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Multiplicative characterization of Hilbert spaces and other interesting classes of Banach spaces.

A. RODRÍGUEZ PALACIOS

Dedicated to Professor Baltasar Rodríguez Salinas

Abstract

For a Banach space X, we show how the existence of a norm-one element u in X and a norm-one continuous bilinear mapping $f : X \times X \longrightarrow X$ satisfying $f(x,u) = f(u,x) = x$ for all x in X, together with some more intrinsic conditions, can be utilized to characterize X as a member of some relevant subclass of the class of Banach spaces.

0 Introduction

Some Banach spaces arise naturally enjoying the following property: they are endowed with a norm-one (continuous bilinear) product together with a distinguished norm-one element which acts as a unit for that product. This is the case for example for the Banach space $L(X)$ of all bounded linear operators on any Banach space X, or the Banach space $C(\Omega)$ of all continuous complex-valued functions on any Hausdorff compact topological space Ω. In the first case the product is nothing but the composition of elements of $L(X)$ as mappings on X, and the distinguished element is the identity operator on X. In the second case the product is the one defined point-wise and, consequently, the distinguished element is the constant function equal to one on Ω.

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In this paper we deal with partial sides of the next two questions:

Question 1. How abundant are the Banach spaces satisfying the property commented above?

Question 2. If a Banach space X enjoys the property we are considering, can this property (together with some more intrinsic ones) be utilized to characterize X as a member of some relevant subclass of the class of Banach spaces?

We are aware of the ambiguity of both questions, and we must recognize that there is not a well-organized theory relative to them. However, in some particular directions, there are interesting answers to the questions that will be either reviewed (if they are previously known) or presented with a proof (if they are new). Most of these results arise in the setting of Hilbert spaces and Banach spaces of the form $\mathcal{C}(\Omega)$.

1 The case of Hilbert spaces: known results

By a product on a Banach space X we mean a continuous bilinear mapping from $X \times X$ into X. A product f on the Banach space X will be called *unit-admissible* if there exists a (unique) norm-one element (say u) in X satisfying $f(u,x) = f(x,u) = x$ for all x in X. When the element u above should be emphasized (for instance, if u is previously prefixed) we will say that the product f is *u-admissible.* Now, Question 1 can be rephrased in the following terms: how abundant are the Banach spaces admitting a norm-one unit-admissible product?. If we restrict the question to complex Hilbert spaces, the following partial answer becomes really disappointing.

Proposition 1.1 *Let H be a complex Hilbert space, and assume there exists a norm-one unit-admissible* Associative *product on H. Then H is one-dimensional.*

For the case of real Hilbert spaces the situation is not much more promising.

Proposition 1.2 *Let H be a real Hilbert space, and assume there exists a norm-one unit-admissible* Associative *product on H. Then H has dimension 1, 2, or 4.*

Propositions 1.1 and 1.2 are due to L. Ingelstam (see [I; Corollary 2] and [I2], respectively). Because of the simplicity of their assertions, they are quite famous and, consequently, they are known in the literature as "the (complex and real, respectively) Ingelstam theorems". In fact the original formulation of Ingelstam's real theorem precisely determines up to algebraic isomorphisms the norm-one unit-admissible associative product whose existence is assumed (depending of course of the dimension, cf. Theorem 1.8 below) in such a way that Proposition 1.1 can be regarded as a consequence of Proposition 1.2.

Thinking again about the restriction of Question 1 to Hilbert spaces we could expect that, if in Ingelstam's theorems we drop the assumption of associativity for the product, then less obstructive results can be obtained. In the real case this idea becomes completely successful, as the following observation shows.

Observation 1.3 *Every non-zero real Hilbert space has norm-one unit-admissible products. More precisely, for every non-zero real Hilbert space H and for every norm-one element u in H, the mapping f from $H \times H$ into H defined by*

$$f(x,y) := (x \mid u)y + (y \mid u)x - (x \mid y)u$$

is a norm-one u-admissible product on H.

Proof. (a particular easy case of the proof of [R2; Proposition 24]). Let H, u, and f be as in the statement. Clearly f is an u-admissible product on H and therefore, as for every unit-admissible product on any Banach space, we have $\| f \| \geq 1$. To prove the converse inequality, let x, y be in H, and decompose x and y in the form $x = \lambda u + z$ and $y = \mu u + t$ with λ, μ in $\mathbb{R}$ and z, t in the orthogonal complement of $\mathbb{R}u$ in H. Then we have $f(x,y) = [\lambda\mu - (z \mid t)]\, u + \lambda t + \mu z$, hence

$$\begin{aligned} \| f(x,y) \|^2 &= [\lambda\mu - (z \mid t)]^2 + \| \lambda t + \mu z \|^2 = \\ &= \lambda^2\mu^2 + (z \mid t)^2 + \lambda^2 \| t \|^2 + \mu^2 \| z \|^2 \leq \\ &\leq \lambda^2\mu^2 + \| z \|^2 \| t \|^2 + \lambda^2 \| t \|^2 + \mu^2 \| z \|^2 = \\ &= (\lambda^2 + \| z \|^2)(\mu^2 + \| t \|^2) = \| x \|^2 \| y \|^2 . \end{aligned}$$

■

To study what happens if in Ingelstam's complex theorem we drop the assumption of associativity of the product we need a celebrated result of H. F. Bohnenblust and S. Karlin [BK]. The statement of the Bohnenblust-Karlin theorem involves the concept of a vertex of (the closed unit ball of) a given Banach space. By definition a *vertex* of a Banach space X is a norm-one element of X which is not a smooth point in any two-dimensional subspace of X containing it. Thanks to the Hahn-Banach theorem, this can be equivalently reformulated in the following more familiar (but also more technical) way. For a Banach space X and a norm-one element u in X denote by $D(X, u)$ the set of all *states of* X *relative to* u, namely

$$D(X, u) := \{\phi \in X^* : \phi(u) = 1 = \| \phi \|\}.$$

Then u is a vertex of X if and only if the conditions $x \in X$ and $\phi(x) = 0$ for all ϕ in $D(X, u)$ imply $x = 0$.

Proposition 1.4 (Bohnenblust-Karlin theorem). *Let X be a complex Banach space, and u be a norm-one element in X. If there exists a norm-one u-admissible* **Associative** *product on X, then u is a vertex of X.*

Since vertices cannot be smooth points except for the case of one-dimensional spaces, and Hilbert spaces are smooth at any point of their unit spheres, it follows that Ingelstam's complex theorem (Proposition 1.1) is a direct consequence of the Bohnenblust-Karlin theorem (Proposition 1.4). This is really curious because the second result is seven years older than the former. The above comment does not mean any criticism to L. Inglestam, who in [I] not only knows and enlarges in several directions the Bohnenblust-Karlin theorem, but also reproves it, and obtains his complex therem as an almost immediate corollary. The Bohnenblust-Karlin theorem is not only stronger than Ingelstam's complex one, but also conceptually easier to "unassociativize", as we show in the next result.

Theorem 1.5 ("non-associative Bohnenblust-Karlin theorem"). *Let X be a complex Banach space, and u be a norm-one element in X. If there exists a norm-one u-admissible product on X, then u is a vertex of X.*

Proof. [MMPR]. Let f denote the norm-one u-admissible product on X whose existence has been assumed, and for x in X consider the bounded linear operator L_x^f on X defined by $L_x^f(y) := f(x,y)$ for all y in X. Easily we see that the mapping $\Phi : x \longrightarrow L_x^f$ from X into $L(X)$ is a linear isometry sending u to the identity mapping (say I_X) on X. Therefore u is a vertex of X if and only if I_X is a vertex of $\Phi(X)$. But, by the (associative) Bohnenblust-Karlin theorem, I_X is a vertex of $L(X)$, and the vertex property is clearly hereditary.

■

From the incompatibility of the vertex property with the smoothness we obtain:

Corollary 1.6 (multiplicative characterization of the complex field). *Let X be a complex Banach space admitting a norm-one u-admissible product for some (norm-one) smooth point u in X. Then $X = \mathbb{C}$.*

As a consequence, we see that, contrarily to what happens in the real case, Ingelstam's complex theorem remains true if the assumption of associativity of the product is dropped.

Corollary 1.7 ("non-associative Ingelstam's complex theorem"). *Let H be a complex Hilbert space, and assume there exists a norm-one unit-admissible product on H. Then $H = \mathbb{C}$.*

In removing the associativity in Ingelstam's complex theorem we have been obliged to pass through more general results replacing Hilbert spaces by Banach spaces possessing a point of smoothness. Ingelstam's real theorem has been also extended in this direction:

Theorem 1.8 *Let X be a real Banach space admitting a norm-one u-admissible* **Associative** *product for some smooth point u in X. Then X has dimension 1, 2 or 4. More precisely, every norm-one u-admissible associative product on X converts X into an algebraic and isometric copy of either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the algebra of Hamilton quaternions).*

It seems rather difficult to settle the paternity of the above theorem. A proof is given in [St2] (1963) under the additional assumption that X is finite-dimensional. Without any restriction on the dimension of X, the theorem is a direct consequence of [I3; Remark in p. 234] (1964) and [I; Example 3(c)] (1962). However, concerning the proof, the remark in [I3] sounds rather imprecise. According to our news, the first complete proof and explicit formulation of Theorem 1.8 arise in [St3] (1966) and, since then, it has been rediscovered several times (see for example [BD] and [Sp]).

From Observation 1.3 we know that associativity cannot be removed in the first assertion of Theorem 1.8. This fact is really fortunate because it grants the life to the next theorem first proved in [R2] and reproved later, with relevant simplifications, in [R4].

Theorem 1.9 (multiplicative characterization of real Hilbert spaces). *A non-zero real Banach space X is a Hilbert space if (and only if), for some (equivalently, every) norm-one element u in X, X is smooth at u and there exists a norm-one u-admissible product on X. Moreover, if H is a non-zero real Hilbert space, and if u is a norm-one element in H, then the mapping f from $H \times H$ into H defined by*

$$f(x,y) := (x \mid u)y + (y \mid u)x - (x \mid y)u$$

is the unique norm-one u-admissible commutative product on H.

Note that, if X is a Banach space, if u is a norm-one element in X, and if X has a norm-one u-admissible product (say f), then X also has a norm-one u-admissible commutative product, namely the mapping

$$(x,y) \longrightarrow \frac{1}{2}\,[f(x,y) + f(y,x)]$$

from $X \times X$ into X. Taking this into account, Theorem 1.8 can be deduced from the above theorem by applying Frobenius' theorem (see [EHHKMNPR]). It is easy to verify that, if H is a real Hilbert space, if u is a norm-one element in H, and if f denotes the unique norm-one u-admissible commutative product on H, then, except for $Dim(H) = 2$, the centre of the algebra (H, f) reduces to the real multiples of u. Therefore Corollary 1.6 is also a consequence of Theorem 1.9.

The characterization of real Hilbert spaces given by Theorem 1.9 has been improved later in [R6], by replacing the assumption of existence of norm-one u-admissible products by a simple "numerical" condition. Let X be a Banach space and u be a norm-one element of X. We define the *multiplicative index* $m(X,u)$ of the couple (X,u) as the infimum of the set of numbers of the form $\| f \|$ when f runs over the set of all u-admissible products on X. Note that we can always construct u-admissible products on X with relatively small norm. For instance, choosing a state of X relative to u (say ϕ), the mapping f from $X \times X$ into X defined by

$$f(x,y) := \phi(x)y + \phi(y)x - \phi(x)\phi(y)u$$

is an u-admissible (associative and commutative) product on X satisfying $\| f \| \leq 3$. According to this we have $1 \leq m(X,u) \leq 3$.

