j}.$$

For each positive integer j, we take x_j in U_{n_j}. Then, there is

$$y_j \in B_{m_1,m_2,\cdots,m_j}, z_j \in \frac{1}{j}B,\ y_j + z_j \in m\ B,$$

$$\| x_j - (y_j + z_j) \| < \frac{1}{j},\ j = 1, 2, \cdots$$

Since (z_j) converges to zero in X, there is an element $x_\circ$ in X^{**} which is weak-star adherent to both sequences (x_j) and (y_j). Let us suppose that $x_\circ$ does not belong to $\varphi(m_1, m_2, \cdots, m_q, \cdots)$. We find u in X^* such that

$$< x_\circ, u >> 1, |< x, u >|< 1, x \in \varphi(m_1, m_2, \cdots, m_q, \cdots)$$

We determine a positive integer r such that

$$|< x, u >|< 1,\ x \in \varphi(b_1, b_2, \cdots, b_q, \cdots),$$

$$(b_1, b_2, \cdots, b_q, \cdots) \in P,\ b_i = m_i,\ i = 1, 2, \cdots, r.$$

Hence, $|< y_j, u >|< 1, \geq r$, and $|< x_\circ, u >|\leq 1$, a contradiction. Thus, $\bigcap_{j=1}^{\infty} U_{n_j}$ is weakly compact. Consider now

$$v \in X^*,\ v \notin \bigcup_{j=1}^{\infty} U_{n_j}^{\circ}$$

For each positive integer j, we select

$$x_j \in U_{n_j}, |< x_j, v >|> 1$$

We proceed as before to find $x_\circ$ in $\varphi(m_1, m_2, \cdots, m_q, \cdots)$ weakly adherent to (x_j). Then, $|< x_\circ, v >| \geq 1$, and thus

$$\left\{ w \in X^* : |< x, w >| < 1, x \in \bigcap_{j=1}^{\infty} U_{n_j} \right\} \subset \bigcup_{j=1}^{\infty} U_{n_j}^\circ. \tag{3}$$

Since $\bigcap_{j=1}^{\infty} U_{n_j}$ is absolutely convex and weakly compact, the conclusion now follows from (3).

$(b) \Rightarrow (a)$. We arrange in a sequence, (B_n), the sets

$$(\cup \{U_n : n \in J\})^\circ$$

with J ranging over the family of non-empty finite subsets of $\mathbb{N}$. These sets are absolutely convex and compact in X^*_σ. Let W be a zero-neighbourhood in this space. There is a finite number of points $x_1, x_2, \cdots, x_r$ in X such that

$$\{x_1, x_2, \cdots, x_r\}^\circ \subset W.$$

We find, for each x_i, $1 \leq i \leq r$, a decreasing subsequence $(U_{n_j,i})$ of (U_n) such that x_i is in $\bigcap_{j=1}^{\infty} U_{n_j,i}$ and $\bigcup_{j=1}^{\infty} U_{n_j,i}^\circ$ is a zero-neighbourhood in X^*_μ. From the sequence

$$(\bigcup \left\{ U_{n_j,i} : i = 1, 2, \cdots, r \right\})^\circ, j = 1, 2, \cdots$$

we may find a subsequence at the same-time being a subsequence (B_{n_j}) of (B_n). Obviously, (B_{n_j}) is increasing and

$$\bigcup_{j=1}^{\infty} B_{n_j} \subset W.$$

It can be easily seen that

$$\bigcap_{i=1}^{r} \left(\bigcup_{j=1}^{\infty} U_{n_j,i}^\circ \right) = \bigcup_{j=1}^{\infty} B_{n_j},$$

therefore this set is a zero-neighbourhood in X^*_μ.

$(c) \Rightarrow (a)$. Let P stand for the subspace of $\mathbb{N}^{\mathbb{N}}$ consisting on all elements $(n_1, n_2, \cdots, n_q, \cdots)$ such that $n_j < n_{j+1}, j = 1, 2, \cdots$, and $\bigcup_{j=1}^{\infty} B_{n_j}$ be a zero-neighbourhood in X_μ^*. We write

$$\varphi(n_1, n_2, \cdots, n_q, \cdots) := \bigcap_{j=1}^{\infty} B_{n_j}^{\circ}.$$

We have that φ maps P on the weakly compact absolutely convex parts of X.

