

## Complemented subspaces of sums and products of copies of $L^1[0, 1]$ .

A. A. ALBANESE and V. B. MOSCATELLI\*

### Abstract

We prove that the direct sum and the product of countably many copies of  $L^1[0, 1]$  are primary locally convex spaces. We also give some related results.

For a while it was an open problem whether a complemented subspace of a countable product of Banach spaces can be written as a product of Banach spaces. This question has been solved in negative by M. I. Ostrovskii [12], but it is still open for  $X^N$  where  $X$  is a classical Banach space. The only countable products of classical Banach spaces whose complemented subspaces have been fully described are:  $\omega; (l^p)^N$ ,  $1 \leq p \leq \infty$ , and  $(c_0)^N$  ([5], [9]) and for these the answer is positive. Moreover, in [1] it was shown that, for  $1 < p < \infty$ ,  $(L^p[0, 1])^N$  is *primary*, i.e. if  $(L^p[0, 1])^N = F \oplus G$ , then either  $F$  or  $G$  is isomorphic to  $(L^p[0, 1])^N$ ; it follows, by reflexivity, that also the direct sum of countably many copies of  $L^p[0, 1]$  is primary. The purpose of this note is to extend these results to the case  $L^1[0, 1]$ , i.e. we will prove that the direct sum and the product of countably many copies of  $L^1[0, 1]$  are also primary spaces. However it remains an open problem whether both the complements  $F$  and  $G$  of a direct decomposition of  $(L^p[0, 1])^N$ , with  $1 \leq p < \infty$ , are isomorphic to a product of Banach spaces. Note that  $(L^p[0, 1])^N$  is isomorphic to  $L^p_{loc}(\mathbf{R})$ ,  $1 \leq p \leq \infty$ .

Our proof is completely different from the one in [1]: the technique of that proof cannot be applied to the case when  $p = 1$ , as it based

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on some special features of the spaces  $L^p[0, 1]$ ,  $1 < p < \infty$ , and on the fact that the Haar-system is an unconditional basis in such spaces (the Haar-system is only a basis of  $L^1[0, 1]$ ; there is no unconditional basis in  $L^1[0, 1]$ !). Actually, in order to obtain our results we will use some known facts about a special class of operators on  $L^1[0, 1]$ , the so-called  $E$ -operators (see [6]), together with a method given in [9].

For other examples of primary non-Banach Fréchet spaces, we refer the reader to [1], [2], [4], [5], [9] and [10].

We will use standard terminology (like e.g. [7], [8] and [9]). In particular, for two locally convex spaces  $E$  and  $F$ , we write  $E \simeq F$  and  $E < F$  to mean respectively that  $E$  is topologically isomorphic to  $F$  or to a complemented subspace of  $F$ . Finally, we put  $L^1 = L^1[0, 1]$ .

## 1 Preliminaries

We recall some definitions and facts which will be used later on.

**Definition 1** ([6]). *A bush is a sequence  $(E_i^n)$ ,  $i = 1, \dots, M_n$ ,  $n = 0, 1, \dots$ , of Lebesgue measurable subsets of  $[0, 1]$  such that*

- (a)  $M_0 = 1$  and  $|E_1^0| > 0$ ,
- (b) for each  $n$   $\cup_{i=1}^{M_n} E_i^n = E_1^0$ ,
- (c) for each  $n$   $E_i^n \cap E_j^n = \emptyset$  if  $i \neq j$ ,
- (d) for each  $n$  and each  $j$ ,  $1 \leq j \leq M_{n+1}$ , there exists an  $i$ ,  $1 \leq i \leq M_n$ , with  $E_j^{n+1} \subset E_i^n$ ,
- (e)  $\max_{1 \leq i \leq M_n} |E_i^n| \rightarrow 0$  as  $n \rightarrow \infty$ .

Here  $|E|$  denotes the Lebesgue measure of a measurable subset  $E \subset [0, 1]$ .

**Definition 2** ([6]). *Let  $T : L^1 \rightarrow L^1$  be a bounded linear operator.  $T$  is called an  $E$ -operator if there exist  $\delta > 0$  and a bush  $(E_i^n)$  with*

$$\frac{1}{|E_1^0|} \int_0^1 \max_{1 \leq i \leq M_n} |T(\chi_{E_i^n})| dx > \delta$$

for each  $n$ , where  $\chi_E$  denotes the characteristic function of a measurable subset  $E \subset [0, 1]$ .

*Enflo and Starbird [6] proved the following useful fact:*

**Theorem 0.** *Let  $T : L^1 \rightarrow L^1$  be a bounded linear operator.  $T$  is an  $E$ -operator if and only if there exists a subspace  $Y$  of  $L^1$  with  $Y$  isomorphic to  $L^1$ , with  $T|_Y$  an isomorphism onto, and with  $TY$  complemented in  $L^1$ .*

**Remark.** (1) If  $T_1 + T_2$  is an  $E$ -operator, then either  $T_1$  or  $T_2$  must be an  $E$ -operator. (2) Obviously, the identity map of  $L^1$  is an  $E$ -operator.