Theorem 1.10 [R6] (approximately multiplicative characterization of real Hilbert spaces). *A non-zero real Banach space X is a Hilbert space if (and only if), for some (equivalently, every) norm-one element u in X, X is smooth at u and $m(X,u) = 1$.*

We conclude this section by referring the reader to some works not previously quoted but that are closely related to the material reviewed above. These are [B], [BD2], [CZ], [F], [L], [N], [S], [St], and [Z]. Concerning the Bonsall-Duncan monograph [BD2], the reader can find in it rather stronger versions of Proposition 1.4 and Theorem 1.8 (see [BD2; Theorem 4.1] and [BD2; Theorem 5.16], respectively).

2 The case of Hilbert spaces: new results

The proof given in [R6] of the approximately multiplicative characterization of real Hilbert spaces provided by Theorem 1.10 consists essentially of techniques of duality theory: it is shown that all conditions assumed for X in that theorem are also satisfied by its bidual X^{**}, with the advantage that the "approximate" requirement $m(X,u) = 1$ converts into the "exact" fact that X^{**} has a norm-one u-admissible product. In this way the proof is concluded by applying Theorem 1.9. Very recently we observed that an alternative reduction of the "approximate" case to the "exact" one can be made replacing the duality theory by the methods of Banach ultraproducts. We happily realized that the new techiques are more appropriate than the old ones, allowing to improve the result itself. Such an improvement will be proved in this section.

First of all we briefly summarize those aspects of the theory of ultraproducts needed for our purpose, referring the reader to the paper of S. Heinrich [H] for deeper information about the topic. Given an ultrafilter $\mathcal{U}$ on a non-empty set I and a family $\{X_i\}_{i \in I}$ of Banach spaces, we may consider the Banach space $\overset{\ell_\infty}{\underset{i \in I}{\oplus}} X_i$ ℓ_∞-sum of this family and the closed subspace $N_{\mathcal{U}}$ of $\overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i$ given by $N_{\mathcal{U}} := \left\{\{x_i\} \in \overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i : \lim_{\mathcal{U}} \| x_i \| = 0\right\}$. *The (Banach) ultraproduct of the family* $\{X_i\}_{i\in I}$ (with respect to the ultrafilter $\mathcal{U}$) is defined as the quotient Banach space $(X_i)_{\mathcal{U}} := (\overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i)/N_{\mathcal{U}}$. If we denote by (x_i) the element in $(X_i)_{\mathcal{U}}$ containing a given family $\{x_i\} \in \overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i$, then it is easy to verify that $\| (x_i) \| = \lim_{\mathcal{U}} \| x_i \|$. If, for all i in I, X_i is equal to a given Banach space X, then the ultraproduct $(X_i)_{\mathcal{U}}$ is called *the ultrapower* of X (with respect to $\mathcal{U}$) and is denoted by $X_{\mathcal{U}}$. In this case $\overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i$ is nothing but the familiar Banach space $B(I, X)$ of all bounded mappings from I to X, and the mapping $x \longrightarrow \widehat{x}$ from X into $X_{\mathcal{U}}$, where $\widehat{x} = (x_i)$ with $x_i = x$ for all i in I, becomes an isometric linear embedding.

Let X be a Banach space and u be a norm-one element of X. For x in X, the *numerical range* $V(X, u, x)$ of x relative to u is defined by

$$V(X, u, x) := \{\phi(x) : \phi \in D(X, u)\},$$

the number $\lim_{\alpha\to 0^+} \frac{\| u + \alpha x \| - 1}{\alpha}$ (which always exists because the mapping $\alpha \longrightarrow \| u + \alpha x \|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex) is usually denoted by $\tau(u, x)$, and it is well-known that the equality

$$\tau(u, x) = Max\{Re(\lambda) : \lambda \in V(X, u, x)\}$$

holds (see for example [DS; Theorem V.9.5]). We say that the *norm of X is strongly subdifferentiable at u* if $\lim_{\alpha\to 0^+} \frac{\| u + \alpha x \| - 1}{\alpha} = \tau(u, x)$ uniformly for x in the closed unit ball of X. The reader is referred to the paper of C. Franchetti and R. Payá [FP] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces. Concerning our purpose, we only need the "only if part" of the next proposition that, by the way, is not collected in the Franchetti-Payá paper.

Proposition 2.1 [AOPR]. *Let X be a Banach space, and u be a norm-one element in X. Then the norm of X is strongly subdifferentiable at u if and only if, for every non-empty set I and for every bounded mapping*

Φ from I to X, the equality

$$V(B(I,X),u^{\#},\Phi)=\overline{\mathrm{co}}\bigcup_{i\in I}V(X,u,\Phi(i))$$

holds (where $\overline{\mathrm{co}}$ denotes closed convex hull and $u^{\#}$ stands for the constant mapping equal to u on I).

Let X,Y be Banach spaces, u,v norm-one elements in X and Y, respectively, and F be a linear contraction from X to Y satisfying $F(u)=v$. Then it is easy to show that, for all x in X, the inclusion $V(Y,v,F(x))\subseteq V(X,u,x)$ is true. Since quotient mappings are linear contractions, this applies successfully to directly derive the next corollary from Proposition 2.1.

Corollary 2.2 *Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, X a Banach space, and u be a norm-one element in X. If the norm of X is strongly subdifferentiable at u, then, for every (x_i) in the ultrapower $X_{\mathcal{U}}$, we have*

$$V(X_{\mathcal{U}},\widehat{u},(x_i))\subseteq\overline{\mathrm{co}}\bigcup_{i\in I}V(X,u,x_i).$$

Corollary 2.3 *Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, X a Banach space, and u be a norm-one element in X. If X is smooth at u, and if the norm of X strongly subdifferenciable at u, then $X_{\mathcal{U}}$ is smooth at $\widehat{u}$.*

Proof. Let (x_i) be an arbitrary element in $X_{\mathcal{U}}$. Denoting by ϕ the unique state of X relative to u, for each i in I we can decompose x_i in the form $x_i=\lambda_i u+y_i$ with λ_i in the base field $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) and y_i in $Ker(\phi)$, so that $\{\lambda_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are bounded families of elements of $\mathbb{K}$ and X, respectively. Since $V(X,u,y_i)=\{0\}$ for all i in I, Corollary 2.2 gives $V(X_{\mathcal{U}},\widehat{u},(y_i))=\{0\}$. Writing $\lambda:=\lim_{\mathcal{U}}\lambda_i$, we have $(x_i)=\lambda\widehat{u}+(y_i)$, hence

$$V(X_{\mathcal{U}},\widehat{u},(x_i))=\lambda+V(X_{\mathcal{U}},\widehat{u},(y_i))=\lambda+\{0\}=\{\lambda\}.$$

In this way we have proved that the numerical range of every element of $X_{\mathcal{U}}$ relative to $\widehat{u}$ is a singleton. In other words, $X_{\mathcal{U}}$ is smooth at $\widehat{u}$.

■

Again let X be a Banach space and u be a norm-one element of X. We define the *small multiplicative index* $sm(X,u)$ of the couple (X, u) as the infimum of the set of numbers of the form $Max\left\{\| f \|, 1+ \| L_u^f - I_X \|, 1+ \| R_u^f - I_X \|\right\}$ when f runs over the set of all products on X. Here, for a given product f on X, L_u^f and R_u^f denote the operators on X given by $x \longrightarrow f(u, x)$ and $x \longrightarrow f(x, u)$, respectively, and I_X stands for the identity on X. We clearly have $1 \leq sm(X, u) \leq m(X, u)$, so that the improvement of Theorem 1.10 we are obtaining will consist in replacing in that theorem the condition $m(X, u) = 1$ by the formally weaker one $sm(X, u) = 1$. Note that $m(X, u) = 1$ means that for every $\varepsilon > 0$ there exists an u-admissible product f on X with $\| f \| \leq 1 + \varepsilon$, whereas $sm(X, u) = 1$ means that for every $\varepsilon > 0$ there exists a product f on X with $\| f \| \leq 1 + \varepsilon$ which need not be u-admissible but only "nearly" u-admissible in the sense that $Max\{\| f(u, x) - x \|, \| f(x, u) - x \|\} \leq \varepsilon \| f \|$ for all x in X.

It is known that, if there exists a norm-one u-admissible product on X, then the norm of X is strongly subdifferentiable at u (see [MMPR; Proposition 4.5] together with [AOPR; Theorem 5.1]). Our next proposition improves this result and even the generalization obtained in [R8; Corollary 2.5] where the assumption of existence of norm-one u-admissible products on X is replaced by $m(X, u) = 1$.

Proposition 2.4 *Let X be a Banach space, and u be a norm-one element in X. If $sm(X, u) = 1$, then the norm of X is strongly subdifferentiable at u. If in addition X is complex, then u is a vertex of X.*

Proof. For each natural number n, choose a product f_n on X satisfying $\| f_n \| \leq 1 + \frac{1}{n}$ and $Max\{\| f_n(u, x) - x \|, \| f_n(x, u) - x \|\} \leq \frac{1}{n} \| x \|$ for all x in X, and choose also an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all natural numbers refining the Fréchet filter (off all cofinite subset of $\mathbb{N}$). Then it is easy to see that

$$((x_n); (y_n)) \longrightarrow (f_n(x_n, y_n))$$

becomes a (well-defined) norm-one $\widehat{u}$–admissible product on the ultrapower $X_{\mathcal{U}}$. Therefore the norm of $X_{\mathcal{U}}$ is strongly subdifferentiable at $\widehat{u}$ and, if X is complex, then $\widehat{u}$ is a vertex of $X_{\mathcal{U}}$ (Theorem 1.5). Since the vertex property and the strong subdifferentiability of the norm are hereditary conditions, also the norm of $\widehat{X}$ is strongly subdifferentiable at $\widehat{u}$ and, in the complex case, $\widehat{u}$ is a vertex of $\widehat{X}$. Finally recall that $\widehat{X}$ is an isometric copy of X.

■

Now we state and conclude the proof of the main result in this section.

Theorem 2.5 *A non-zero real Banach space X is a Hilbert space if (and only if), for some (equivalently, every) norm-one element u in X, X is smooth at u and $sm(X, u) = 1$.*

Proof. Let X be a real Banach space containing a smooth point u which satisfies $sm(X, u) = 1$. By Proposition 2.4 and its proof we know that the assumption $sm(X, u) = 1$ implies the existence of some ultrafilter $\mathcal{U}$ such that the corresponding ultrapower $X_{\mathcal{U}}$ possesses a norm-one $\widehat{u}-$admissible product, and therefore the norm of X is strongly subdifferentiable at u. Since X is smooth at u, it follows from Corollary 2.3 that $X_{\mathcal{U}}$ is smooth at $\widehat{u}$. Now Theorem 1.9 gives us that $X_{\mathcal{U}}$ is a Hilbert space. In view of the natural embedding $X \hookrightarrow X_{\mathcal{U}}$, X is also a Hilbert space.