Given x in X, there is an increasing subsequence (B_{n_j}) of (B_n) such that $\bigcup_{j=1}^{\infty} B_{n_j}$ is contained in $\{x\}^{\circ}$ and is also a zero-neighbourhood in X_μ^*. Then $(n_1, n_2, \cdots, n_q, \cdots)$ belong to P and

$$x \in \varphi(n_1, n_2, \cdots, n_q \cdots).$$

Henceforth,

$$\bigcup \{\varphi(n_1, n_2, \cdots, n_q, \cdots) : n_1, n_2, \cdots, n_q, \cdots) \in P\} = X.$$

Assuming that φ is not upper semicontinuous, there are a point $(n_1, n_2, \cdots, n_q, \cdots)$ in P, a neighbourhood V of $\varphi(n_1, n_2, \cdots, n_q, \cdots)$ in X_σ and a sequence in P,

$$(n_1^{(j)}, n_2^{(j)}, \cdots, n_q^{(j)}, \cdots),\ j = 1, 2, \cdots,$$

such that, for each positive integer $j, n_s^{(j)} = n_s$, $s = 1, 2, \cdots, j$, and

$$\varphi(n_1^{(j)}, n_2^{(j)}, \cdots, n_q^{(j)}, \cdots) \not\subset V$$

Choosing

$$x_j \in \varphi(n_1^{(j)}, n_2^{(j)}, \cdots, n_q^{(j)} \cdots) \backslash V,$$

we have

$$x_j \in \bigcap_{s=1}^{j} B_{n_s}^{\circ}.$$

For a given element u of X^*, there are $\alpha > 0$ and a positive integer m such that αu is in B_{n_m}. Thus

$$|< x_j, u >| \leq \frac{1}{\alpha},\ j \geq m,$$

and so (x_j) is a bounded sequence in X. Now, let $x_\circ$ be weak-star adherent to (x_j) in X^{**}. Obviously,

$$x_\circ \notin \varphi(n_1, n_2, \cdots, n_q, \cdots).$$

Let v be an element of X^* such that

$$|< x_\circ, v >| > 1, |< x, v >| < 1, x \in \varphi(n_1, n_2, \cdots, n_q, \cdots).$$

Since $\overset{\infty}{\underset{j=1}{\cup}} B_{n_j}$ is a zero-neighbourhood in X^*_μ, we have that v belongs to such a neighbourhood. Hence, there is a positive integer k such that v is in B_{n_k}. Thus, x_j is in $B^\circ_{n_k}$ for $j \geq k$, which implies that $< x_\circ, v > \leq 1$, a contradiction.

q.e.d.

Note 2 It is shown in [13] that if X is a weakly countably determined Banach space, there is a sequence (A_n) of absolutely convex closed and bounded zero-neighbourhood in X such that, for each x in X, there is a subsequence (A_{n_j}) of (A_n) for which

$$x \in \bigcap_{j=1}^{\infty} \widetilde{A}_{n_j} \subset X,$$

where $\widetilde{A}_{n_j}$ denotes the weak-star closure of A_{n_j} in $X^{**}, j = 1, 2, \cdots$ This result can be used to obtain condition (b) of proposition 8 when X is a Banach space.

3 Markushevich bases in weakly countably convex-determined normed spaces

It is shown in [12], using a method originally introduced by D. Amir and J. Lindenstrauss to study certain properties of weakly compactly generated Banach spaces, [1], that a weakly countably determined Banach space admits a projective resolution of the identity operator. This result is extend in [11] for certain metrizable locally convex spaces by means of a somewhat simpler method than that of [1].

Following a standard procedure, Markushevich bases for certain Banach spaces, such as those weakly countably determined, may be obtained by means of projective resolutions of the identity operator.

In this section we shall use some ideas of [11] to construct total biorthogonal systems in certain Banach spaces, consequently obtains Markushevich bases, with some additional properties, in weakly countably determined Banach spaces with no mention of projective resolutions of the identity operator.

In the next four propositions, we consider a Banach space X and a sequence (V_n) of absolutely convex closed and bounded zero-neighbourhood in X, with $\|\cdot\|_n$ being Minkowski's functional of $V_n, n = 1, 2, \cdots$ Let Y be a normed subspace of X. We assume that, for each y in Y, there is a decreasing subsequence (V_{n_j}) of (V_n) such that y belongs to $\bigcap_{j=1}^{\infty} V_{n_j}$ and $\bigcup_{j=1}^{\infty} V_{n_j}^{\circ}$ is a zero-neighbourhood of $X^*\,[\mu(X^*, Y)]$.