For more about such operators the reader is referred to [6].

## 2 Complemented subspaces of $(L^1)^N$

We denote by  $(L^1)^N$  the product of countably many copies of  $L^1$ . In particular, the space  $(L^1)^N$  can be represented as the projective limit of the Banach spaces  $\prod_{i=1}^n L^1$  with respect to the linking maps

$$p_{n-1,n} : \prod_{i=1}^n L^1 \rightarrow \prod_{i=1}^{n-1} L^1, (x_1, \dots, x_{n-1}, x_n) \rightarrow (x_1, \dots, x_{n-1}),$$

which are surjective. It is clear that, for each increasing sequence  $(k(n)) \subset \mathbf{N}$ , we have  $(L^1)^N = \text{proj}_n \left( \prod_{i=1}^{k(n)} L^1, p_{k(n-1),k(n)} \right)$ , where  $p_{k(n-1),k(n)} = p_{k(n-1),k(n-1)+1} \cdots p_{k(n)-1,k(n)}$ .

Now, let  $p_n : (L^1)^N \rightarrow \prod_{i=1}^n L^1$  be the canonical projection  $(x_i)_i \rightarrow (x_1, \dots, x_n)$ . Then  $p_{n,n+1} p_{n+1} = p_n$ .

Now we are ready to prove

**Theorem 1.** *The space  $(L^1)^N$  is primary.*

**Proof.** Suppose that  $(L^1)^N = F \oplus G$  with  $P$  projection from  $(L^1)^N$  onto  $F$  and  $\ker P = G$ . Put  $Q = I - P$ .

Because  $F$  and  $G$  are closed subspaces of  $(L^1)^N$ , by Lemma 1.1 of [9], we may write  $F = \text{proj}_n(F_n, p_{n-1,n})$  and  $G = \text{proj}_n(G_n, p_{n-1,n})$ , where  $F_n$  (resp.  $G_n$ ) denotes the closure of  $p_n(F)$  (resp.  $p_n(G)$ ) in  $\prod_{i=1}^n L^1$  and  $p_{n-1,n}$  also denotes the restriction of  $p_{n-1,n}$  to  $F_n$  (resp.  $G_n$ ). Moreover, since  $F_n$  (resp.  $G_n$ ) is Banach every map  $p_n P$  (resp.  $p_n Q$ ) factors canonically through  $\prod_{i=1}^{k(n)} L^1$ . Therefore, we can find two sequence  $(k(n))_n$  and  $(h(n))_n$  of integer numbers with  $1 = h(1) < k(1) <$

$h(2) < \dots < h(n) < k(n) < h(n + 1) < \dots$  such that the diagrams

$$\begin{array}{ccc}
 (L^1)^N \xrightarrow{p_{h(n)}^P} F_{h(n)} & & (L^1)^N \xrightarrow{p_{h(n)}^Q} G_{h(n)} \\
 p_{k(n)} \downarrow \nearrow r_n & \text{and} & p_{k(n)} \downarrow \nearrow s_n \\
 \prod_{i=1}^{k(n)} L^1 & & \prod_{i=1}^{k(n)} L^1
 \end{array} \tag{1}$$

commute, where  $r_n$  (resp.  $s_n$ ) denotes the map associated with  $p_{h(n)}^P$  (resp.  $p_{h(n)}^Q$ ).

Put  $E_{0,1} = \prod_{i=1}^{k(1)} L^1$ ,  $E_{n-1,n} = \{0\}^{k(n-1)} \times \prod_{i=k(n-1)+1}^{k(n)} L^1$ , and  $p_{h(n),k(n)} = p_{h(n),h(n)+1} \cdots p_{k(n)-1,k(n)}$ . Then, by (1), as it is easy to verify, we obtain that, for each  $x \in E_{n-1,n}$ ,  $(r_n + s_n)(x) = p_{h(n),k(n)}(x)$ , i.e.

$$r_n + s_n = p_{h(n),k(n)|E_{n-1,n}} : E_{n-1,n} \rightarrow \prod_{i=1}^{h(n)} L^1$$

is the canonical projection ( $\neq 0$  as  $k(n-1) < h(n) < k(n)$ ) and hence is an  $E$ -operator as it follows from Theorem 0. This implies that, by Remark 1, either  $r_{n|E_{n-1,n}}$  or  $s_{n|E_{n-1,n}}$  is an  $E$ -operator for each  $n$ . Therefore, we can suppose that  $r_{n|E_{n-1,n}}$  is an  $E$ -operator for infinite indices  $n$ .

