Chaos made visual.

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Dedicated to Professor Baltasar Rodríguez Salinas

Abstract

In this paper we show how the main properties of chaos can be fully visualized at the light of a very easy to handle object, the tent function. Although very concrete, this case is representative of a very large number of examples, with more or less the same properties.

Discrete chaotic dynamical systems

If (M, ρ) is a metric space and $F : M \to M$ is a continuous function, F is called chaotic on M, according to R.Devaney, when the following three conditions are satisfied:

1) The function F is topologically transitive, i.e., for each two open sets U, V of M, there is some iteration F^k of F such that $F^k(U) \cap V \neq \phi$. (One can express this fact by saying that the iterations of F mix the open sets of M).

2) There is sensitive dependence with respect to small perturbations, i.e. there is some $\alpha > 0$ such that for each open set U of M, no matter how small, there are two points x, y in U and there exists some k such that $\rho(F^k(x), F^k(y)) > \alpha$.(One calls this property the butterfly effect).

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3) The periodic points of F are dense in M.

One now knows that if M is an interval, as we are going to consider, property (1) implies the other two. But our purpose here is to show how all three properties and some other very interesting ones can be detected in a very easy way which could be appropriate at an elementary mathematical level.

The tent function and its iterations

The tent function $t : [0, 1] \to [0, 1]$ is defined in the following way:

$$t(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x), & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

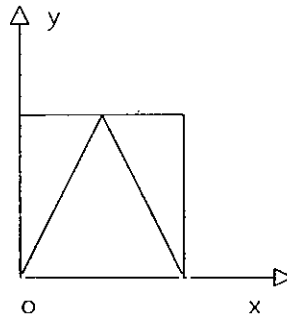Its graph, which will also be denoted by t, is of the following form:

Fig. 1

Many of the considerations we are going to make are based in the shape of the graphs of the iterations t^k of the function t, which have the following form:
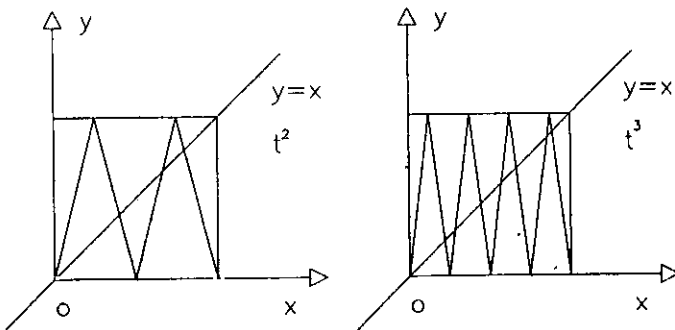

Fig. 2

As we shall show, just by looking at the graphs of these functions, which tend to fill densely the unit square, the properties of chaos appear immediately.

The mixing property for the tent function

The following figure makes evident the mixing property for t. Given any two intervals I and J, we form the rectangle IxJ. It is obvious that infinitely many of the graphs t^k intersect IxJ and this makes clear that the mixing property is present here.
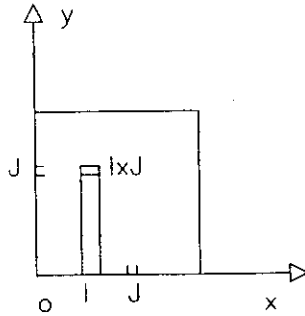

Fig. 3

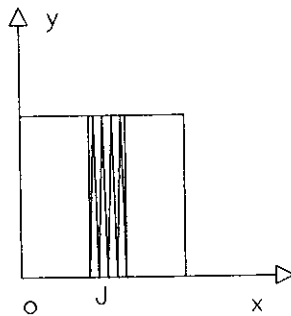## The butterfly effect for the tent function

Look at the following figure

Fig. 4

If we take any non-empty interval J on [0,1] and we consider the infinitely many dyadic points there are in it, it is quite clear that there exist infinitely many t^k with peaks and valleys at these points. If t^m is

one such iteration, $x \in J$ is a peak of t^m and $y \in J$ is a valley, then $\mid t^m(x) - t^m(y) \mid = 1$. This puts in evidence the butterfly effect.

The periodic points of the tent function are dense in $[0, 1]$. For each k the tent function has periodic points of period k

It is clear that a periodic point of t of period k is a fixed point of t^k which is not a fixed point of any previous iteration of t. Also it is quite obvious that the fixed points of t^k are given by the intersections of the graph of t^k with the line y = x. From the shape of the graphs of the iterations one immediately can count:

t has 2 fixed points

t^2 has 2^2 fixed points

. . .

t^k has 2^k fixed points.