Proposition 9 *Let $A_\circ$ and $B_\circ$ be two infinite subsets of Y and X^*, respectively, and let λ be a cardinal number such that $|A_\circ| \leq \lambda$, $|B_\circ| \leq \lambda$. There are two closed subspaces E and F of Y and X^*_σ, respectively, such that*

(a) $E \supset A_\circ$, $\operatorname{dens} E \leq \lambda$.

(b) $F \supset B_\circ$, $\operatorname{dens} F \leq \lambda$.

(c) *If $\overline{E}$ denotes the closure of E in $X, F_\perp$ is an orthogonal complement of $\overline{E}$ in $\overline{E} + F_\perp$ respect to $\|\cdot\|_n, n = 1, 2, \cdots$*

(d) $E + F_\perp \cap Y = Y$.

Proof. If u is an element of X^* and m is a positive integer, we put $|u|_m$ for the norm of $u|_Y$ when we consider Y provided with the norm restriction of $\|\cdot\|_m$. For every positive integer r ad x in X, we choose $u(x,r)$ in X^* such that

$$\| u(x,r) \|_r = 1, \| x \|_r = < x, u(x,r) > .$$

Given two positive integers r, s and a vector v in X^* we find $x(v, r, s)$ in Y such that

$$\| x(v,r,s) \|_r = 1, |< x(v,r,x>, v>| \geq \frac{s-1}{s} \mid v \mid_r$$

Proceeding inductively, let us assume that, for a non-negative integer n, we have subset A_n and B_n of Y and X^*, respectively, with $|A_n| \leq \lambda, |B_n| \leq \lambda$. Denote by C_n and D_n the linear span over the field of rational numbers of A_n and B_n, respectively. We set

$$A_{n+1} := C_n \bigcup \{x(v,r,x) : v \in D_n,\ r,s = 1,2,\cdots\},$$
$$B_{n+1} := D_n \bigcup \{u(x,r) : x \in C_n,\ r = 1,2,\cdots\},$$

If E and F are the closures of $\overset{\infty}{\underset{n=0}{\cup}} A_n$ and $\overset{\infty}{\underset{n=1}{\cup}} B_n$ in Y and X^*_σ, respectively, with the topologies induced by those spaces, then E and F are linear spaces such that

$$\operatorname{dens} E \leq \lambda,\ \operatorname{dens} F \leq \lambda,\ E \supset A_\circ,\ F \supset B_\circ.$$

Given x in $\overline{E}, z$ in $F_\perp, r$ in $I\!N$ and $\delta > 0$, we choose a positive integer n and an element t in A_n with $\| x - t \|_r < \delta$. Then

$$\| x \|_r \leq \| x - t \|_r + \| t \|_r < \delta + < t, u(t,r) >= \delta + < t + z, u(t,r) >$$

$$\leq \delta + |< x + z,\ u(t,r) >| + |< t - x,\ u(t,r) >|$$

$$\leq \delta + \| x + z \|_r + \| t - x \|_r < \| x + z \|_r + 2\delta$$

and consequently

$$\| x \|_r \leq \| x + z \|_r,\ r = 1,2,\cdots,$$

whence we have that $\overline{E} \cap F_\perp = \{0\}$ and $F_\perp$ is an orthogonal complement of $\overline{E}$ in $\overline{E} + F_\perp$ respecto to $\| \cdot \|_r$, $r = 1,2,\cdots$ Suppose now that $\overline{E} + F_\perp$ does not contain Y. We find w in X^* whose restriction to Y does not vanish while it does in the Banach space $\overline{E} + F_\perp$. Take $y_\circ$ in Y with $< y_\circ, v >= 3$. We find a decreasing subsequence (V_{n_j}) of (V_n) such that

$$y_\circ \in M := \bigcap_{j=1}^{\infty} V_{n_j},$$

and

$$\bigcup_{j=1}^{\infty} V^\circ_{n_j} \tag{4}$$

is a zero-neighbourhood in $X^* [\mu(X^*, Y)]$. We claim that

$$(v + M^\circ) \cap M^\circ = \phi, \tag{5}$$

otherwise, if u were in $(v + M^\circ) \cap M^\circ$, then $v - u$ and u would both belong to M°, thus yielding the clear contradiction