Now, for the sake of simplicity, we assume that for each  $n$   $r_{n|E_{n-1,n}}$  is an  $E$ -operator and  $k(n) = n + 1$ ,  $h(n) = n$ . Thus, we have that the following diagram

$$\begin{array}{ccc}
 \prod_{i=1}^{n+2} L^1 & \xrightarrow{p_{n+1,n+2}} & \prod_{i=1}^{n+1} L^1 \\
 r_{n+1} \downarrow & & \downarrow r_n \\
 F_{n+1} & \xrightarrow{p_{n,n+1}} & F_n
 \end{array}$$

commutes for each  $n$ .

Because  $r_{1|E_{0,1}}$  is an  $E$ -operator, by Theorem 0 there exists a closed subspace  $M_1 \subset E_{0,1} = L^1$  with  $r_{1|M_1}$  an isomorphism into, with  $H_1 =$

$r_1(M_1) < L^1$  and, with  $H_1 \simeq L^1$ . Since  $p_{n,n+1}r_{n+1} = r_n p_{n+1,n+2}$ , it is clear that also the maps

$$p_{n,n+1} : r_{n+1}(M_1) \rightarrow r_n(M_1)$$

are isomorphism onto.

Now, note that  $p_{1,2}(x_1, x_2) = x_1$  and hence  $N = \ker p_{1,2} = F_2 \cap (\{0\} \times L^1)$ . Because  $p_{1,2} : r_2(M_1) \rightarrow r_1(M_1) = H_1$  is an isomorphism onto, there exists a continuous linear map  $A : H_1 \rightarrow \{0\} \times L^1$  with  $r_2(M_1) = \{(x, Ax) : x \in H_1\}$ . It follows that, if  $t_1 : L^1 \rightarrow H_1$  is a projection, then the map  $r : L^1 \times L^1 \rightarrow r_2(M_1)$  defined by  $r(x_1, x_2) = (t_1 x_1, A t_1 x_1)$  is a projection onto  $r_2(M_1)$  with  $\ker r = \{(x_1, x_2) : t_1 x_1 = 0\} = \ker t_1 \times L^1$ . Now, we observe that  $r_2^{-1}(N) \subset E_{1,2}$  and  $r_{2|_{E_{1,2}}}$  is an  $E$ -operator. Then, again Theorem 0 gives that there exists a closed subspace  $M_2 \subset E_{1,2}$  with  $r_{2|_{M_2}}$  an isomorphism into, with  $H_2 = r_2(M_2) < \{0\} \times L^1$  and with  $H_2 \simeq L^1$ . As before, all the maps  $p_{n,n+1} : r_{n+1}(M_2) \rightarrow r_n(M_2)$  are isomorphism onto. If  $I$  is the identity map of  $L^1 \times L^1$  and  $q : \{0\} \times L^1 \rightarrow H_2$  is a projection onto  $H_2$ , we consider the diagram

$$L^1 \times L^1 \xrightarrow{I-r} \ker t_1 \times L^1 \xrightarrow{I-p_{1,2}} \{0\} \times L^1 \xrightarrow{q} H_2.$$

Then the map

$$s = qp_{1,2}(I - r) : L^1 \times L^1 \rightarrow H_2$$

is a projection onto  $H_2$  and  $rs = 0 = sr$ . It follows that  $r_2(M_1) + H_2$  is a closed subspace of  $F_2$ , hence equal to  $r_2(M_1) \oplus r_2(M_2) \simeq L^1 \oplus L^1$ , and the map  $t_2 = r + s$  is clearly a projection from  $L^1 \times L^1$ , hence from  $F_2$ , onto  $r_2(M_1) \oplus r_2(M_2)$  such that  $p_{1,2}t_2 = t_1 p_{1,2}$ .

Continuing in this way, we inductively obtain that for each  $n$  there exists a closed subspace  $X_n = \bigoplus_{i=1}^n r_n(M_i) \simeq \prod_{i=1}^n L^1$  of  $F_n$  and a projection  $t_n : \prod_{i=1}^n L^1 \rightarrow X_n$  such that

$$p_{n,n+1}t_{n+1} = t_n p_{n,n+1} \quad (2)$$

so that  $p_{n,n+1}(X_{n+1}) = X_n$ . Now, if we form the projective limit  $X$  of the spaces  $X_n$  with respect to the restriction maps  $p_{n-1,n} : X_n \rightarrow X_{n-1}$ ,

we see that  $X \subset F, X \simeq (L^1)^N$ . Moreover, by using (2), we see that the map

$$t : (L^1)^N \rightarrow X, x = (x_n)_n \rightarrow (t_n p_n(x_n))_n$$

is a projection onto  $X$ . Therefore, we have the situation  $(L^1)^N < F < (L^1)^N$  which gives, by using Pelczyński's decomposition method, that  $F \simeq (L^1)^N$  and hence the proof is complete.