■

We conclude this section by applying the methods of ultraproducts to improve the last assertion in Theorem 1.9. This improvement implies that, if H is a non-zero real Hilbert space and if u is a norm-one element in H, then the product f in Observation 1.3 is not only the unique norm-one $u-$admissible commutative product on H (cf. Theorem 1.9) but also a (clearly unique) product on H with the property that sequences $\{f_n\}$ of commutative products on H satisfying $\{\| f_n \|\} \longrightarrow 1$ and $\left\{\| L_u^{f_n} - I_X \|\right\} \longrightarrow 0$ are norm-convergent to f (as a consequence, sequences $\{f_n\}$ of $u-$admissible commutative products on H satisfying $\{\| f_n \|\} \longrightarrow 1$ are also non-convergent to f). For the proof we must take into account that the class of Hilbert spaces is closed under ultraproducts. Indeed, if $\{X_i\}_{i\in I}$ is a family of Banach spaces, and if for each i in I the norm of X_i derives from a inner product $(\cdot \mid \cdot)_i$, then the norm of the ultraproduct $(X_i)_{\mathcal{U}}$ derives from the inner product $(\cdot \mid \cdot)$ given by

$$((x_i) \mid (y_i)) := lim_{\mathcal{U}}(x_i \mid y_i)_i.$$

Theorem 2.6 *For each $\varepsilon > 0$ there exists $\delta > 0$ such that, for every non-zero real Hilbert space H, for every norm-one element u in H, and* *for every commutative product g on H satisfying $\| g \| \leq 1 + \delta$ and $\| L_u^g - I_H \| \leq \delta$, we have $\| g - f \| \leq \varepsilon$, where f denotes the (norm-one $u-$admissible commutative) product on H given by*

$$f(x, y) := (x \mid u)y + (y \mid u)x - (x \mid y)u.$$

Proof. First of all, we recall the result we are extending, namely: given a non-zero real Hilbert space H and a norm-one element u in H, there exists a unique norm-one $u-$admissible commutative product f on H, and this is given by the equality at the end of the statement of the theorem. Now assume the theorem is not true. Then there exists $\varepsilon > 0$ such that, for each n in $\mathbb{N}$ we can find a Hilbert space H_n, together with norm-one elements u_n, v_n, w_n in H_n and a commutative product g_n on H_n, satisfying $\| g_n \| \leq 1 + \frac{1}{n}$, $\| L_{u_n}^{g_n} - I_X \| \leq \frac{1}{n}$, and $\| g_n(v_n, w_n) - f_n(v_n, w_n) \| \geq \varepsilon$, where f_n denotes the unique norm-one u_n-admissible commutative product on H_n. If we choose an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all natural numbers refining the Fréchet filter, then we realize that $((x_n); (y_n)) \longrightarrow (g_n(x_n, y_n))$ and $((x_n), (y_n)) \longrightarrow (f_n(x_n, y_n))$ are (well-defined) norm-one $(u_n)-$admissible commutative products on the ultraproduct $(H_n)_{\mathcal{U}}$. Since $(H_n)_{\mathcal{U}}$ is a Hilbert space, the uniqueness of such products gives us that, for all $(x_n), (y_n)$ in $(H_n)_{\mathcal{U}}$, the equality $(g_n(x_n, y_n)) = (f_n(x_n, y_n))$ is true. As a consequence, we have

$$lim_{\mathcal{U}} \| g_n(v_n, w_n) - f_n(v_n, w_n) \| = 0.$$

But this is incompatible with the condition

$$\| g_n(v_n, w_n) - f_n(v_n, w_n) \| \geq \varepsilon \text{ for all } n \text{ in } \mathbb{N}.$$

in the choice of the sequences $\{v_n\}$ and $\{w_n\}$.

■

Corollary 2.7 *For each $\varepsilon > 0$ there exists $\delta > 0$ such that, for every non-zero real Hilbert space H, for every nom-one element u in H, and for every $u-$admissible commutative product g on H satisfying $\| g \| \leq 1 + \delta$, we have $\| g - f \| \leq \varepsilon$, where f denotes the product on H given by*

$$f(x, y) := (x \mid u)y + (y \mid u)x - (x \mid y)u.$$

Theorem 2.6 can be rephrased in the following terms.

Theorem 2.6 (bis) *There exists an increasing function ρ, from the set of all non-negative real numbers into itself, continuous at zero, with $\rho(0) = 0$, and such that, for every non-zero real Hilbert space H, for every norm-one element u in H, and for every commutative product g on H, we have*

$$\| g - f \| \leq \rho(Max\{\| g \| -1, \| L_u^g - I_H \|\}),$$

where f denotes the product on H given by

$$f(x, y) := (x \mid u)y + (y \mid u)x - (x \mid y)u.$$

Of course, we could state a "Corollary 2.7 bis", replacing in Theorem 2.6 bis *commutative product* by *$u-$admissible commutative product* and, consequently, the inequality $\| g - f \| \leq \rho(Max\{\| g \| -1, \| L_u^g - I_X \|\})$ by the one $\| g - f \| \leq \rho(\| g \| -1)$.

3 The case of spaces $\mathcal{C}(\Omega)$: known results

The natural examples of Banach spaces possessing norm-one unit-admissible products that we provided in the introduction, namely $L(X)$ (X, any Banach space) and $\mathcal{C}(\Omega)$ (Ω, any Hausdorff compact topological space), are not totally independent. Actually, up to isometric unit-preserving algebra isomorphisms, the Banach algebras $\mathcal{C}(\Omega)$ are nothing but the norm-closed self-adjoint commutative subalgebras of $L(H)$ containing I_H, when H runs in the class of all complex Hilbert spaces. In one direction this is straightforward: given the Hausdorff compact space Ω, we can build the complex Hilbert space $H := \ell_2(\Omega)$ (of all families of complex numbers $\{\lambda_t\}_{t\in\Omega}$ satisfying $\sum_{t\in\Omega} \mid \lambda_t \mid^2 < \infty$) and identify each element x in $\mathcal{C}(\Omega)$ with the (bounded linear) operator $x^{\#}$ on H given by $x^{\#}(\{\lambda_t\}) := \{x(t)\lambda_t\}$, obtaining in this way that the mapping $x \longrightarrow x^{\#}$ from $\mathcal{C}(\Omega)$ to $L(H)$ is an isometric unit-preserving algebraic homomorphism. In the converse direction our assertion above is the famous "commutative Gelfand-Naimark theorem".

Norm-closed selfadjoint subalgebras of $L(H)$, for some complex Hilbert space H, are usually known in the literature with the name of

C^*−algebras. In view of the fact commented in the above paragraph, characterizations and properties of spaces $C(\Omega)$ are often obtained as specializations of more general results for unital C^*−algebras. Let us therefore briefly review the nice geometric characterization of unital C^*−algebras obtained by T. W. Palmer [P] and known in the literature as "the Vidav-Palmer theorem".

Let X be a complex Banach space, and u be a norm-one element in X. We denote by $H(X,u)$ the closed real subspace of X consisting of all elements h in X satisfying $V(X,u,h) \subseteq \mathbb{R}$. The Vidav-Palmer theorem asserts that X is the Banach space of a C^*−algebra with unit u if and only if $X = H(X,u) + iH(X,u)$ and there exists a norm-one u−admissible associative product on X. If this is the case, then each norm-one u−admissible associative product on X converts X into a C^*−algebra. The Vidav-Palmer theorem, together with a famous result of M. H. Stone on unit-preserving isometries of the spaces $C(\Omega)$, implies the next proposition.

Proposition 3.1 *Let X be a complex Banach space, and u be a norm-one element in X. Then $X = C(\Omega)$ and u is the constant function equal to one on Ω (for some Hausdorff compact space Ω) if and only if $X = H(X,u) + iH(X,u)$ and there exists a norm-one u−admissible* **associative and commutative** *product on X. Moreover, if this is the case, then there is a unique norm-one u−admissible* **associative and commutative** *product on X.*

Now the question is if in the above proposition the associativity and/or the commutativity can be either altogether dropped or at least replaced by more intrinsic conditions. It is clear that both requirements cannot be simultaneously dropped (take X equal to the Banach space of a not commutative unital C^*−algebra). In presence of associativity, commutativity can be in fact "numerically" settled. This is a consequence of a result of T. Huruya [Hu] on "numerical indices" of C^*−algebras.

Let X be a Banach space, and u be a norm-one element in X. For x in X, the *numerical radius of x relative to u*, $v(X,u,x)$, is defined by

$$v(X,u,x) := Max\{|\lambda| : \lambda \in V(X,u,x)\} ,$$

and the *numerical index of X relative to u*, $n(X,u)$, is the number given by

$$n(X,u) := Max\{r \geq 0 : r \| x \| \leq v(X,u,x), \text{ for all } x \text{ in } X\}.$$

Now, if X is a Banach space (no norm-one element in X is distinguished), then we define the *Banach space numerical index of* X, $N(X)$, by the equality

$$N(X) := n(L(X), I_X).$$

Huruya's theorem asserts that the Banach space numerical index of a C^*−algebra A is 1 or $\frac{1}{2}$ depending on whether or not A is commutative. Therefore we have:

Proposition 3.2 *Let X be a complex Banach space, and u be a norm-one element in X. Then $X = C(\Omega)$ and u is the constant function equal to one on Ω (for some Hausdorff compact space Ω) if and only if $X = H(X,u) + iH(X,u)$, $N(X) = 1$, and there exists a norm-one $u-$admissible* **associative** *product on X. Moreover, if this is the case, then there is a unique norm-one $u-$admissible* **associative** *product on X.*

Since the date of Huruya's paper [Hu], much work has been done in the field of non-associative Banach algebras, showing in particular that the associativity in the above proposition is completely superfluous [R]. Even, according to [R; Corollary 32], the condition $N(X) = 1$ above can be alternatively replaced by the one $n(X,u) = 1$ that, in our context, is conceptually weaker (note that all numerical indices are less than or equal to 1, and that, if X is a Banach space, if u is a norm-one element in X, and if there exists a norm-one $u-$admissible product on X (say f), then, in view of the isometric embedding $x \longrightarrow L_x^f$, we have $N(X) = n(L(X), I_X) \leq n(X,u)$). Therefore we have the following "multiplicative characterization of the spaces $C(\Omega)$".

Theorem 3.3 *Let X be a complex Banach space, and u be a norm-one element in X. Then $X = C(\Omega)$ and u is the constant function equal to one on Ω (for some Hausdorff compact space Ω) if and only if $X = H(X,u) + iH(X,u)$, there exists a norm-one $u-$admissible product on X, and either $N(X) = 1$ or $n(X,u) = 1$. Moreover, if this is the case, then there is a unique norm-one $u-$admissible product on X.*

Very recently, the above theorem has been improved in the spirit of Theorem 1.10.

Theorem 3.4 [R8] (approximately multiplicative characterization of spaces $C(\Omega)$) *Let X be a complex Banach space, and u be a norm-one element in X. Then $X = C(\Omega)$ and u is the constant function equal to one on Ω (for some Hausdorff compact space Ω) if and only if $X = H(X,u)+iH(X,u)$, $m(X,u) = 1$, and either $N(X) = 1$ or $n(X,u) = 1$.*

From the material reviewed until now in this section, the reader can have felt that neither Theorem 3.4 non even Theorem 3.3 would have seen the light if a general non-associative Vidad-Palmer type theorem would not have been obtained. Such a feeling is completely right. But, what can be expected about the assertions in such a theorem. In other words, which are the non-associative counterparts of unital C^*-algebras ?.