$$1 \geq |< y_\circ, v - u >| \geq |< y_\circ, v >| - |< y_\circ, u >| \geq 2.$$

Since (4) is a zero-neighbourhood in $X^* [\mu(X^*, Y)]$ and v is in F, there is a term V_r of (V_n) and an element w in X^* such that

$$w \in (v + V_r^\circ) \cap (\bigcup_{n=0}^{\infty} B_n). \tag{6}$$

It follows from (5) and (6) that w is not in M° and $\| w - v \|_r \leq 1$. But M, being the polar of (4) in X, is a weakly compact absolutely convex subset of Y, hence

$$M = \bigcap_{j=1}^{\infty} (V_{n_j} \cap Y).$$

It is now plain that w is not in $(V_r \cap Y)^\circ$ and so $| w |_r > 1$. We take a positive integer s such that $\frac{s-1}{s} | w |_r > 1$. Then $x(w, r, x)$ is in E and thus

$$1 \geq \| w - v \|_r \geq |< x(w, r, s), w - v >| = |< x(w, r, s), w >| \geq \frac{s-1}{s} | w |_r > 1,$$

again a clear contradiction. it follows then that $\overline{E} + F_\perp \supset Y$. Finally, take a point z in Y. We find a decreasing subsequence (V_{m_j}) of (V_m) such that

$$z \in P := \bigcap_{j=1}^{\infty} V_{m_j}$$

and

$$\overset{\infty}{\underset{j=1}{\cup}} V_{m_j}^\circ \tag{7}$$

is a zero-neighbourhood in $X^* [\mu(X^*, Y)]$. Then P is the polar set of (7) and thus it is a weakly absolutely convex subset of Y. Let T be denote the projection of $E + F_\perp$ on E along $F_\perp$. Then

$$Tz \in T(P) \subset \bigcap_{j=1}^{\infty} T(V_m) \subset \bigcap_{j=1}^{\infty} V_{m_j} = P \subset Y,$$

and so $z - Tz$ is also in Y. Hence

$$z = Tz + (z - Tz),\ Tz \in E,\ z - Tz \in F_\perp \cap Y,$$

and we have $E + F_\perp \cap Y = Y$.

q.e.d.

Proposition 10 *If* $\operatorname{dens} Y \geq \operatorname{dens} X_\sigma^*$*, then* $\operatorname{dens} Y = \operatorname{dens} X_\sigma^*$.

Proof. Assume $\lambda := \operatorname{dens} X_\sigma^* < \operatorname{dens} Y$. Let $A_\circ$ be an infinite subset of Y with $| A_\circ | \leq \lambda$, and $B_\circ$ a dense subset of X_σ^* such that $| B_\circ | = \lambda$. We apply our last proposition to obtain E and F with all the properties there is stated. Since $\operatorname{dens} E \leq \lambda$ and $E + F_\perp \cap Y = Y$, it follows that $F_\perp$ has infinite dimension . It is also clear that F coincides with X^*, thus attaining a contradiction.

q.e.d.

For the two coming propositions we assume X is infinite dimensional and $\operatorname{dens} Y = \operatorname{dens} X_\sigma^*$. Let μ denote ther first ordinal of $\operatorname{dens} Y$. The sets

$$\{y_\eta : 0 \leq \eta < \mu\},\ \{v_\eta \leq \eta < \mu\}$$

will be two dense subsets of Y and X_σ^*, respectively. We shall represent by $\mathcal{H}$ the family of all pairs

$$((x_i, u_i))_{i \in I}, \alpha)$$

satisfying the following conditions:

1. $(x_i, u_i)_{i \in I}$ *is a biorthogonal system for* X.
2. $\{u_i : i \in I\}_\perp$ *is an orthogonal complement of* $[x_i : i \in I]$ *in* $[x_i : i \in I] + \{u_i : i \in I\}_\perp$ *for the norms* $\|\cdot\|_n$, $n = 1, 2, \cdots$
3. α *is an infinite ordinal not greater than* μ *and such that* $| \alpha | = | I |$,

$$y_\eta \in [x_i : i \in I],\ 0 \leq \eta < \alpha,$$

and if F_I *denotes the closure of* $\operatorname{lin}\{u_i : i \in I\}$ *in* X_σ^*,

4. $x_i \in Y$, $i \in I$, *and* $[x_i : i \in I] \cap Y + \{u_i : i \in I\}_\perp \cap Y = Y$.