Moreover

**Proposition 1.** *If  $F < (L^1)^N$  then one of the following cases occurs: (i)  $F$  is a complemented subspace of  $L^1$ . (ii)  $F \simeq \omega \cdot$  (iii)  $F \simeq \omega \oplus X$  where  $X$  is a complemented subspace of  $L^1$ . (iv)  $F'_\beta \simeq (l^\infty)^{(N)}$ , moreover in this case  $F$  contains a complemented copy of  $(l^1)^N$ .*

In order to prove Proposition 1, we need the following Lemma

**Lemma.** *Let  $E$  be a quojection (i.e.,  $E$  is a projective limit of a projective sequence  $(E_n, r_{n,n+1})$  of Banach spaces  $E_n$  and surjective linking maps  $r_{n,n+1} : E_{n+1} \rightarrow E_n$ ). If  $E'_\beta$  has a subspace isomorphic to  $(l^\infty)^{(N)}$ , then  $E$  contains a complemented copy of  $(l^1)^N$ .*

**Proof.** First, we write  $E'_\beta = \text{ind } E'_n$ , where the increasing sequence  $(E'_n)$  of Banach spaces is strict since  $E$  is a quojection.

Now, we assume that  $E'_\beta$  contains a copy of  $(l^\infty)^{(N)}$ . Put  $X_n = l^\infty$  for all  $n$ ,  $(l^\infty)^{(N)} = \bigoplus X_n$ . Then there is a  $k(1)$  such that  $X_1 \subset E'_{k(1)}$  since  $X_1$  is Banach. By Proposition 2.e.8 of [8] it follows that  $E_{k(1)}$  contains a complemented copy of  $l^1$ , i.e. there is a subspace  $G_1$  of  $E'_{k(1)}$  with  $G_1 \simeq l^1$  and a projection  $t_1 : E_{k(1)} \rightarrow G_1$ . We denote by  $(e_j)$  the unit vectors basis of  $G_1$ : because  $E$  is a quojection there is a bounded sequence  $(x_j) \subset E$  such that  $r_{k(1)} x_j = e_j$  (for each  $n$ ,  $r_n$  denotes the map  $r_n : E \rightarrow E_n$  defined by  $r_n x = x_n$ ). Therefore, the map  $s_1 : G_1 \rightarrow E, \sum_{j=1}^\infty a_j e_j \rightarrow \sum_{j=1}^\infty a_j x_j$  is an isomorphism onto  $G_1 = [x_j]$ . Actually,  $s_1 = (r_{k(1)|_{G_1}})^{-1}$ . It follows that the composition map

$$\tilde{t}_1 = s_1 t_1 r_{k(1)} : E \rightarrow E_{k(1)} \rightarrow G_1 \rightarrow \tilde{G}_1$$

is also a projection from  $E$  onto  $\tilde{G}_1 \simeq l^1$ . So,  $E = \tilde{G}_1 \oplus \text{ker } \tilde{t}_1 \simeq l^1 \oplus \text{ker } \tilde{t}_1$  and, hence,  $E'_\beta = \tilde{G}_1 \oplus (\text{ker } \tilde{t}_1)'_\beta$ , where  $F = \text{ker } \tilde{t}_1$  is also a quojection as a quotient of a quojection (see Proposition 3 of [3]).

In order to complete the proof, we observe that  $(l^\infty)^{(N)}$  is also a complemented subspace of  $E'_\beta$  (it is an easy consequence of the fact that  $l^\infty$  is injective (see Proposition 2.f.2 of [8]) and that  $E'_\beta$  is a strict LB-space). Then, we denote by  $p$  a projection from  $E'_\beta$  onto  $(l^\infty)^{(N)}$ : because  $\tilde{G}'_1$  is a Banach subspace of  $E'_\beta$  there is a  $k \in N$  such that  $q_k p \left( \tilde{G}'_1 \right) = 0$ , where  $q_k$  denotes the canonical  $k$ -th projection from  $(l^\infty)^{(N)} = \bigoplus_n X_n$  onto  $\bigoplus_{n>k} X_n$ . By noting that  $q_k p$  is a projection from  $E'_\beta$  onto  $\bigoplus_{n>k} X_n$ , it follows that, for  $x \in \bigoplus_{n>k} X_n$ ,  $x = (id_E - \tilde{t}_1)' x + \tilde{t}'_1 x$  and hence  $x = q_k p x = q_k p (id_E - \tilde{t}_1)' x + q_k p \tilde{t}'_1 x = \tilde{q}_k p (id_E - \tilde{t}_1)' x$ , i.e.  $q_k p (id_E - \tilde{t}_1)' \Big|_{\bigoplus_{n>k} X_n} = id_{\bigoplus_{n>k} X_n}$ . Therefore, the composition map

$$(id_E - \tilde{t}_1)' q_k p : F'_\beta \rightarrow \bigoplus_{n>k} X_n \rightarrow (id_E - \tilde{t}_1)' (\bigoplus_{n>k} X_n) \subset F'_\beta$$

is a projection from  $F'_\beta$  onto  $Y = (id_E - \tilde{t}_1)' (\bigoplus_{n>k} X_n)$  and  $Y \simeq (l^\infty)^{(N)} \subset F'_\beta$ .