Since $2 + 2^2 + \ldots + 2^{k-1} = 2^k - 2 < 2^k$, it is clear t has at least 2 periodic points of period k.

For the fact that the periodic points are dense in $[0,1]$ it suffices to look at the intersections of the graphs of the iterations t^k with y = x. Their projections on $[0,1]$ are clearly dense in $[0,1]$.

The existence of dense orbits

Another interesting fact related to the structure of the iteration of the tent function is the existence of points whose orbits by the tent function are dense in the interval $[0,1]$. It would be good to have also a nice visual proof of such a fact, since the following one requires a little more effort.

We consider the dyadic intervals, i.e. those closed intervals which result from dividing by their midpoints the interval $[0,1]$ and all those

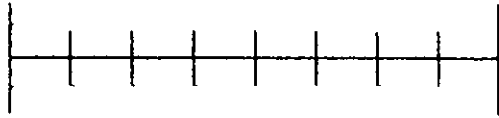rising in this way, as indicated in the following figure

Fig. 5

Each dyadic interval is univocally denoted by a sequence of symbols L (left) and R (right) according to the sequence of its *ancestors*. With this notation the interval RLLRL is the one indicated in the following figure
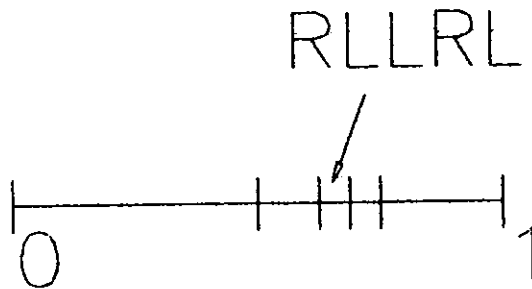

Fig. 6

It is very simple to visualize the effect of applying t to each one of
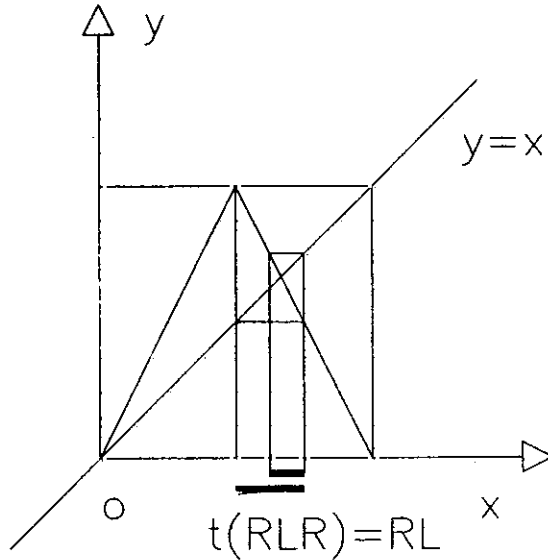
these dyadic intervals:

Fig. 7

*it is clear that $t(I) = t(D) = [0, 1]$

*when the interval has more than one symbol:

if the name of interval starts by L, one takes away that L

if the name of the interval starts by R, one takes away that R

and one changes every other L by R and every other R by L.

For instance, t(LRRLLR)=RRLLR, t(RRLLRRLR)=LRRLLRL

This is a simple consequence of the fact that t is increasing on L (and so it preserves the order on L) and is decreasing on R (and so it reverts the order on R).

Let us now observe that each point of [0,1] is represented by means of an infinite sequence of symbols L and R, so that each partial sequence represents one of the dyadic intervals to which such point belongs. This representation is not unique for the extreme points of dyadic intervals

(except 0 and 1), but this absence of univocity is not going to represent any problem for our purposes here.

Consider now the following point of [0,1]. For clarity's sake we use blanks and parenthesis in order to make more understandable how the sequence of symbols is constructed:

$(LR)(RL)(LLLRRLRR)$(here we write all the variations of the two elements L and R cyclically rotated) (LLL LLR LRL LRR RLL RLR RRL RRR) (here we write all the variations of three elements formed with L and R cyclically rotated) and so on...

If we fix an arbitrary interval, for example, M=LLRLRRLR, it is easy to see that there is a k such that $t^k(p) \in$M. This proves that the orbit of p by t is dense in [0,1].

References

[1]   D. Assaf, IV and S. Gadbois, *Definition of chaos*, letter in American Mathematical Monthly, 99(1992), 865.

[2]   J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, American Mathematical Monthly, 99(1992), 332 − 334.

[3]   R. L. Devaney, *An introduction to chaotic dynamical systems*, (Addison-Wesley, New York, 1989)

[4]   M. Vellekoop and R. Berglund, *On intervals, Transitivity=Chaos*, American Mathematical Monthly, 101(1994), 353 − 355.

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