Proposition 11 *$\mathcal{H}$ is not empty.*

Proof. We take two countably infinite subsets $A_\circ$ and $B_\circ$ of Y and X^*, respectively, such that $\operatorname{lin} A_\circ$ has infinite dimension and

$$y_\eta \in A_\circ,\ v_\eta \in B_\circ, 0 \leq \eta < \omega.$$

Proposition 9 applies to yield E and F with the afore mentioned properties. Using the method described in [8, Prop. 1, f. 3] one can easily construct a biorthogonal system $(x_n, u_n)_{n\in \mathbb{N}}$ for X such that $\operatorname{lin}\{x_n : n \in \mathbb{N}\}$ and $\operatorname{lin}\{u_n : n \in \mathbb{N}\}$ be dense subspaces of E and F, respectively. It follows immediately that

$$((x_n, u_n)_{n\in \mathbb{N}}, \omega) \in \mathcal{H}.$$

q.e.d.

For two elements in $\mathcal{H}$, we write

$$((x_i, u_i)_{i\in I}, \alpha) < ((z_j, v_j)_{j\in J}, \beta)$$

whenever $\{(x_i, u_i) : i \in I\}$ is strictly contained in $\{(z_j, v_j) : j \in J\}$ and $\alpha < \beta$.

Proposition 12 *$(\mathcal{H}, \leq)$ is an inductive ordered set.*

Proof. It is plain that $(\mathcal{H}, \leq)$ is an ordered set. Let $\mathcal{L}$ denote a non-empty linearly ordered subset of $(\mathcal{H}, \leq)$. We denote by $\{(x_k, u_k) : k \in K\}$ and γ the union of all sets $\{(x_i, u_i) : i \in I\}$ and the supremum of all ordinal α, respectively, when $((x_i, u_i)_{i\in I}, \alpha)$ ranges over $\mathcal{L}$.

We then have that $(x_k, u_k)_{k\in K}$ is a biorthogonal system for X such that x_k is in $Y, k \in K$. Clearly, $\{u_k : k \in K\}_\perp$ is an orthogonal complement of $[x_k : k \in K]$ in $[x_k : k \in K] + \{u_k : k \in K\}_\perp$ respect to $\|\cdot\|_n, n = 1, 2, \cdots$ Besides, γ is an infinite ordinal, $\gamma \leq \mu$, such that $|\gamma| = |K|$,

$$y_\eta \in [x_k : k \in K], 0 \leq \eta < \gamma,$$

and, if F_k is the closure of $\operatorname{lin}\{u_k : k \in K\}$ in X^*_σ, then

$$v_\eta \in F_k, 0 \leq \eta < \gamma.$$

Let us take

$$\Lambda := ((x_i, u_i)_{i\in I}, \alpha) \in \mathcal{L}$$

and choose x in Y. Then

$$x = y(\Lambda) + z(\Lambda),\ y(\Lambda) \in [x_i : i \in I] \cap Y,\ z(\Lambda) \in \{u_k : i \in I\}_{\perp} \cap Y.$$

We find a decreasing subsequence (V_{n_j}) of (V_n) such that

$$x \in P := \bigcap_{j=1}^{\infty} V_{n_j}$$

and $\overset{\infty}{\underset{j=1}{\cup}} V_{n_j}^{\circ}$ be a zero-neighbourhood in $X^* [\mu(X^*, Y)]$. Now following a similar argument to that of Proposition 9, we have that $y(\Lambda)$ belongs to the weakly compact absolutely compact subset P of Y. Hence, the net $\{y(\Lambda) : \Lambda \in \mathcal{L}, \leq\}$ has a weakly adherent point y in $P \cap [x_k : k \in K]$. The point $z := x - y$ is therefore weakly adherent to the net $\{z(\Lambda) : \Lambda \in \mathcal{L}, \leq\}$, thus obtaining

$$x = y + z \in [x_k : k \in K] \cap Y + \{u_k : k \in K\}_{\perp} \cap Y$$

and so $[x_k : k \in K] \cap Y + \{u_k : k \in K\}_{\perp} \cap Y = Y$. We may thus conclude that $((x_k, u_k)_{k \in K}, \gamma)$ is an upper bound of $\mathcal{L}$ in $(\mathcal{H}, \leq)$.