In trying to answer this question it could be interesting to notice that, from the point of view of the theory of Banach spaces, the associative product of a unital C^*-algebra is not much interesting: unit-preserving surjective linear isometries between unital C^*-algebras need not be algebra isomorphism. However, a celebrated theorem of R. V. Kadison [K] (extending the one of Stone quoted above) asserts on the contrary that the symmetrization $x \cdot y := \frac{1}{2}(xy + yx)$ of the associative product xy of a C^*-algebra is "unit-spacially" determined. If the C^*-algebra is not commutative, then the new product is not associative: it is of course commutative, but satisfies only the "nearly associative" condition $x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x$. It turns out that every C^*-algebra endowed with this new product is a Jordan algebra. Moreover, the characteristic property $\| x^*x \| = \| x \|^2$ of C^*-algebras ("non-commutative Gelfand-Naimark theorem" [DF]) is equivalent to the one $\| xx^*x \| = \| x \|^3$ and, fortunately, this last equality can be "jordanized". Indeed, for elements x, y in any associative algebra, the equality

$$xyx = 2x \cdot (y \cdot x) - x^2 \cdot y$$

holds. With some effort, even the remaining conditions involving the associative product in the intrinsic Gelfand-Naimark characterization of C^*-algebras, namely $(xy)^* = y^*x^*$ and $\| xy \| \leq \| x \| \| y \|$, also "jordanize" [R3]: they are equivalent to the ones $(x \cdot y)^* = y^* \cdot x^*$ and $\| x \cdot y \| \leq \| x \| \| y \|$.

Following a suggestion of *I.* Kaplansky [Ka2], it is therefore reasonable to consider, as geometric objects extending C^*-algebras, the so called JB^*-algebras. *JB^*-algebras* are defined as those complete normed complex Jordan algebras J with a conjugate-linear algebra involution * satisfying $\| U_x(x^*) \| = \| x \|^3$ for all x in J, where U_x means the operator on J defined by

$$U_x(y) := 2x \cdot (y \cdot x) - x^2 \cdot y$$

for all y in J. Replacing the associative product by the "Jordan" product, every C^*-algebra becomes a JB^*-algebra. More examples of JB^*-algebras (called JC^*-algebras) can be obtained from C^*-algebras by taking $^*-$invariant norm-closed subspaces which are closed under the Jordan product. The space of all symmetric 2×2 complex matrices (regarded as operators on the 2-dimensional complex Hilbert space) is the easiest example of a JC^*-algebra which is not a C^*-algebra endowed with its Jordan product. However, there exist JB^*-algebras which are not JC^*-algebras [W].

Speaking about JB^*-algebras, it is worth to mention their close relation to the question of the classification of bounded symmetric domains in complex Banach spaces. We will not enter this side of the theory of JB^*-algebras, but we refer the interested reader to the fundamental work of W. Kaup [Kau], the crucial Corollary 2 in [FR], [BKU] and [U].

Rougly speaking, the structure theory for JB^*-algebras (see [W], [FGR] and [R7]) shows that the enlargement obtained replacing C^*-algebras by JB^*-algebras is not too big. Therefore, it would seem rather surprising that, replacing C^*-algebras by JB^*-algebras, the condition of associativity in the Vidav-Palmer theorem could be completely removed. However, this is true, as the next theorem shows.

Theorem 3.5 *Let X be a complex Banach space, and u be a norm-one element in X. Then X is the Banach space of a JB^*-algebra with unit u if and only if $X = H(X,u) + iH(X,u)$ and there exists a norm-one $u-$admissible product on X. Moreover, if this is the case, then there is a unique norm-one $u-$admissible commutative product on X.*

It follows that, if X is a complex Banach space, if u a norm-one element in X, if f is a norm-one $u-$admissible commutative product on X, and if $X = H(X,u) + iH(X,u)$, then X, endowed with the product f, becomes a JB^*-algebra. This is "the non-associative Vidav-Palmer theorem" [R2]. Now, the uniqueness of such a product f is one of the main results in [WY] (see also [KMR; Lemma 6]). The non-associative

Vidav-Palmer theorem obtained in [R2] is the culmination of a relatively wide collection of papers, namely [Bo], [Y], [Y2], [M], [R] and [KMR].

Since associative JB^*-algebras and commutative C^*-algebras are the same, Theorem 3.3 is a direct consequence of Theorem 3.5 and the fact proved in [R] that the numerical index of a unital JB^*-algebra J relative to its unit is 1 or $\frac{1}{2}$ depending on whether or not J is associative. In the same way, Theorem 3.4 follows from the next result.

Theorem 3.6 [R8] *Let X be a complex Banach space, and u be a norm-one element in X. Then X is the Banach space of a JB^*-algebra with unit u if and only if $X = H(X,u) + iH(X,u)$ and $m(X,u) = 1$.*

4 The case of spaces $C(\Omega)$: new results

In a similar way as Theorem 1.10 was proved in [R6] from Theorem 1.9 (see the comment at the beginning of Section 2), Theorem 3.6 has been derived in [R8] from Theorem 3.5 by reducing the approximate case to the exact one by means of rather more involved techniques of duality theory. After the success shown by ultraproduct methods in improving Theorem 1.10 (see Theorem 2.5), we retake here these methods in order to obtain a similar improvement of Theorem 3.6. As a consequence, we will refine the approximately multiplicative characterization of spaces $C(\Omega)$ given by Theorem 3.4.

Theorem 2.6, which in the context of Section 2 is merely anecdotic, in our present situation has a parallel that is crucial for our purpose. We state it in the next theorem, which improves [R8; Theorem 1.2] in several directions. Taking into account the uniqueness of the norm-one $u-$admissible commutative product on any JB^*-algebra with unit u (Theorem 3.5) and the almost obvious fact that the class of JB^*-algebras is closed under ultraproducts, the proof is the same as that of Theorem 2.6, and therefore we omit it.

Theorem 4.1 *For each $\varepsilon > 0$ there exists $\delta > 0$ such taht, for every JB^*-algebra J with unit u, and for every commutative product g on J satisfying $\| g \| \leq 1 + \delta$ and $\| L_u^g - I_J \| \leq \delta$, we have $\| g - f \| \leq \varepsilon$, where f denotes the JB^*-product of J.*

As happened with Theorem 2.6, the above theorem admits a "bis" version, as well as direct suggestive corollaries, whose statements are left to the reader. To obtain a new (equally direct) corollary, we introduce the adequate notation.

Let X be a Banach space, u a norm-one element in X, and S be a non-empty subset of X. We denote by $sm(X,u,S)$ the infimum of the set of numbers of the form $Max\left\{\| f \|, 1+ \| L_u^f - I_X \|, 1+ \| R_u^f - I_X \|\right\}$ when f runs over the set of all products on X satisfying $f(S,S) \subseteq S$. A simple argument of symmetrization gives us that, in computing the above infimum, it is enough to move the variable f into the set of all commutative products on X satisfying $f(S,S) \subseteq S$.

Corollary 4.2 *Let J be a JB^*-algebra with unit u, and S be a non-empty norm-closed subset of J satisfying $sm(X,u,S) = 1$. Then S is closed under the JB^*-product of J.*

Proposition 4.3 *Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, X a complex Banach space, and u be a norm-one element in X. If the norm of X is strongly subdifferentiable at u, if u is a vertex, and if $X = H(X,u) + iH(X,u)$, then $X_{\mathcal{U}} = H(X_{\mathcal{U}}, \widehat{u}) + iH(X_{\mathcal{U}}, \widehat{u})$.*

Proof. First note that the vertex property for u implies $H(X,u) \cap iH(X,u) = \{0\}$, hence, since $H(X,u)$ is closed in X and $X = H(X,u) + iH(X,u)$, we actually have $X = H(X,u) \oplus iH(X,u)$ in a topological meaning. Let (x_i) be an arbitrary element in $X_{\mathcal{U}}$. For each i in I we can decompose x_i in the form $x_i = y_i + iz_i$ with y_i and z_i in $H(X,u)$, so that, by the above, the families $\{y_i\}_{i \in I}$ and $\{z_i\}_{i \in I}$ are bounded and satisfy $(x_i) = (y_i) + i(z_i)$. Finally, since the norm of X is strongly subdiffermtiable at u, a direct application of Corollary 2.2 gives us that (y_i) and (z_i) belong to $H(X_{\mathcal{U}}, \widehat{u})$.

■

Now we are ready to state and conlude the proof of the main result in this section.

Theorem 4.4 *Let X be a complex Banach space, and u be a norm-one element in X. Then X is the Banach space of JB^*-algebra with unit u if and only if $X = H(X,u) + iH(X,u)$ and $sm(X,u) = 1$.*

Proof. Assume $X = H(X,u) + iH(X,u)$ and $sm(X,u) = 1$. By Proposition 2.4 and its proof, there exists an ultrafilter $\mathcal{U}$ such that the ultrapower $X_{\mathcal{U}}$ has a norm-one $\widehat{u}$–admissible product, u is a vertex of X, and the norm of X is strongly subdifferentiable at u. Then, by Proposition 4.3, we have

$$X_{\mathcal{U}} = H(X_{\mathcal{U}}, \widehat{u}) + iH(X_{\mathcal{U}} \widehat{u}),$$

and hence Theorem 3.5 gives us that $X_{\mathcal{U}}$ endowed with a suitable product becomes a JB^*–algebra with unit $\widehat{u}$. Retakig the assumption $sm(X, u) = 1$, for each $\varepsilon > 0$ we can find a product f on X satisfying

$$Max\left\{\| f \|, 1+ \| L_u^f - I_X \|, 1+ \| R_u^f - I_X \|\right\} < 1 + \varepsilon,$$

and, by "ultraproducing" such an f in the familiar way

$$((x_i), (y_i)) \longrightarrow (f(x_i, y_i)),$$

we obtain a product $\widehat{f}$ on $X_{\mathcal{U}}$ satisfying

$$Max\left\{\| \widehat{f} \|, 1+ \| L_{\widehat{u}}^{\widehat{f}} - I_{X_{\mathcal{U}}} \|, 1+ \| R_{\widehat{u}}^{\widehat{f}} - I_{X_{\mathcal{U}}} \|\right\} < 1 + \varepsilon$$

and $\widehat{f}(X, X) \subseteq X$. Of course we are viewing X as closed subspace of $X_{\mathcal{U}}$ via the canonical embedding. It follows that $sm(X_{\mathcal{U}} \widehat{u}, X) = 1$, hence, by Corollary 4.2, X is a subalgebra of the JB^*–algebra $X_{\mathcal{U}}$. In other words, the restriction of the JB^*–product of $X_{\mathcal{U}}$ to X is a norm-one u–admissible commutative product on X, so that, since $X = H(X, u) + iH(X, u)$, again by Theorem 3.5, X is a JB^*–algebra with unit u for this product.

■

Remark 4.5 In the above proof, the last application of Theorem 3.5 can be avoided. Indeed, once we know that X is a norm-closed subalgebra of the JB^*–algebra $X_{\mathcal{U}}$, to see that X is a JB^*–algebra with unit u it is enough to show that X is *–invariant in $X_{\mathcal{U}}$. But this is almost straightforward from the well-known geometric characterization of the JB^*–involution on any JB^*–algebra J with unit 1: for x in J, written in the form $x = h + ik$ for suitable h, k in $H(J, 1)$, we have $x^* = h - ik$. The particularization of this fact for C^*–algebras was the inspiration in the search of the Vidav-Palmer theorem [V].

In the same way as Theorem 3.3 can be derived from Theorem 3.5 (see the comment before Theorem 3.6), we can apply Theorem 4.4 to obtain the announced refinement of Theorem 3.4.