Since  $F'_\beta$  contains also a (complemented) copy of  $(l^\infty)^{(N)}$ , as before, we find a subspace  $\tilde{G}_2$  of  $F$  with  $\tilde{G}_2 \simeq l^1$  and a projection  $\tilde{t}_2 : F \rightarrow \tilde{G}_2$  so that  $E = \tilde{G}_1 \oplus F = \tilde{G}_1 \oplus \tilde{G}_2 \oplus \ker \tilde{t}_2 \simeq l^1 \oplus l^1 \oplus \ker \tilde{t}_2$ , where  $\tilde{t}_1 + \tilde{t}_2(id_E - \tilde{t}_1)$  is a projection from  $E$  onto  $\tilde{G}_1 \oplus \tilde{G}_2$ . Iterating this procedure, for each  $n$  we find a subspace  $\tilde{G}_n$  of  $\ker \tilde{t}_{n-1}$  with  $\tilde{G}_n \simeq l^1$  and a projection  $\tilde{t}_n : \ker \tilde{t}_{n-1} \rightarrow \tilde{G}_n$  so that  $E = \bigoplus_{i=1}^n \tilde{G}_i \oplus \ker \tilde{t}_n \simeq \bigoplus_{i=1}^n l^1 \oplus \ker \tilde{t}_n$ . Then, if we form the projective limit  $G$  of the Banach spaces  $\bigoplus_{i=1}^n \tilde{G}_i$  with respect to the maps  $s_n$ , where  $s_n$  is the restriction of the map  $\sum_{i=1}^n \tilde{t}_i (id_E - \tilde{t}_{i-1}) \cdots (id_E - \tilde{t}_1)$  to  $\bigoplus_{i=1}^{n+1} \tilde{G}_i$ , we obtain that  $G \subset E$  and  $G \simeq (l^1)^N$ . Moreover, the map  $s = \sum_{i=1}^\infty \tilde{t}_i (id_E - \tilde{t}_{i-1}) \cdots (id_E - \tilde{t}_1)$  is a projection from  $E$  onto  $G$ . This completes the proof.

### Proof of Proposition 1.

It follows from assumption that  $F$  is a quojection (because it is a quotient of  $(L^1)^N$ ) and  $F'_\beta < (L^\infty)^{(N)} \simeq (l^\infty)^{(N)}$ . Thus Theorem 2.1 of [9] implies that one of the cases (i)  $\div$  (iv) must occur. In particular, when the case (iv) occurs, by the above lemma, we get that  $F$  contains a complemented copy of  $(l^1)^N$ .

**Remark.** We observe that, for a Fréchet space  $E$ , the fact the dual of  $E$  is a countable direct sum of Banach spaces (thus the bidual is a countable product of Banach spaces) does not necessarily imply that  $E$  is a countable product of Banach spaces. The second author and Metafune [11] constructed examples of quojections which are not countable products of Banach spaces but whose duals are countable direct sums of Banach spaces. Thus case (iv) need not imply that the complemented subspace  $F$  is a countable product of Banach spaces.

### 3 Complemented subspaces of $(L^1)^{(N)}$

We denote by  $(L^1)^{(N)}$  the sum of countably many copies of  $L^1$ . In particular, the space  $(L^1)^{(N)}$  can be represented as the inductive limit of the Banach spaces  $\bigoplus_{i=1}^n L^1$  with respect to the linking maps

$$i_{n+1,n} : \bigoplus_{i=1}^n L^1 \rightarrow \bigoplus_{i=1}^{n+1} L^1, (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_n, 0),$$

which are isomorphism into. Clearly, if  $(k(n))$  is an increasing sequence of integer numbers, we have also that  $(L^1)^{(N)} = \text{ind}_n \left( \bigoplus_{i=1}^{k(n)} L^1, i_{k(n+1),k(n)} \right)$ , where  $i_{k(n+1),k(n)} = i_{k(n+1),k(n+1)-1} \dots i_{k(n)+1,k(n)}$ .

Also recall that if  $E$  a complemented subspace of  $(L^1)^{(N)}$ ,  $E$  is an LB-space and hence we may represent it as the strict inductive limit of the Banach spaces  $E_n = E \cap \left( \bigoplus_{i=1}^n L^1 \right)$ .

**Theorem 2.** *The space  $(L^1)^{(N)}$  is primary.*

**Proof.** We suppose that  $(L^1)^{(N)} = F \oplus G$  with  $P$  projection from  $(L^1)^{(N)}$  onto  $F$  and  $\ker P = G$ . Put  $Q = I - P$ . Then,  $F = \text{ind}_n F_n$  (resp.  $G = \text{ind}_n G_n$ ), where  $F_n = F \cap \left( \bigoplus_{i=1}^n L^1 \right)$  (resp.  $G_n = G \cap \left( \bigoplus_{i=1}^n L^1 \right)$ ). Clearly  $(L^1)^{(N)} = \text{ind}_n F_n \oplus G_n$ .