Theorem 3 *Let X be a Banach space. Let Y be a normed subspace of X such that* $\operatorname{dens} Y \leq \operatorname{dens} X_{\sigma}^*$. *If Y is weakly countably convex determined, then there is a total biorthogonal system $(x_i, u_i)_{i \in I}$ for X such that* $\operatorname{lin}\{x : i \in I\}$ *is a dense subspace of Y.*

Proof. We proceed over the density character of Y. If Y is separable, the stated property can be shown via the method of [8, Prop. 1. f. 3]. Assume now that Y is not separable and that for each Banach space W with a normed subspace H such that dens $\geq \operatorname{dens} W_{\sigma}^*, \operatorname{dens} H < \operatorname{dens} Y$, being H weakly countably convex-determined, there is a total system $(z_j, w_j)_{j \in J}$ such that $\operatorname{lin}\{z_j : j \in J\}$ is a dense subset of H. Let (U_n) be a sequence of zero neighbourhoods in Y such that they are absolutely convex closed and bounded and, for each y in Y, there is a decreasing subsequence (U_{n_j}) of (U_n) such that y belongs to $\bigcap_{j=1}^{\infty} U_{n_j}$ and, if W_n is the polar set of U_n in $Y^*, n = 1, 2, \cdots, \overset{\infty}{\underset{j=1}{\cup}} W_{n_j}$ be a zero-neighbourhood in $Y^*[\mu(Y^*, Y)]$. We denote by V_n the closure in X of $U_n + \frac{1}{n}B(X)$. It is clear that $\overset{\infty}{\underset{j=1}{\cup}} U^\circ_{n_j}$ is a zero neighbourhood in $X^*[\mu(X^*, Y)]$. Let u be an element of $\frac{1}{2} \overset{\infty}{\underset{j=1}{\cup}} U^\circ_{n_j}$. We find a positive integer r such that $2u \in U^\circ_{n_r}$ and $2 \parallel u \parallel < n_r$. Let us assume that u is not in $\overset{\infty}{\underset{j=1}{\cup}} V^\circ_{n_j}$. Thus, u is not in $V^\circ_{n_r}$, and hence there is an x in $U_{n_r} + \frac{1}{n_r}B(X)$ such that $< x, u > > 1$. We write

$$x = x_1 + x_2,\ x_1 \in U_{n_r}, x_2 \in \frac{1}{n_r}B(X).$$

We have then

$$\frac{1}{2} \geq < x_1, u > = < x, u > - < x_2, u > > 1 - \frac{\parallel u \parallel}{n_r} > \frac{1}{2},$$

a contradiction. Therefore

$$\frac{1}{2} \overset{\infty}{\underset{j=1}{\cup}} U^\circ_{n_j} \subset \overset{\infty}{\underset{j=1}{\cup}} V^\circ_{n_j},$$

and $\overset{\infty}{\underset{j=1}{\cup}} V^\circ_{n_j}$ is a zero-neighbourhood in $X^*[\mu(X^*, Y)]$.

Now, Proposition 10 may be applied. Let μ denote the first ordinal of dens $Y =$ dens X^*_σ. We choose and set $\{y_\eta \leq \eta < \mu\}$, $\{v_\eta : 0 \leq \eta < \mu\}$ and $(\mathcal{H}, \leq)$ as before. By applying Proposition 12 we obtain a maximal element $((x_i, u_i)_{i \in I}, \alpha)$ in $(\mathcal{H}, \leq)$. Suppose $\alpha < \mu$. Then $\mid \alpha \mid = \mid I \mid < \mid \mu \mid$. We choose a subspace H of $Y \cap \{u_i : i \in I\}_\perp$ with dens $H = \mid \alpha \mid$. We take a dense subspace A of H with $\mid A \mid = \mid \alpha \mid$. Applying Proposition 9 for