Corollary 4.6 *Let X be a complex Banach space, and u be a norm-one element in X. Then $X = C(\Omega)$ and u is the constant function equal to one on Ω (for some Hausdorff compact space Ω) if and only if $X = H(X,u) + iH(X,u)$, $sm(X,u) = 1$, and either $N(X) = 1$ or $n(X,u) = 1$.*

We conclude this section with a not difficult consequence of Theorem 4.1. The proof is left to the reader, who can take as a hint that, if X and Y are Banach spaces, if F is a bounded linear bijection from X to Y, and if f is a product on Y, then $(x_1, x_2) \longrightarrow F^{-1}[f(F(x_1), F(x_2))]$ becomes a product on X.

Corollary 4.7 *For each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all JB^*−algebras J and K with units u and v, respectively, and for every bounded linear bijection F from J to K satisfying $\| F \| \| F^{-1} \| \leq 1 + \delta$ and $\| F(u) - v \| \leq \delta$, e have*

$$\| F(x_1 \cdot x_2) - F(x_1) \cdot F(x_2) \| \leq \varepsilon \| x_1 \| \| x_2 \|$$

for all x_1, x_2 in X.

The above "approximate" version of the "unit-spacial" determination of the JB^*−product on every JB^*−algebra [WY] can be applied to C^*−algebras (endowed with their Jordan products) and in particular to the algebras $C(\Omega)$, providing "approximate" versions of the results of Kadison and Stone already quoted.

5 Some complements

This section will be a miscellany. It is devoted to obtain new results related to the material early developed in the paper.

We begin by improving Ingelstam's complex theorem (Proposition 1.1) in a direction different from the multiplicative characterization of the complex field obtained in Corollary 1.6. Precisely, our result is the following.

Theorem 5.1 *Let H be a complex Hilbert space with $Dim(H) \geq 2$, and*

u be a norm-one element in H. Then, for every $u-$admissible product f on H, we have

$$\| f \| \geq e(e^2-1)^{-1/2}.$$

Equivalently, $m(H,u) \geq e(e^2-1)^{-1/2}$.

The tools for the proof of Theorem 5.1 will be the actual non-associative Bohnenblust-Karlin theorem (a well-known improved version of Theorem 1.5) and the next lemma.

Lemma 5.2 *Let X be a Banach space over the field $\mathbb{K}$ of real or complex numbers, u a norm-one element in X, and M be a real number with $M \geq 1$. Then the equality*

$$\left| \! \left| \! \left| x \right| \! \right| \! \right| := Inf\{ |\lambda| + M \| x - \lambda u \| : \lambda \in \mathbb{K} \}$$

defines an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X satisfying $|\!|\!|u|\!|\!| = 1$ and

$$Max\left\{ Re(\lambda) \, : \, \lambda \in V^{|\!|\!|\cdot|\!|\!|}(X,u,x) \right\} = Inf\{ Re(\mu) + M \| x - \mu u \| : \mu \in \mathbb{K} \}$$

for all x in X, where the symbol $V^{|\!|\!|\cdot|\!|\!|}$ means numerical range relative to the new norm $|\!|\!|\cdot|\!|\!|$. Moreover, if f is any $u-$admissible product on X with $\| f \| \leq M$, then $|\!|\!|f|\!|\!| = 1$.

Proof. That $|\!|\!|\cdot|\!|\!|$ is an equivalent norm on X satisfying $|\!|\!|u|\!|\!| = 1$ is of straightforward verification. Let x be in X. Since the function $\beta \longrightarrow \beta + |\!|\!|x - \beta u|\!|\!|$ from $\mathbb{R}$ to $\mathbb{R}$ is increasing, we have

$$Inf\left\{ \beta + |\!|\!|x - \beta u|\!|\!| \, : \, \beta \in \mathbb{R} \right\} = \lim_{\substack{\beta<0 \\ \beta \to -\infty}} (\beta + |\!|\!|x - \beta u|\!|\!|).$$

The chage of variable $\beta = -\frac{1}{\alpha}$ in the above limit, together with the equality

$$Max\left\{ Re(\lambda) \, : \, \lambda \in V^{|\!|\!|\cdot|\!|\!|}(X,u,x) \right\} = \lim_{\alpha \to 0^+} \frac{|\!|\!|u + \alpha x|\!|\!| - 1}{\alpha},$$

quoted in Section 2, gives us

$$Max\left\{Re(\lambda)\ :\ \lambda\in V^{|\!|\!|\cdot|\!|\!|}(X,u,x)\right\}=Inf\left\{\beta+|\!|\!|x-\beta u|\!|\!|\ :\ \beta\in\mathbb{R}\right\}.$$

Now, the definition of the norm $|\!|\!|\cdot|\!|\!|$, allows us to write

$$Max\left\{Re(\lambda)\ :\ \lambda\in V^{|\!|\!|\cdot|\!|\!|}(X,u,x)\right\}=$$

$$Inf\{\beta+\mid\nu\mid+M\parallel x-(\beta+\nu)u\parallel:\ \beta\in\mathbb{R},\ \nu\in\mathbb{K}\}=$$

$$Inf\{\beta+\mid\mu-\beta\mid+M\parallel x-\mu u\parallel:\ \beta\in\mathbb{R},\ \mu\in\mathbb{K}\}.$$

Since, for μ in $\mathbb{K}$, the equality $Inf\{\beta+\mid\mu-\beta\mid:\ \beta\in\mathbb{R}\}=Re(\mu)$ is true, we finally have

$$Max\left\{Re(\lambda)\ :\ \lambda\in V^{|\!|\!|\cdot|\!|\!|}(X,u,x)\right\}=Inf\{Re(\mu)+M\parallel x-\mu u\parallel:\ \mu\in\mathbb{K}\}.$$

Let x,y be in X. Then, for arbitrary λ,μ in $\mathbb{K}$, we have

$$|\!|\!|f(x,y)|\!|\!|\leq\mid\lambda\mu\mid+M\parallel f(x,y)-\lambda\mu u\parallel=$$

$$\mid\lambda\mu\mid+M\parallel f(x-\lambda u,y-\mu u)+\lambda(y-\mu u)+\mu(x-\lambda u)\parallel\leq$$

$$\mid\lambda\mid\mid\mu\mid+M^2\parallel x-\lambda u\parallel\parallel y-\mu u\parallel+M\mid\lambda\mid\parallel y-\mu u\parallel+M\mid\mu\mid\parallel x-\lambda u\parallel=$$

$$(\mid\lambda\mid+M\parallel x-\lambda u\parallel)(\mid\mu\mid+M\parallel y-\mu u\parallel).$$

Now, taking infimum in λ,μ, we obtain $|\!|\!|f(x,y)|\!|\!|\leq|\!|\!|x|\!|\!||\!|\!|y|\!|\!|$

■

Let X be a Banach space, and u be a norm-one element in X. The condition $n(X,u)>0$ implies clearly the vertex property for u. As commented in [BD2; p.34], Bohnenblust and Karlin actually prove in their paper [BK] that, if X is complex, and if there exists a norm-one $u-$admissible asociative product on X, then $n(X,u)\geq e^{-1}$. Then the proof of Theorem 1.5 shows that this result remains true if the assumption of associativity of the product is removed. This should we taken into account in what follows.

Proof of Theorem 5.1 Let f be an $u-$admissible product on H, and consider the equivalent norm $\|\cdot\|$ on H given by

$$\|x\| := Inf\{|\lambda| + \| f \| \| x - \lambda u \| : \lambda \in \mathbb{C}\}$$

for all x in H. By Lemma 5.2 we have $\|f\| = 1$, and therefore $n^{\|\cdot\|}(X, u) \geq e^{-1}$. Now choose a norm-one element y in the orthogonal complement of $\mathbb{C}u$ in H. Then $\|y\| = \| f \|$ and hence $v^{\|\cdot\|}(X, u, y) \geq e^{-1} \| f \|$. Multiplying the chosen element y by a unimodular complex number, if necessary, we may obtain such an y satisfying in addition

$$v^{\|\cdot\|}(X, u, y) = Max\left\{Re(\lambda) : \lambda \in V^{\|\cdot\|}(X, u, y)\right\}.$$

Then we invoke Lemma 5.2 to obtain

$$v^{\|\cdot\|}(X, u, y) = Inf\left\{Re(\mu) + \| f \| (1 + |\mu|^2)^{1/2} : \mu \in \mathbb{C}\right\} =$$

$$Inf\left\{t + \| f \| (1 + t^2)^{1/2} : t \in \mathbb{R}\right\}.$$

But a classical study of the function $t \longrightarrow t + \| f \| (1+t^2)^{1/2}$ from $\mathbb{R}$ to $\mathbb{R}$ shows that it actually attains its minimum at $t = -(\| f \|^2 - 1)^{-1/2}$. Therefore

$$v^{\|\cdot\|}(X, u, y) = (\| f \|^2 - 1)^{1/2}.$$

If follows $(\| f \|^2 - 1)^{1/2} \geq e^{-1} \| f \|$. Equivalently, $\| f \| \geq e(e^2 - 1)^{-1/2}$.

■

Theorem 5.1 remains true with $sm(H, u)$ instead of $m(H, u)$ whenever the number $e(e^2 - 1)^{-1/2}$ is replaced by a suitable (unfortunately unknown) universal constant $C > 1$. As the reader can suspect, this will be proved by ultraproduct techniques.

Theorem 5.3 *There exists a real number $C > 1$ such that, for every complex Hilbert space H with $Dim(H) \geq 2$, and for every norm-one element u in H, we have $sm(H, u) \geq C$.*

Proof. Assume the theorem is not true. Then, for each n in $\mathbb{N}$, we can find a complex Hilbert space H_n, mutually orthogonal norm-one elements u_n, v_n in H_n, and a product f_n on H_n satisfying $\| f_n \| \leq 1 + \frac{1}{n}$ and $Max\{\| f_n(u_n, x_n) - x_n \|, \| f_n(x_n, u_n) - x_n \|\} \leq \frac{1}{n} \| x_n \|$ for all x_n in H_n. Taking an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all natural numbers which refines the Fréchet filter, $((x_n), (y_n)) \longrightarrow (f_n(x_n, y_n))$ is a norm-one (u_n)−admissible product on the ultraproduct $(H_n)_{\mathcal{U}}$. Since $(H_n)_{\mathcal{U}}$ is a complex Hilbert space, Corollary 1.6 gives us that $(H_n)_{\mathcal{U}}$ is one-dimensional. This is a contrdiction because (u_n) and (v_n) are mutually orthogonal non-zero elements in $(H_n)_{\mathcal{U}}$.

■

Let X be a Banach space, and u be a norm-one element in X. We denote by $\delta(X, u)$ the diameter of $D(X, u)$ regarded as a subset of the metric space X^*. Note that $\delta(X, u) = Sup\{Diam[V(X, u, x) : x \in X, \| x \| = 1\}$. G. Lummer [L] proved the existence of a universal constant $k > 0$ such that, if X is real, if $\delta(X, u) < k$, and if there exists a norm-one u−admissible associative product on X, then X has dimension 1, 2, or 4 and, more precisely, X endowed with such a product is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. This extension of Theorem 1.8 is reproved (with ultraproduct techniques) and even improved in the next theorem.