Now, let  $P_1 = P|_{L^1}$  (resp.  $Q_1 = Q|_{L^1}$ ) be. Then there exists an  $h(1) > 1$  such that the maps  $P_1 : L^1 \rightarrow F_{h(1)}$ ,  $Q_1 : L^1 \rightarrow G_{h(1)}$  are bounded and  $F_{h(1)} \oplus G_{h(1)} \supset L^1$ . Put  $P_2 = P|_{\bigoplus_{i=1}^{h(1)+1} L^1}$  and  $Q_2 = Q|_{\bigoplus_{i=1}^{h(1)+1} L^1}$ , we also find an  $h(2) > h(1) + 1$  such that the maps  $P_2 : \bigoplus_{i=1}^{h(1)+1} L^1 \rightarrow F_{h(2)}$  and  $Q_2 : \bigoplus_{i=1}^{h(1)+1} L^1 \rightarrow G_{h(2)}$  are bounded and  $F_{h(2)} \oplus G_{h(2)} \supset \bigoplus_{i=1}^{h(1)+1} L^1$ .



Continuing in this way, we inductively find a sequence  $(h(n))$  of integer numbers with  $h(n) > h(n-1) + 1$ ,  $h(0) = 1$ , such that the maps

$$P_n = P \Big|_{\bigoplus_{i=1}^{h(n-1)+1} L^1} : \bigoplus_{i=1}^{h(n-1)+1} L^1 \rightarrow F_{h(n)}$$

and

$$Q_n = Q \Big|_{\bigoplus_{i=1}^{h(n-1)+1} L^1} : \bigoplus_{i=1}^{h(n-1)+1} L^1 \rightarrow G_{h(n)}$$

are bounded and  $F_{h(n)} \oplus G_{h(n)} \supset \bigoplus_{i=1}^{h(n-1)+1} L^1$  for each  $n \geq 1$ .

Now, we note that the following diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{h(n-1)+1} L^1 \xrightarrow{P_n + Q_n} F_{h(n)} \oplus G_{h(n)} \xrightarrow{q_n} \frac{F_{h(n)} \oplus G_{h(n)}}{F_{h(n-1)} \oplus G_{h(n-1)}} \simeq \frac{F_{h(n)}}{F_{h(n-1)}} \oplus \frac{G_{h(n)}}{G_{h(n-1)}} \\ \tilde{p}_{n-1} \downarrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow p_n \\ \frac{\bigoplus_{i=1}^{h(n-1)+1} L^1}{\bigoplus_{i=1}^{h(n-1)} L^1} \simeq L^1 \qquad \qquad \xrightarrow{j_n} \qquad \qquad \frac{F_{h(n)} \oplus G_{h(n)}}{\bigoplus_{i=1}^{h(n-1)} L^1} \end{array}$$

commutes for each  $n > 1$ , where  $q_n, p_n$  and  $\tilde{p}_{n-1}$  are the quotient maps and  $j_n$  is the canonical isomorphism into. Moreover, for  $n = 1$ ,

$$P_1 + Q_1 = i_{h(1),1} : L^1 \rightarrow \bigoplus_{i=1}^{h(1)} L^1$$

is the canonical inclusion. By Remark 1  $i_{h(1),1}$  is an  $E$ -operator and, hence, either  $P_1$  or  $Q_1$  is an  $E$ -operator. Also,  $j_n \tilde{p}_{n-1}$  is an  $E$ -operator and, as follows from the above diagram,  $p_n q_n (P_n + Q_n)$  is an  $E$ -operator. Then, by Remark 1 either  $p_n q_n P_n$  or  $p_n q_n Q_n$  is an  $E$ -operator, where, clearly,

$$\bigoplus_{i=1}^{h(n-1)+1} L^1 \xrightarrow{P_n} F_{h(n)} \xrightarrow{q_n} \frac{F_{h(n)}}{F_{h(n-1)}} \xrightarrow{p_n} \frac{F_{h(n)} \oplus G_{h(n)}}{\bigoplus_{i=1}^{h(n-1)} L^1}$$

and

$$\bigoplus_{i=1}^{h(n-1)+1} L^1 \xrightarrow{Q_n} G_{h(n)} \xrightarrow{q_n} \frac{G_{h(n)}}{G_{h(n-1)}} \xrightarrow{p_n} \frac{F_{h(n)} \oplus G_{h(n)}}{\bigoplus_{i=1}^{h(n-1)} L^1}.$$

Therefore, we can suppose that  $p_n q_n P_n$  (for  $n = 0$   $q_0$  denotes the identity map of  $F_{h(1)} \oplus G_{h(1)}$ ) is an  $E$ -operator for infinite indices  $n$ .

For the sake of simplicity, we assume that  $p_n q_n P_n$  is an  $E$ -operator for each  $n$ .