$$A_\circ := \{x_i : i \in I\} \cup A \cup \{y_\alpha\}$$

$$B_\circ := \{u_i : i \in I\} \cup \{v_\alpha\}$$

and $\lambda := \mid \alpha \mid$, we get two subspaces E and F with all the properties there mentioned. Let G represent the closure of lin$\{u_i : i \in I\}$ in X^*_σ. Let φ be the canonical mapping from X onto $X/F_\perp$. If $L := \varphi(E)$, then L is isomorphic to E and thus is weakly countably convex-determined. We identify F, in the usual manner, with the topological dual of $X/F_\perp$. Let M denote the orthogonal complement of G in $X/F_\perp$. Then, $M \cap L$ is a topological complement of $\varphi([x_i : i \in I] \cap Y)$ in L. We have thus the following situation: M is a Banach space whose topological dual is

F/G; $M \cap L$ is a normed subspace of M such that it is weakly countably convex-determined and

$$| \alpha | = \text{dens}(M \cap L) \geq \text{dens}(F/G).$$

Hence, there is a total biorthogonal system $(t_j, w_j)_{j \in J}$ for M, J disjoint from I, such that $\text{lin}\{t_j : j \in J\}$ is a dense subset of $M \cap L$. Let ψ denote the canonical mapping from F onto F/G. For every j in J we take x_j in $E \cap \{u_i : i \in I\}_\perp$ and u_j in $F \cap [x_i : i \in I]^\perp$ such that $\varphi(x_j) = t_j$ and $\psi(u_j) = w_j$. Setting $K := I \cup J$, we have that $(x_i, u_i)_{i \in K}$ is a biorthogonal system for X such that $[x_i : i \in K] \cap Y = E$, and $\text{lin}\{u_i : i \in K\}$ is dense in F. It is easy to see that $((x_i, u_i)_{i \in K}, \alpha + 1)$ belongs to $(\mathcal{H}, \leq)$ and it is clearly strictly posterior to $((x_i, u_i)_{i \in I}, \alpha)$. We have thus attained a contradiction. Hence $\alpha = \mu$, $(x_i, u_i)_{i \in I}$ satisfies the required properties.

q.e.d.

Corollary *Let Y be a dense subspace of a Banach space X. If Y is weakly countably convex determined there is a Markushevich basis $(x_i, u_i)_{i \in I}$ for X such that $x_i \in Y$, $i \in I$.*

Note 3 Let X be a weakly countably determined Banach space. Proposition 6 and the former corollary allow us to assert that there is a Markushevich basis $(x_i, u_i)_{i \in I}$ for X. Let M be the subset of X^* consisting on all elements u such that

$$\{i \in I : < x_i, u > \neq 0\}$$

is countable. Clearly, M is a linear subspace of X^*. Let (B_n) denote the sequence of weakly compact absolutely convex subsets of X_σ^* defined by means of condition (c) of Proposition 8. Let v be a point in the closure of $M \cap B(X^*)$ in X_σ^*. Given a positive integer n, if

$$(v + B_n) \cap M \cap B(X^*)$$

is not void, we take v_n in such a set; on the other hand, if it is empty, we take v_n to be the zero vector of X^*. Let V be a zero-neighbourhood in X_σ^*. We find an increasing subsequence (B_{n_j}) of (B_n) such that

$$B_{n_j} \subset V, \; j = 1, 2, \cdots,$$

and $\bigcup_{j=1}^{\infty} B_{n_j}$ is a zero-neighbourhood in X^*_μ. Since $M \cap B(X^*)$ is convex, its closures in X^*_σ and X^*_μ, respectively, coincide, and hence we have that $v + \bigcup_{j=1}^{\infty} B_{n_j}$ meets $M \cap B(X^*)$. Thus, there is a positive integer j such that

$$v + B_{n_j} \cap M \cap B(X^*) \neq \phi,$$

and so v_{n_j} belongs to $v + V$. Consequently, (v_n) has v as an adherent point in X^*_σ, and we conclude that v is in M. Then, $M \cap B(X^*)$ is weak-star closed. By Krein-Smulian theorem, [5, p. 246], we have that M coincides with X^*. A consequence of this is the well known result which states that, for a weakly countably determined Banach space X, the closed unit ball of its dual is a Corson compact for the weak-star topology, [12].

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Universidad de Valencia
Facultad de CC. Matemáticas
Dpto. de Análisis Matemático
Dr. Moliner, 50
46100 Burjasot (Valencia)
Spain

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