Theorem 5.4 *There exists a positive constant k such that, if X is any real Banach space, if u is a norm-one element in X satisfying $sm(X, u) = 1$ and $\delta(X, u) < k$ and if f is any associative product on X with*

$$Max\left\{\| f \|, 1 + \| L_u^f - I_X \|, 1 + \| R_u^f - I_X \|\right\} < 1 + k,$$

then X endowed with the product f is algebraically isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Among other tools, the proof of this theorem needs to lightly improve the "only if part" of Proposition 2.1 as well as its Corollary 2.2. Since the proofs of these improvements only involve minor changes in the original arguments, we omit then. If $\{X_i\}_{i \in I}$ is a family of Banach spaces, and if, for each i in I, u_i is a norm-one element in X_i, then we say that the couple $(\{X_i\}, \{u_i\})$ satisfies the *uniform property* (for the subdifferentiability of the norm) whenever

$$\lim_{\alpha\to 0^+} \frac{\| u_i + \alpha x_i \|}{\alpha} = \tau(u,x) \text{ uniformly for } i \text{ in } I \text{ and } x_i \text{ in } B_{X_i}.$$

(where, for every Banach space X, B_X denotes the closed unit ball of X).

Proposition 5.5 *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces, and for each i in I, let u_i be a norm-one element in X_i. If the couple $(\{X_i\}, \{u_i\})$ satisfies the uniform property, then, for every $\{x_i\}$ in $\overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i$, the equality*

$$V(\overset{\ell_\infty}{\underset{i\in I}{\oplus}} X_i, \{u_i\}, \{x_i\}) = \overline{co} \bigcup_{i\in I} (X_i, u_i, x_i)$$

holds.

Corollary 5.6 *Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, $\{X_i\}_{i\in I}$ be a family Banach spaces, and, for each i in I, let u_i be a norm-one element in X_i. If the couple $(\{X_i\}, \{u_i\})$ satisfies the uniform property, then, for every (x_i) in the ultraproduct $(X_i)_{\mathcal{U}}$, we have*

$$V((X_i)_{\mathcal{U}}, (u_i), (x_i)) \subseteq \overline{co} \bigcup_{i\in I} V(X_i, u_i, x_i).$$

Corollary 2.3 has to be rather deeper improved.

Corollary 5.7 *Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, $\{X_i\}_{i\in I}$ be a family of Banach spaces, and, for each i in I, let u_i be a norm-one element in X_i. If the couple $(\{X_i\}, \{u_i\})$ satisfies the uniform property, then*

$$\delta((X_i)_{\mathcal{U}}, (u_i)) \leq \lim_{\mathcal{U}} \delta(X_i, u_i).$$

Proof. Let (x_i) be an arbitrary norm-one element in $(X_i)_{\mathcal{U}}$ (so that we may actually assume that $\| x_i \| = 1$ for all i in I). For every U in $\mathcal{U}$, we have $(x_i) = (y_i)$, where (y_i) denotes the element in $(X_i)_{\mathcal{U}}$ defined by $y_i = x_i$, if $i \in U$, and $y_i = x_{i_0}$ for a fixed $i_0 \in U$, otherwise. It follows from Corollary 5.6 that

$$V((X_i)_{\mathcal{U}}, (u_i), (x_i)) \subseteq \overline{co} \bigcup_{i\in I} V(X_i, u_i, y_i) = \overline{co} \bigcup_{i\in U} V(X_i, u_i, x_i).$$

Therefore

$$Diam[V((X_i)_{\mathcal{U}},(u_i),(x_i))] \leq Sup\{Diam[V(X_i,u_i,x_i)] : i \in U\} \leq$$

$$Sup\{\delta(X_i,u_i) : i \in U\},$$

and, since U is an arbitrary element in $\mathcal{U}$, we obtain

$$Diam[V((X_i)_{\mathcal{U}},(u_i),(x_i))] \leq Inf\{Sup\{\delta(X_i,u_i) : i \in U\} : U \in \mathcal{U}\} =$$

$$lim_{\mathcal{U}}\,\delta(X_i,u_i).$$

Since (x_i) is an arbitrary norm-one element in $(X_i)_{\mathcal{U}}$, we finally have

$$\delta((X_i)_{\mathcal{U}},(u_i)) \leq lim_{\mathcal{U}}\,\delta(X_i,u_i).$$

■

Now, it is also crucial for the proof of Theorem 5.5 the following Claim.

Claim 5.8 *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and, for each i in I, let u_i be a norm-one element in X_i. If $sm(X_i,u_i)=1$ for all i in I, then the couple $(\{X_i\},\{u_i\})$ satisfies the uniform property.*

Proof. We must prove that, for arbitrary $\varepsilon>0$, there exists $\delta>0$ such that, for every Banach space X, for every norm-one element u in X with $sm(X,u)=1$, and for every x in the closed unit ball of X, we have

$$\frac{\| u+\alpha x \| -1}{\alpha} - \tau(u,x) < \varepsilon$$

whenever $0<\alpha<\delta$. Let us therefore fix $\varepsilon>0$. Then, according ro the proof of [MMPR; Proposition 4.5], for every Banach space X, for every norm-one element u in X such that there exists a norm-one $u-$admissible product on X, for every norm-one element y in X with $\| y-u \| < \varepsilon$, and for every ϕ in $D(X,y)$, there exists ψ in $D(X,u)$ satisfying $\| \phi-\psi \| < \varepsilon$. Now, we follow the proof of $(i) \Longrightarrow (ii)$ in [AOPR; Theorem 5.1], so that the above fact implies that, for such X and u, we have

$$\frac{\| u+\alpha x \| -1}{\alpha} - \tau(u,x) < \varepsilon$$

whenever x is in the closed unit ball of X and $0 < \alpha\, Min\{\frac{\varepsilon}{4}, \frac{1}{2}\}$. With the arguments in the proof of Proposition 2.4, the last assertion remains true if the assumption of existence of norm-one u−admissible products on X is replaced by $sm(X, u) = 1$.

■

Proof of Theorem 5.4 Again we argue by contradiction. If the theorem is not true, then, for each n in $\mathbb{N}$, we can choose a real Banach space X_n, a norm-one element u_n in X_n with $sm(X_n, u_n) = 1$ and $\delta(X_n, u_n) \leq \frac{1}{n}$, and an associative product f_n on X_n satisfying $\| f_n \| \leq 1 + \frac{1}{n}$ and

$$Max\{\| f_n(u_n, x_n) - x_n \|, \| f_n(x_n, u_n) - x_n \|\} \leq \frac{1}{n} \| x_n \|$$

for all x_n in X_n, but not converting X_n into an algebraic copy of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. By a theorem of Kaplansky [Ka; Theorem 3.1] (see also [CR]), this last pathology implies that, again for each n in $\mathbb{N}$, there exists norm-one elements v_n, w_n in X_n satisfying $\| f_n(v_n, w_n) \| \leq \frac{1}{n}$. Taking an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ which refines the Fréchet filter, the mapping

$$f : ((x_n), (y_n)) \longrightarrow (f_n(x_n, y_n))$$

is a norm-one (u_n)−admissible associative product on the ultraproduct $(X_n)_{\mathcal{U}}$. On the other hand, since for all n in $\mathbb{N}$ the conditions $sm(X_n, u_n) = 1$ and $\delta(X_n, u_n) \leq \frac{1}{n}$ are true, Claim 5.8 and Corollary 5.7 show that $\delta((X_n)_{\mathcal{U}}, (u_n)) = 0$, and therefore $(X_n)_{\mathcal{U}}$ is smooth at (u_n). It follows from Theorem 1.8 that $(X_n)_{\mathcal{U}}$, endowed with the product f, is an algebraic copy of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. But this is a contradiction because (v_n) and (w_n) are non-zero elements in $(X_n)_{\mathcal{U}}$ satisfying $f((v_n), (w_n)) = 0$.

■

Theorem 5.4, together with Observation 1.3, has the following immediate consequence (compare Proposition 1.2).

Corollary 5.9 *Let H be a real Hilbert space of dimension different from 1, 2 and 4. Then, for every norm-one element u in H, and for every associative product f on H, we have*

$$Max\left\{\| f \|, 1 + \| L_u^f - I_X \|, 1 + \| R_u^f - I_X \|\right\} \geq 1 + k,$$

where k denotes the positive universal constant given by Theorem 5.4. As a consequence, for every norm-one element u in H, and for every $u-$admissible associative product on H, the inequality $\| f \| \geq 1 + k$ is true.

Tue associative ingredients in the proof of Theorem 5.4, namely Kaplansky's result in [Ka] and Theorem 1.8, have been extended to the case of alternative products (see [MM], [CR] and [St3], [N], respectively). Therefore we have:

Theorem 5.10 *There exists a positive constant k such that, if X is any real Banach space, if u is a norm-one element in X satisfying $sm(X, u) = 1$ and $\delta(X, u) < k$, and if f is any alternative product on X with*

$$Max\left\{\| f \|, 1+ \| L_u^f - I_X \|, 1+ \| R_u^f - I_X \|\right\} < 1 + k,$$

then X endowed with the product f is algebraically isomorphism to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (the Cayley algebra of real octonions [EHHKMNPR]).

If in Theorem 5.4 (respectively, 5.10) we replace *associative* (respectively, *alternative*) *product* by *Jordan product*, then the finite-dimensionality of X cannot be expected. Indeed, the norm-one unit-admissible products constructed in Observation 1.3 on arbitrary non-zero real Hilbert spaces actually are Jordan products. Therefore, the result for Jordan products is more involved.

Theorem 5.11 *There exists a positive constant k such that, if X is any real Banach space, if u is a norm-one element in X satisfying $sm(X, u) = 1$ and $\delta(X, u) < k$, and if f is any Jordan product on X with*

$$Max\left\{\| f \|, 1+ \| L_u^f - I_X \|\right\} < 1 + k,$$

then there exists an inner product $(\cdot \mid \cdot)$ on X with $(u \mid u) = 1$ and satisfying

$$f(x, y) := (x \mid u)y + (y \mid u)x - (x \mid y)u$$

and

$$k \| x \| \leq (x \mid x)^{1/2} \leq \| f \| \| x \|$$

for all x, y in X.

Let J be a real Jordan-Banach algebra with a unit 1 satisfying $\| 1 \| = 1$, and let k be the universal positive constant given by the above theorem. It follows that, if either the Banach space of J is not isomorphic to a Hilbert space or the Jordan algebra J is not a division algebra (relative to the Jacobson notion of inverse [KS; Definition 1]), then $\delta(J,1) \geq k$.

Proof of Theorem 5.11 If we replace Kaplansky's result in [Ka] by [KS; Theorem 2 and Remark 2], and Theorem 1.8 by Theorem 1.9, then minor changes in the proof of Theorem 5.4 gives us the following

Claim *There exists a positive constant k' such that, if X is any real Banach space, if u is a norm-one element in X satisfying $sm(X,u) = 1$ and $\delta(X,u) < k'$, and if f is any Jordan product on X with*

$$Max\left\{\| f \|, 1+ \| L_u^f - I_X \|\right\} < 1 + k',$$

then there is an inner product $(\cdot \mid \cdot)$ on X satisfying $(u \mid u) = 1$ and

$$f(x,y) := (x \mid u)y + (y \mid u)x - (x \mid y)u$$

for all x, y in X.