Because  $P_1$  is an  $E$ -operator, by Theorem 0 there exists a closed subspace  $M_1 \subset L^1$  with  $P_1|_{M_1}$  an isomorphism into, with  $P_1(M_1) = H_1 \simeq L^1$  and, with  $H_1 < \bigoplus_{i=1}^{h(1)} L^1$ . Also  $p_2 q_2 P_2$  is an  $E$ -operator and, hence, by Theorem 0 there exists a closed subspace  $M_2 \subset \bigoplus_{i=1}^{h(1)+1} L^1$ , with  $M_2 \simeq L^1$ , on which  $p_2 q_2 P_2$  is an isomorphism onto a complemented subspace of  $\frac{F_{h(2)} \oplus G_{h(2)}}{\bigoplus_{i=1}^{h(1)} L^1}$ . Putting  $H_2 = P_2(M_2)$ , we then have  $H_2 \subset F_{h(2)}$ ,  $H_2 \simeq L^1$  and  $p_2 q_2(H_2) < \frac{F_{h(2)} \oplus G_{h(2)}}{\bigoplus_{i=1}^{h(1)} L^1}$ ,  $P_2|_{M_2}$  an isomorphism into and  $H_2 \cap F_{h(1)} = \{0\}$ ,  $H_1 + H_2$  is closed in  $F_{h(2)}$ , hence equal to  $H_1 \oplus H_2 \simeq L^1 \oplus L^1$ .

Continuing in this way, we inductively obtain for each  $n$  a closed subspace  $M_n \subset \bigoplus_{i=1}^{h(n-1)+1} L^1$  with  $P_n|_{M_n}$  an isomorphism into,  $P_n(M_n) = H_n \subset F_{h(n)}$ , with  $H_n \simeq L^1$  and  $p_n q_n(H_n) < \frac{F_{h(n)} \oplus G_{h(n)}}{\bigoplus_{i=1}^{h(n-1)} L^1}$  and  $H_n \cap F_{h(n-1)} = \{0\}$ , with  $H_n + F_{h(n-1)}$  closed subspace of  $F_{h(n)}$ , hence  $H_n + H_{n-1} = H_n \oplus H_{n-1}$  closed subspace of  $F_{h(n)}$ .

Clearly, if we now form the inductive limit  $X$  of the Banach spaces  $X_n = \bigoplus_{i=1}^n H_i$  with respect to the canonical inclusions  $X_n \rightarrow X_{n+1}$ , we see that  $X \subset F$  and  $X \simeq (L^1)^{(N)}$ .

To conclude the proof we have to show that  $X < F$  and again to apply Pelczynski's decomposition method. Then we proceed as follows.

Let  $r_i : \bigoplus_{i=1}^{h(1)} L^1 \rightarrow H_1$  be a projection. Now, recall that  $p_2 q_2(H_2)$  is a complemented subspace of  $\frac{F_{h(2)} \oplus G_{h(2)}}{\bigoplus_{i=1}^{h(1)} L^1}$  and  $p_2 q_2(H_2) \simeq H_2$ . Moreover, the following diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{h(2)} L^1 & \xrightarrow{\bar{p}_2} & \frac{\bigoplus_{i=1}^{h(2)} L^1}{\bigoplus_{i=1}^{h(1)} L^1} \\ s_{2,1} \downarrow & & \nearrow t_{2,1} \\ \bigoplus_{i=h(1)+1}^{h(2)} L^1 & & \end{array}$$

commutes, where  $s_{2,1}$  denotes the canonical inclusion,  $t_{2,1}$  denotes the canonical isomorphism and  $\bar{p}_2$  denotes the quotient map (we note that  $\bar{p}_2|_{F_{h(2)} \oplus G_{h(2)}} = p_2 q_2$ ). Then  $s_{2,1}(H_2) \simeq H_2$  and  $s_{2,1}(H_2) < \bigoplus_{i=h(1)+1}^{h(2)} L^1$ . It follows that there exists a continuous linear map  $A : s_{2,1}(H_2) \rightarrow \bigoplus_{i=1}^{h(1)} L^1$  with  $H_2 = \{(Ay, y) : y \in s_{2,1}(H_2)\}$ . Moreover, if  $r_2 :$

$\bigoplus_{i=h(1)+1}^{h(2)} L^1 \rightarrow s_{2,1}(H_2)$  is a projection, then the map  $\tilde{r}_2 : \bigoplus_{i=1}^{h(2)} L^1 \rightarrow H_2$  defined by  $\tilde{r}_2(x, y) = (Ar_2y, r_2y)$  is a projection onto  $H_2$  with  $\ker \tilde{r}_2 = \bigoplus_{i=1}^{h(1)} L^1 \oplus \ker r_2$ . Now, if  $I$  is the identity map of  $\bigoplus_{i=1}^{h(2)} L^1$ , we consider the diagram

$$\bigoplus_{i=1}^{h(1)} L^1 \oplus \bigoplus_{i=h(1)+1}^{h(2)} L^1 \xrightarrow{I-\tilde{r}_2} \bigoplus_{i=1}^{h(1)} L^1 \oplus \ker r_2 \xrightarrow{I-s_{2,1}} \bigoplus_{i=1}^{h(1)} L^1 \oplus \{0\} \xrightarrow{r_1} H_1.$$