We note the changes required in the proof of Theorem 5.4 to obtain the claim. In the present situation, for n in $\mathbb{N}$, the product f_n is not associative but Jordan, and, according to [KS; Remark 2 and Theorem 2], the norm-one elements v_n, w_n in X_n can and must be chosen satisfying $\| U_{v_n}^{f_n}(w_n) \| \leq \frac{1}{n}$ instead of $\| f_n(v_n, w_n) \| \leq \frac{1}{n}$, where $U_{v_n}^{f_n}$ denotes the familiar operator U_{v_n} on the Jordan algebra (X_n, f_n). Consequently, the equality $f((v_n),(w_n)) = 0$ at the end of the proof has to be replaced by $U_{(v)_n}^f((w_n)) = 0$. This leads to a contradiction that we explain in what follows. By Theorem 1.9, $(X_n)_{\mathcal{U}}$ is a Hilbert space and, since the $(u_n)-$admissible product f on $(X_n)_{\mathcal{U}}$ is commutative, we must have

$$f((x_n),(y_n)) = ((x_n) \mid (u_n)) + ((y_n) \mid (u_n))(x_n) - ((x_n) \mid (y_n))(u_n)$$

for all $(x_n),(y_n)$ in $(X_n)_{\mathcal{U}}$. With this determination of f, the equality $U_{(v_n)}^f((w_n)) = 0$ implies either $(u_n) = 0$ or $(w_n) = 0$.

We note also that the condition $(x \mid x)^{1/2} \leq \| f \| \| x \|$, in the statement of our theorem, is an automatic consequence of the previous ones $(u \mid u) = 1$ and $f(x,y) := (x \mid u)y + (y \mid u)x - (x \mid y)y$ (see Remark 5.12 below), and therefore, concerning the proof, can be forgotten.

Now we formally attack the proof of the theorem. Assume the theorem is not true. Then, for each n in $\mathbb{N}$, we may consider the number $k_n := Min\left\{k', \frac{1}{n}\right\}$ and apply the claim to obtain a Banach space X_n, a norm-one element u_n in X_n satisfying $sm(X_n, u_n) = 1$ and $\delta(X_n, u_n) \leq k_n$, and an inner product $(\cdot \mid \cdot)_n$ on X_n with $(u_n \mid u_n)_n = 1$, such that, if f_n denotes the product on X_n defined by

$$f_n(x_n, y_n) := (x_n \mid u_n)_n u_n + (y_n \mid u_n)_n x_n - (x_n \mid y_n)_n u_n,$$

then we have $\| f_n \| \leq 1 + k_n$ and

$$\| f_n(u_n, x_n) - x_n \| \leq k_n \| x_n \|$$

for all x_n in X_n, but there is a norm-one element v_n in X_n satisfying $k_n > (v_n \mid v_n)^{1/2}$. Once more, we take an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ which refines the Fréchet filter, so that

$$f : ((x_n), (y_n)) \longrightarrow (f_n(x_n, y_n))$$

is a norm-one $(u_n)-$admissible commutative product on the ultraproduct $(X_n)_{\mathcal{U}}$, and therefore, as in the proof of Theorem 5.4, $(X_n)_{\mathcal{U}}$ is smooth at (u_n). By Theorem 1.9, $(X_n)_{\mathcal{U}}$ is a Hilbert space and

$$f((x_n), (y_n)) = ((x_n) \mid (u_n))(y_n) + ((y_n) \mid (u_n))(x_n) - ((x_n) \mid (y_n))(u_n)$$

for all $(x_n), (y_n)$ in $(X_n)_{\mathcal{U}}$. As a consequence,

$$(2(u_n \mid v_n)_n v_n - (v_n \mid v_n)_n u_n) = (f_n(v_n, v_n)) =$$

$$f((v_n), (v_n)) = 2((u_n) \mid (v_n))(v_n) - ((v_n) \mid (v_n))(u_n).$$

But, since $k_n > (v_n \mid v_n)^{1/2}$ for all n in $\mathbb{N}$, we have $lim_{\mathcal{U}}(u_n \mid v_n) = lim_{\mathcal{U}}(v_n \mid v_n) = 0$. It follows that $2((u_n) \mid (v_n))(v_n) = ((v_n) \mid (v_n))(u_n)$, that implies either $(u_n) = 0$ or $(v_n) = 0$, a contradiction.

■

Remark 5.12 It is well known that, if $\|\cdot\|$ is an algebra norm on $\mathbb{C}$ (regarded as a real algebra), then we have $| \lambda | \leq \|\lambda\|$ for all λ in $\mathbb{C}$. This is a straightforward consequence of the basic spectral theory of

Banach algebras, but can be also proved with elemental tolls (see for instance [R5; p. 53]). Now let X be a real Banach space, u a non-zero element in X, and $(\cdot \mid \cdot)$ be an inner product on X satisfying $(u \mid u) = 1$. Denote by f the mapping from $X \times X$ to X given by $f(x,y) := (x \mid u)y + (y \mid u)x - (x \mid y)u$, and assume that f is continuous. Then, for y in $X \backslash \mathbb{R}u$, the subspace Y of X generated by u and y is closed under the product f and, endowed with this product, converts into a copy of $\mathbb{C}$. Moreover, in this identification, the mapping $z \longrightarrow (z \mid z)^{1/2}$ from Y to $\mathbb{R}$ converts into the usual absolute value on $\mathbb{C}$. Since $\| f \| \| \cdot \|$ is an algebra norm on the algebra (Y, f), it follows that $(z \mid z)^{1/2} \leq \| f \| \| z \|$ for all z in Y. Therefore, $(x \mid x)^{1/2} \leq \| f \| \| x \|$ for all x in X.

Now we pass to drastically extend the multiplicative characterization of the complex field (corollary 1.6) in the spirit of Theorem 5.4, 5.10 and 5.11.

Theorem 5.13 *Let X be a complex Banach space with $Dim(X) \geq 2$, and u be a norm-one element in X satisfying $sm(X,u) = 1$. Then $\delta(X,u) \geq 3^{1/2}e^{-1}$.*

This result will be an easy consequence of the (improved) nonassociative Bohnenblust-Karlin theorem and the next lemma. For the sake of completeness, we state and prove the lemma covering also the case of real spaces.

Lemma 5.14 *Let X be a Banach space over the field $\mathbb{K}$ of real or complex number with $Dim(X) \geq 2$, and u be a norm-one element in X. If $\mathbb{K} = \mathbb{R}$, then $2n(X,u) \leq \delta(X,u)$. If $\mathbb{K} = \mathbb{C}$, then $3^{1/2}n(X,u) \leq \delta(X,u)$.*

Proof. it is enough to consider the case $n(X,u) > 0$. Also note that numerical ranges of elements of X relative to u are non-empty closed convex subsets of $\mathbb{K}$. First assume $\mathbb{K} = \mathbb{R}$. Since $Dim(X) \geq 2$ and $n(X,u) > 0$, we can find a norm-one element x in X satisfying that its numerical range relative to u is equal to the closed real interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then we have

$$2n(X,u) = 2n(X,u) \| x \| \leq 2v(X,u,x) =$$

$$2\varepsilon = Diam[V(X,u,x)] \leq \delta(X,u) \| x \| = \delta(X,u).$$

Now assume $\mathbb{K} = \mathbb{C}$. Then there exists a norm-one element x in X and a positive number ε such that the closed disk with center at zero an radius ε is the smallest closed disk in $\mathbb{C}$ containing $V(X,u)$. By [E; Theorem 49], we have $3^{1/2}\varepsilon \leq Diam[V(X,u,x)]$ hence, arguing as in the real case, we obtain $3^{1/2}n(X,u) \leq \delta(X,u)$.

■

Proof of Theorem 5.13 If actually there exists a norm-one u−admissible product on X, then, by the extended non-associative Bohnenblust-Karlin theorem, we have $n(X,u) \geq e^{-1}$. But the arguments in the proof of Proposition 2.4 show that this remains true if the assumption of the existence of norm-one u−admissible products is relaxed to the one $sm(X,u) = 1$. Finally, apply Lemma 5.14.

■

We conclude the paper by determining, for any complex Hilbert space H, the norm-one unit-admissible products on $L(H)$. Let X be a complex Banach space, and u be a norm-one element in X. We know that, if there exists a norm-one u−admissible product on X, then u is a vertex of X (Theorem 1.5). When X is (the Banach space of) a C^*−algebra, the converse is also true. Indeed, a C^*−algebra A possesses vertices if and only if it has a unit (say 1) [Sa; Proposition 1.6.1], and, if this is the case, then the vertices of A are nothing but the *unitary* elements in A (those elements u in A satisfying $u^*u = uu^* = 1$) [BK; Example 4.1]. Therefore, if A is a C^*−algebra, and if there exists a vertex u in A, then A has a unit 1 and, from the obvious existence of norm-one 1-admissible products on A, and the fact that the mapping $x \longrightarrow ux$ from A to A is a surjective linear isometry sending 1 to u, it follows that there also exist norm-one u−admissible products on A. More precisely, there is a natural one-to-one correspondence from the set of norm-one 1-admissible products on A onto the set of norm-one u-admissible products on A. In this way, the determination of norm-one unit-admissible products on C^*−algebras centers in the determination of 1-admissible products on C^*−algebras with a unit 1.

Lemma 5.15 *Let A be a C^*−algebra with a unit* 1, *and f be a norm-one* 1−*admissible product on A. Then, for every x, y in A, we have*

$$f(x,y)^* = f(y^*,x^*).$$

Moreover, for every x in A, the mapping

$$y \longrightarrow f(x,y) - f(y,x)$$

from A to A is a derivation of A.

Proof. Let x be in $H(A,1)$, and y be in A. Then $L_x^f - R_x^f$ belongs to $H(L(A), I_A)$ and vanishes at 1. By [Si; Remark 3.5], $L_x^f - R_x^f$ is a derivation of A satisfying $[(L_x^f - R_x^f)(y)]^* = -(L_x^f - R_x^f)(y^*)$. But, by Theorem 3.5, we have

$$[f(x,y) + f(y,x)]^* = f(x,y^*) + f(y^*,x).$$

Therefore

$$f(x,y)^* = f(y^*,x).$$

Now take into account that $A = H(A,1) + iH(a,1)$.

■

Lemma 5.16 *Let A be a C^*–algebra with a unit, and u, v be non central unitary elements in A. Then there exists a unitary element w in A satisfying $uw - wu \neq 0$ and $vw - wv \neq 0$.*

Proof. For x, y in A, denote $[x,y] := xy - yx$. Since A is the linear hull of the set of its unitary elements. we may choose unitary elements t, z in A satisfying $[u,t] \neq 0$ and $[v,z] \neq 0$. If $[v,t] \neq 0$, take $w = t$. If $[u,z] \neq 0$, take $w = z$. In the remaining case, take $w = tz$.

■

Theorem 5.17 *Let H be a complex Hilbert space. Then the norm-one I_H–admissible products on $L(H)$ are the mappings f of the form*

$$f(x,y) = \alpha xy + (1-\alpha)yx$$

for a suitable real number α with $0 \leq \alpha \leq 1$.

Proof. Let f be a norm-one I_H–admissible product on $L(H)$. By Theorem 3.5, we have

$$f(x,y) + f(y,x) = xy + yx$$

for all x, y in $L(H)$. By Lemma 5.15 and [Sa; Theorem 4.1.6], for each x in $L(H)$, there exists some element $\vartheta(x)$ in $L(H)$ satisfying

$$f(x,y) - f(y,x) = \vartheta(x)y - y\vartheta(x)$$

for all y in $L(H)$. (Note that, since the center of $L(H)$ reduces to $\mathbb{C}I_H$, each x in $L(H)$ determines $\vartheta(x)$ up to a sum of a complex multiple of I_H). It follows

$$f(x,y) = \frac{x + \vartheta(x)}{2}y + y\frac{x - \vartheta(x)}{2}$$

for all x, y in $L(H)$. By [Sta; Theorem 8], for x in $L(H)$, we have

$$\| x \| = \| L_x^f \| = \frac{1}{2} Inf\{\| x + \vartheta(x) + \lambda I_H \| + \| x - \vartheta(x) - \lambda I_H \| : \lambda \in \mathbb{C}\},$$

so that, taking a point λ_0 in $\mathbb{C}$ where the function

$$\lambda \longrightarrow \| x + \vartheta(x) + \lambda I_H \| + \| x - \vartheta(x) - \lambda I_H \|$$

from $\mathbb{C}$ to $\mathbb{R}$ attains its minimum, and replacing $\vartheta(x)$ by $\vartheta(x) + \lambda_0 I_H$ if necessary, we may assume that the equality

$$\| x \| = \frac{1}{2}(\| x + \vartheta(x) \| + \| x - \vartheta(x) \|)$$

holds for all x in $L(H)$.

Now, let u be an arbitrary unitary element in $L(H)$. Since u is a vertex of $L(H)$, and vertices of any Banach space are extreme points of its closed unit ball, the last equality implies the existence of some real number $\alpha(u)$ satisfying $0 \leq \alpha(u) \leq 1$ and $u + \vartheta(u) = 2\alpha(u)u$ (hence $u - \vartheta(u) = 2(1 - \alpha(u))u$). It follows

$$f(u,y) = \alpha(u)uy + (1 - \alpha(u))yu$$

for all y in $L(H)$.

Now since $L(H)$ is the linear hull of the set of its unitary elements, to conclude the proof it is enough to show that the number $\alpha(u)$ above can be chosen the same with independence of the unitary element u in $L(H)$. To see this, note first that, if such an element u belongs to $\mathbb{C}I_H$, then every number in the closed real interval [0,1] can be taken as an admissible value of $\alpha(u)$, whereas, if u is not in $\mathbb{C}I_H$, then $\alpha(u)$ is uniquely determined by u (again apply that the center of $L(H)$ reduces to $\mathbb{C}I_H$). Therefore it is enough to prove $\alpha(u) = \alpha(v)$ whenever u and v are unitary elements in $L(H)$ not belonging to $\mathbb{C}I_H$. Let u and v be in such a situation. Assume $[u, v] \neq 0$. Then, by the definition of α, we have

$$f(u, v^*) = \alpha(u)uv^* + (1 - \alpha(u))v^*u,$$

and by Lemma 1.15,

$$f(u, v^*) = f(v, u^*)^* = \alpha(v)uv^* + (1 - \alpha(v))v^*u.$$

Therefore $(\alpha(u) - \alpha(v))[u, v^*] = 0$, hence $\alpha(u) = \alpha(v)$. Now assume $[u, v] = 0$. Then we invoke Lemma 5.16 to obtain a unitary element w in A with $[u, w] \neq 0$ and $[v, w] \neq 0$. By the above, $\alpha(u) = \alpha(w) = \alpha(v)$.

■

References

[AOPR] C. Aparicio, F. Ocaña, R. Paya and A. Rodríguez, *A non-smooth extension of Fréchet differentiability of the norm with applications to numerical ranges*, Glasgow Math. J. **28** (1986), 121-137.

[B] A. Bensebah, *JV−algèbres et JH*−algèbres*, Canad. Math. Bull. **34** (1991), 447-455.

[BK] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit ball of Banach algebras*, Ann. Math. **62** (1955), 217-229.

[Bo] F. F. Bonsall, *Jordan algebras spanned by hermitian elements of a Banach algebra*, Math. Proc. Cambridge Philos. Soc. **81** (1977), 3-13.

[BD] F. F. Bonsall and J. Duncan, *Dually irreducible representations of Banach algebras*, Quart. J. Math. Oxford **19** (1968), 97-111.

[BD2] F. F. Bonsall and J. Duncan, *"Numerical ranges of operators on normed spaces and of elements of normed algebras"*, London Math. Soc. Lecture Note Series **2**, Cambridge University Press, 1971.

[BKU] R. B. Braun, W. Kaup and H. Upmeier, *A holomorphic characterization of Jordan C^*-algebras*, Math. Z. **161** (1978), 277-290.

[CR] M. Cabrera and A. Rodríguez, *A new simple proof of the Gelfand-Mazur-Kaplansky theorem*, Proc. Amer. Math. Soc. **123** (1995), 2663-2666.

[CZ] A. Cedilnik and B. Zalar, *Nonassociative algebras with submultiplicative bilinear form*, Acta. Math. Univ. Comenianae **63** (1994), 285-301.

[DF] R. S. Doran and V. A. Belfi, *"Characterizations of C^*-algebras: The Gelfand-Naimark theorems"*, Marcel Dekker, New York and Basel, 1986.

[DS] N. Dunford and J. T. Schwartz, *"Linear operators, Part I"*, Interscience Publishers, New York, 1958.

[EHHKMNPR] H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel and R. Remmert, *"Numbers"*, Graduate texts in Mathematics **123**, Springer-Verlag, New York, 1990.

[E] H. G. Eggleston, *"Convexity"*, Cambridge University Press, Cambridge, 1958.

[FGR] A. Fernández, E. García and A. Rodríguez, *A Zelmanov prime theorem for JB^*-algebras*, J. London Math. Soc. **46** (1992), 319-335.

[FP] C. Franchetti and R. Paya, *Banach spaces with strongly subdifferentiable norm.* Bolletino U. M. I. **7-B** (1993), 45-70.

[FR] Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for JB^*-triples*, Duke Math. J. **53** (1986), 139-148.

[F] J. Froelich, *Unital multiplications on a Hilbert space*, Proc. Amer. Math. Soc. **117** (1993), 757-759.

[H] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72-104.

[Hu] T. Huruya, *The normed space numerical index of C^*−algebras*, Proc. Amer. Math. Soc. **63** (1977), 289-290.

[I] L. Ingelstam, *A vertex property for Banach algebras with identity*, Math. Scand. **11** (1962), 22-32.

[I2] L. Ingelstam, *Hilbert algebras with identity*, Bull. Amer. Math. Soc. **69** (1963), 794-796.

[I3] L. Ingelstam, *Non-associative normed algebras and Hurwitz' problem*, Ark. Mat. **5** (1964), 231-238.

[K] R. V. Kadison, *Isometries of operator algebras*, Ann. Math. **54** (1951), 325-338.

[KMR] A. M. Kaidi, J. Martínez and A. Rodríguez, *On a nonassociative Vidav-Palmer theorem* Quart. J. Math. Oxford **32** (1981), 435-442.

[KS] A. M. Kaidi and A. Sánchez, *J−diviseurs topologiques de zéro dans une algèbre de Jordan n.c. normée, In "Non-associative algebra and its applications"* (Oviedo, 1993), 193-197, Kluwer Academic Publishers, Dordrecht-Boston-London, 1994.

[Ka] I. Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399-418.

[Ka2] I. Kaplansky, *"Algebraic and analytic properties of operator algebras"*, Regional Conference Series in Math. **2**, Providence, Amer. Math. Soc., 1970.

[Kau] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503-529.

[L] G. Lumer, *Complex methods and the estimation of operator norms and spectra from real numerical ranges*, J. Functional Analysis **10** (1972), 482-495.

[MM] M. L. El-Mallah and A. Micali, *Sur les algèbres normées sans diviseurs topologiques de zéro*, Boletín de la Sociedad Matemática Mexicana **25** (1980), 23-28.

[M] J. Martínez, *JV−algebras*, Math. Proc. Cambridge Philos. Soc. **87** (1980), 47-50.

[MMPR] J. Martínez, J. F. Mena, R. Payá and A. Rodríguez, *An approach to numerical ranges without Banach algebra theory*, Illinois J. Math. **29** (1985), 609-626.

[N] J. I. Nieto, *Gateaux differentials in Banach algebras*, Math. Z. **139** (1974), 23-34.

[P] T. W. Palmer, *Characterization of C^*-algebras*, Bull. Amer. Math. Soc. **74** (1968), 538-540.

[R] A.Rodríguez, *A Vidav-Palmer theorem for Jordan C^*-algebras and related topics*, J. London Math. Soc. **22** (1980), 318-332.

[R2] A. Rodríguez, *Nonassociative normed algebras spanned by hermitian elements*, Proc. London Math. Soc. **47** (1983), 258-274.

[R3] A. Rodríguez, *Jordan axioms for C^*-algebras*, Manuscripta Math. **61** (1988), 297-314.

[R4] A. Rodríguez, *An approach to Jordan-Banach algebras from the theory of nonassociative complete normed algebras*, Ann. Sci. Univ. "Blaise Pascal", Clermont II, Sér. Math. **27** (1991), 1-57.

[R5] A. Rodríguez, *"Números hipercomplejos en dimensión infinita"*. Discurso de ingreso en la Academia de Ciencias Matemáticas, Físico-Químicas y Naturales de Granada, Granada 1993.

[R6] A. Rodríguez, *Nonassociative normed algebras: Geometric aspects, In "Functional analysis and operator theory"* (Warsaw, 1992), 299-311, Banach Center Publications **30**, Warsaw, 1994.

[R7] A. Rodríguez, *Jordan structures in Analysis. In "Jordan algebras"*, (Oberwolfach, 1992), 97-186, Walter de Gruyter, Berlin-New York, 1994.

[R8] A. Rodríguez, *A Banach space characterization of JB^*-algebras*, Math. Japon. (to appear).

[Si] A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209-214.

[Sa] S. Sakai, *"C^*-algebras and W^*-algebras"*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[S] M. F. Smiley, *Real Hilbert algebras with involution*, Proc. Amer. Math. Soc. **16** (1965), 440-441.

[Sp] I. N. Spatz, *Smooth Banach algebras*, Proc. Amer. Math. Soc. **22** (1969), 328-329.

[Sta] J. G. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), 737-747.

[St] E. Strzelecki, *Metric properties of normed algebras*, Bull. Acad. Pol. Sci. **10** (1962), 341-342.

[St2] E. Strzelecki, *Metric properties of normed algebras*, Studia Math. **23** (1963), 41-51.

[St3] E. Strzelecki, *Power-associative regular real normed algebras*, J. Austral. Math. Soc. **6** (1966), 193-209.

[U] H. Upmeier, *"Symmetric Banach Manifolds and Jordan C^*-algebras"*, North Holland, Amsterdam, 1985.

[V] I. Vidav, *Eine metrische Kennzeichnung der selbstad-jungierten Operatoren*, Mat. Z. **66** (1956), 121-128.

[W] J. D. M. Wright, *Jordan C^*-algebras*, Michigan Math. J. **24** (1977), 291-302.

[WY] J. D. M. Wright and M. A. Youngson, *On isometries of Jordan algebras*, J. London Math. Soc. **17** (1978), 339-344.

[Y] M. A. Youngson, *A Vidav theorem for Banach Jordan algebras*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 263-272.

[Y2] M. A. Youngson, *Hermitian operators on Banach Jordan algebras*, Proc. Edinburgh Math. Soc. **22** (1979), 93-104.

[Z] B. Zalar, *On Hilbert spaces with unital multiplication*, Proc. Amer. Math. Soc. **123** (1995), 1497-1501.

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071-Granada (Spain)

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