It is immediate to verify that the composition map  $v_2 = r_1(I - s_{2,1})(I - \tilde{r}_2)$  is a projection onto  $H_1$ ,  $v_2\tilde{r}_2 = 0 = \tilde{r}_2v_2$  and  $v_2|_{\bigoplus_{i=1}^{h(1)} L^1} = r_1 = (v_2 + \tilde{r}_2)|_{\bigoplus_{i=1}^{h(1)} L^1}$ . Therefore,  $v_2 + \tilde{r}_2$  is a projection from  $\bigoplus_{i=1}^{h(2)} L^1$  onto  $H_1 \oplus H_2$  which extends  $r_1$ .

Also the diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{h(3)} L^1 & \xrightarrow{\bar{p}_3} & \frac{\bigoplus_{i=1}^{h(3)} L^1}{\bigoplus_{i=1}^{h(2)} L^1} \\ s_{3,2} \downarrow & & \nearrow \nearrow t_{3,2} \\ & & \bigoplus_{i=h(2)+1}^{h(3)} L^1 \end{array}$$

commutes, where  $s_{3,2}$  denotes the canonical inclusion,  $t_{3,2}$  denotes the canonical isomorphism and  $\bar{p}_3$  denotes the quotient map ( $\bar{p}_3|_{F_{h(3)} \oplus G_{h(3)}} = p_3q_3$ ). Then  $s_{3,2}(H_3) \simeq H_3$  and  $s_{3,2}(H_3) < \bigoplus_{i=h(2)+1}^{h(3)} L^1$ . As before, it follows that there exists a continuous linear map (which, for simplicity, we again denotes by  $A$ )  $A : s_{3,2}(H_3) \rightarrow \bigoplus_{i=1}^{h(2)} L^1$  with  $H_3 = \{(Ay, y) : y \in s_{3,2}(H_3)\}$ . Moreover, if  $r_3 : \bigoplus_{i=h(2)+1}^{h(3)} L^1 \rightarrow s_{3,2}(H_3)$  is a projection, then the map  $\tilde{r}_3 : \bigoplus_{i=1}^{h(3)} L^1 \rightarrow H_3$  defined by  $\tilde{r}_3(x, y) = (Ar_3y, r_3y)$  is a projection onto  $H_3$  with  $\ker \tilde{r}_3 = \bigoplus_{i=1}^{h(2)} L^1 \oplus \ker r_3$ . Then, again denoting by  $I$  the identity map of  $\bigoplus_{i=1}^{h(3)} L^1$ , the composition map  $v_3 = (v_2 + \tilde{r}_2)(I - s_{3,2})(I - \tilde{r}_3)$  is a projection from the space  $\bigoplus_{i=1}^{h(3)} L^1$  onto  $H_1 \oplus H_2$  such that  $v_3\tilde{r}_3 = 0 = \tilde{r}_3v_3$ ,  $v_3|_{\bigoplus_{i=1}^{h(2)} L^1} = v_2 + \tilde{r}_2 = (v_3 + \tilde{r}_3)|_{\bigoplus_{i=1}^{h(2)} L^1}$ .

Therefore,  $v_3 + \tilde{r}_3$  is a projection from  $\bigoplus_{i=1}^{h(3)} L^1$  onto  $H_1 \oplus H_2 \oplus H_3$  which extends  $v_2 + \tilde{r}_2$ .

Continuing in this way, for each  $n$  we find a projection  $t_n$  from  $\bigoplus_{i=1}^{h(n)} L^1$  onto  $X_n$  satisfying  $t_n|_{\bigoplus_{i=1}^{h(n-1)} L^1} = t_{n-1}$ . To complete the proof

it is enough to notice that the map  $t : (L^1)^{(N)} \rightarrow X \simeq (L^1)^{(N)}$ , defined by the sequence  $(t_n)$ , is the desired projection.

Moreover

**Proposition 2.** ([9]). *If  $F < (L^1)^{(N)}$  then one of the following cases occurs: (i)  $F$  is a complemented subspace of  $L^1$ . (ii)  $F \simeq \varphi$ . (iii)  $F \simeq \varphi \oplus X$  where  $X$  is a complemented subspace of  $L^1$ . (iv)  $F'_\beta \simeq (l^\infty)^N$ , moreover in this case  $F$  contains a complemented copy of  $(l^1)^{(N)}$ .*

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Dipartimento di Matematica  
Università di Lecce  
C. P. 193  
73100 Lecce, Italy.

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